Optimal Heating on Parallelogram Tori

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Abstract

If N heat sources are placed onto a closed torus and taken away at time $t = 0$, surface temperature will initially spike at these N points and eventually converge to some equilibrium. An interesting question to consider is, what is the optimal placement of heat sources so that surface temperature reaches equilibrium at the quickest rate? Solutions to this question are known for square tori S. We instead explore solutions for various parallelogram tori, represented by lattice grids on which the fundamental domain is a parallelogram $P = A(S)$ for $A \in GL(2, \mathbb{Z})$. In particular, we use Fourier series to show that if $\{(z_n, w_n)_{n=1}^N\}$ is an optimal solution on S, then $\{A(z_n, w_n)_{n=1}^N\}$ is an optimal solution on P, where decay rate is preserved up to a change in constant.

Summary

A number of heat sources are placed onto a closed torus (or equivalently, the surface of a donut) and then taken away. Initially, surface temperature will spike at the points where the heat sources where placed, then eventually level off everywhere to some equilibrium. An interesting question to consider is, what is the optimal placement of heat sources so that surface temperature spreads evenly at the quickest rate? Answers to this question are known for the square torus, obtained by gluing together opposite sides of a square. In this paper, we explore solutions on parallelogram tori, obtained by gluing together opposite sides of various parallelograms. In particular, we show that analogous solutions to those on square tori preserve decay rate up to a constant.

1 Introduction

What is the most efficient way to cook a donut? Consider the mathematical version of this question: what is the optimal placement of heat sources onto a torus so that surface temperature converges to equilibrium at the quickest rate? Intuitively, if we place N identical heat radiators on the surface of a torus and take them away at time $t = 0$, initial temperature will spike at the points where the radiators were placed. As hotter points become cooler and vice versa, the surface of the torus will converge to some equilibrium temperature.

We first introduce some background. All heat flow must follow the governing physical law of the free heat equation, which states that if we denote the temperature of (x, y) at time t as $u(t, x, y)$, then

$$\frac{\partial u}{\partial t} = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

This equation will later allow us to explicitly solve for $u(t, x, y)$ in terms of Fourier series.

Gluing opposite edges of a square together produces a *square torus*, which motivates representing such a surface as an infinite grid of squares of side 2π, such that the point (z, w) is equivalent to $(z + 2\pi, w)$ is equivalent to $(z, w + 2\pi)$. To our question of optimal radiator placement, Pausinger and Steinerberger [1] offer a family of solutions on the square torus. They present a placement of N points which guarantees that the torus reaches an even heat distribution at a rate of at least $e^{-\frac{N}{4}t}$. For a prime number N, an integer p satisfying $\sqrt{N}/2 < p \leq \sqrt{N}$ and an arbitrary $q \in \mathbb{N}$, they define the point set $\{(z_n, w_n)_{n=1}^{N}\}$ by

$$z_n = 2\pi \frac{n}{N} \quad \text{and} \quad w_n = 2\pi \frac{(pn + q) \mod N}{N}. \tag{1}$$

Figure 1 shows an example for $N = 7, p = 2$, and $q = 3$.

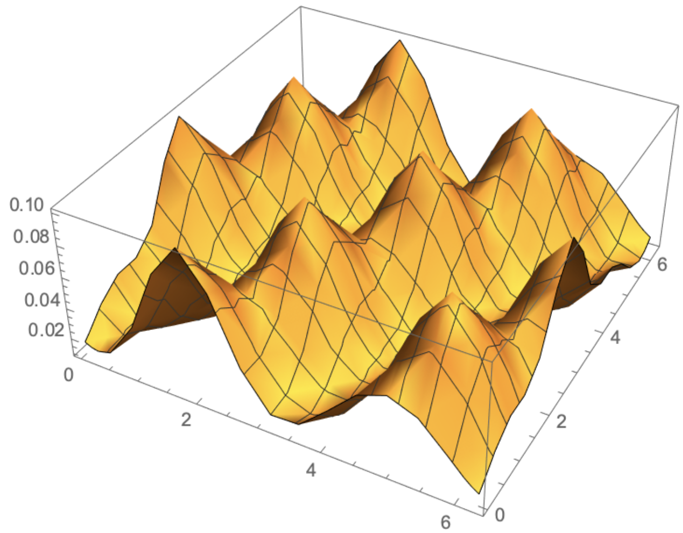

Figure 1: Plot of an optimal point set on S

Theorem 1 (Pausinger and Steinerberger [1]). *For every smooth heat distribution $\phi : \mathbb{T}^2 \to \mathbb{R}$, the initial heat distribution $v(0, z, w)$ given by*

$$v(0, z, w) = \sum_{n=1}^{N} \phi(z - z_n, w - w_n)$$

converges to equilibrium with speed at least

$$\max_{(z,w)\in\mathbb{T}^2} |v(t, z, w) - c_{0,0}| = ce^{-\alpha t},$$

where $c_{0,0}$ is the constant term of the Fourier series of $v(0, z, w)$, c is a constant independent of N and t, and $\alpha = \lfloor\sqrt{N}/2\rfloor^2 + 2\lfloor\sqrt{N}/2\rfloor + 1 \geq N/4$.

The proof of this theorem partly motivates the proof of our main result, Theorem 4. In particular, Pausinger and Steinerberger [1] use Fourier series to model functions of heat distribution and consider their Fourier coefficients. The temperature function $v(t, z, w)$ involves an $e^{-(k^2+m^2)t}$ term which decays slowly when the integers k, m are small. Thus, they attempt to make as many of the early Fourier coefficients $c_{k,m}$ vanish in their placement of heat sources.

In this paper, we consider optimal placement of heat sources on a *parallelogram torus*, which can be represented as an infinite grid of repeating parallelograms. In particular, we show that solutions analogous to those of the form in Eq. (1) give the same decay rate up to a change in constant. In Section 2, we compute the Fourier series of a function periodic over P and use it to derive an explicit formula for the temperature function. In Section 3, we prove our main theorem on the decay rate and optimality of analogous solutions on P. In Section 4, we examine several examples of particularly interesting tori. Finally, we suggest future directions in Section 5.

2 Preliminaries

Each unique torus can be represented as a lattice grid associated with one or more fundamental domains [2]. For instance, a square torus can be obtained by gluing opposite edges of a square S together and is consequently represented by the lattice grid on which S is a fundamental domain. The parallelogram $P = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)S$, however, is also a fundamental domain of the same lattice grid, and thus the torus obtained by gluing opposite edges of P together is equivalent to the square torus. We define a *parallelogram torus* as one represented by the lattice grid on which the fundamental domain is a general parallelogram $P = A(S)$, for matrices $A \in GL(2, \mathbb{R})$. Such a surface is denoted as the torus *associated with the matrix* A.

We build upon the results in [1] by exploring various parallelogram tori. Since the temperature distribution on the torus must be periodic over P at any point in time, it is natural to determine a Fourier series expression for such a function. See Figure 2 for a visualization of one heat source on $P = \left(\begin{smallmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{smallmatrix}\right)S$.

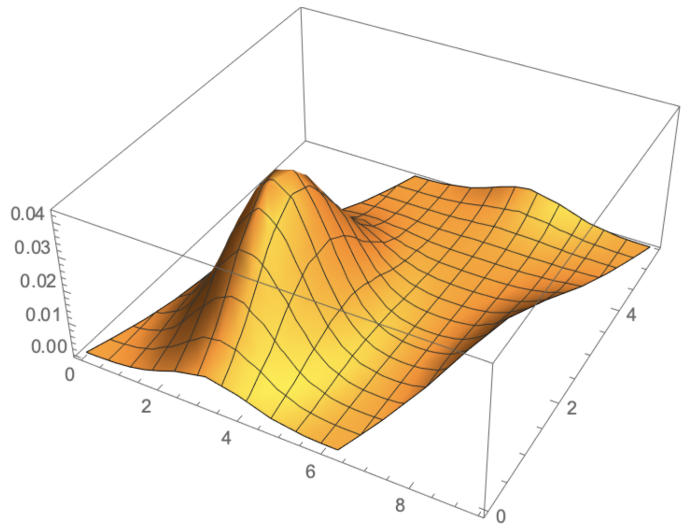

Figure 2: Plot of a single heat source on P

Now, let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If we let a P-periodic function be $f(x, y)$ and a point on S be (z, w), then we have that $f \circ A(z, w)$ is S-periodic, so

$$\begin{aligned} f(x, y) &= f \circ A(z, w) \\ &= \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{i(k,m)\cdot(z,w)} \\ &= \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{i(k,m)\cdot[A^{-1}(x,y)]} \\ &= \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{i(A^T)^{-1}(k,m)\cdot(x,y)}, \end{aligned}$$

and because

$$(A^T)^{-1}\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right) = \frac{1}{\det(A)}\left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right) = \frac{1}{\det(A)}\left(\begin{smallmatrix} dk-cm \\ am-bk \end{smallmatrix}\right),$$

the Fourier series of $f(x, y)$ is

$$f(x, y) = \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{i\frac{1}{\det(A)}[(dk-cm)x+(am-bk)y]}.$$

If we place our N heat sources at $\{(x_n, y_n)_{n=1}^N\}$ and let

$$\phi(x,y) = \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{i\frac{1}{\det(A)}[(dk-cm)x+(am-bk)y]}$$

denote the heat distribution of a single heat source, then our initial temperature distribution is given by

$$\begin{aligned}
u(0,x,y) &= \sum_{n=1}^N \phi(x-x_n, y-y_n) \\
&= \sum_{n=1}^N \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{i\frac{1}{\det(A)}[(dk-cm)(x-x_n)+(am-bk)(y-y_n)]} \\
&= \sum_{n=1}^N \sum_{(k,m)\in\mathbb{Z}^2} \underbrace{a_{k,m} e^{i\frac{1}{\det(A)}[-(dk-cm)x_n-(am-bk)y_n]}}_{:=c_{k,m}} e^{i\frac{1}{\det(A)}[(dk-cm)x+(am-bk)y]}.
\end{aligned}$$

We now compute an expression for the temperature function $u(t,x,y)$ on P from the heat equation:

$$u(t,x,y) = \sum_{(k,m)\in\mathbb{Z}^2} c_{k,m} e^{i\frac{1}{\det(A)}[(dk-cm)x+(ak-bm)y]} e^{-\left(\frac{1}{\det(A)}\right)^2[(dk-cm)^2+(am-bk)^2]t}. \tag{2}$$

3 Optimal Solutions on P

Now, we explore a family of optimal solutions on P. The surface temperature of the torus decays only as quickly as the slowest term in Eq. (2), which must be of the form $ce^{-\alpha t}$, where c is a constant and α is what we call the *decay rate*. We define an *optimal* placement of points to be one whose decay rate α grows linearly with N. Results by Montgomery [3, 4] confirm that N heat sources on a torus can never decay faster than $e^{-\alpha t}$, where α is linear in N.

One possibility to consider is the family of analogous solutions to the square torus in

[1]. More precisely, if $\{(z_n, w_n)\}_{n=1}^N$ is an optimal placement of heat sources on S of the form in Eq. (1), then consider $\{A(z_n, w_n)\}_{n=1}^N$ on P. See, for example, Figure 3, which is a transformation of the solutions in Figure 1 onto $P = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} S$.

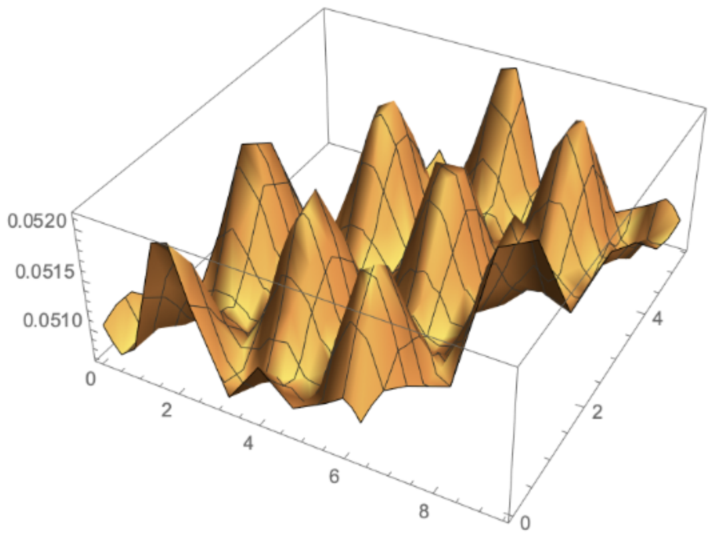

Figure 3: Plot of an optimal point set on P

Question 2. *For what matrices A is the solution $\{A(z_n, w_n)\}_{n=1}^N$ on P optimal up to a change in constant? What are the corresponding decay rates α?*

First, we show that in the Fourier series of the temperature function $u(t, x, y)$ on P, the terms that vanish are the same as the corresponding terms in $v(t, z, w)$ on S. To do this, it is sufficient to show that corresponding coefficients are equivalent.

Theorem 3. *Let $\{(z_w, w_n)_{n=1}^N\}$ be an optimal placement of heat sources on S of the form in Eq. (1). Then the coefficients of the temperature function $v(t, z, w)$ on S are the same as the coefficients of the temperature function $u(t, x, y)$ on $P = A(S)$ with heaters placed at $\{A(z_n, w_n)_{n=1}^N\}$.*

Proof. On S, fix a heat kernel $\phi = \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{2\pi i(k,m)\cdot(z,w)}$ for each source. Then,

$$v(0,z,w) = \sum_{n=1}^{N} \phi(z - z_n, w - w_n) = \sum_{n=1}^{N} a_{k,m} \left(e^{-2\pi i(k,m)\cdot(z_n,w_n)}\right) e^{2\pi i(k,m)\cdot(z,w)}.$$

On P, take the heat kernel $\phi \circ A^{-1}(x,y)$. This has Fourier series

$$\phi \circ A^{-1}(x,y) = \sum_{n=1}^{N} a_{k,m} e^{2\pi i[(A^T)^{-1}(k,m)]\cdot(x,y)}.$$

Let $A(z_n, w_n) = (x_n, y_n)$. Then, we have

$$\begin{aligned}
u(0,x,y) &= \sum_{n=1}^{N} \phi \circ A^{-1}(x - x_n, y - y_n) \\
&= \sum_{n=1}^{N} \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{i[(A^T)^{-1}(k,m)]\cdot(x-x_n,y-y_n)} \\
&= \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} \left(\sum_{n=1}^{N} e^{-i[(A^T)^{-1}(k,m)]\cdot(x_n,y_n)} \right) e^{i[(A^T)^{-1}(k,m)]\cdot(x,y)} \\
&= \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} \left(\sum_{n=1}^{N} e^{-i[(A^T)^{-1}(k,m)]\cdot(x_n,y_n)} \right) e^{i[(A^T)^{-1}(k,m)]\cdot A(z,w)} \\
&= \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} \left(\sum_{n=1}^{N} e^{-i[(k,m)]A^{-1}A\cdot(z_n,w_n)} \right) e^{i[(k,m)A^{-1}]\cdot A(z,w)} \\
&= \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} \left(\sum_{n=1}^{N} e^{-i(k,m)\cdot(z_n,w_n)} \right) e^{i(k,m)\cdot(z,w)} \\
&= v(0,z,w).
\end{aligned}$$

So, regardless of the matrix A, identical coefficients will vanish in $u(0,x,y)$ as in $v(0,z,w)$.

$\square$

However, an important distinction to make is that although corresponding terms vanish, the decay rate may not necessarily remain the same. We explain why in the following example. Let a curve in k and m for which the decay rate remains constant be called a *decay rate level curve*. Then, the decay rate level curves on a square torus S are circles of the form

$k^2 + m^2 = \alpha$. However, the decay rate level curves on a parallelogram torus P are ellipses of the form $\left(\frac{dk-cm}{\det(A)}\right)^2 + \left(\frac{am-bk}{\det(A)}\right)^2 = \alpha$. This follows from Eq. (2). Now, solutions of the form in Eq. (1) on S guarantee that all terms for which $|k|, |m| \le \ell = \lfloor\sqrt{N}/2\rfloor$ disappear, so that the next slowest term decays at a rate corresponding to $(k, m) = (0, \lfloor\sqrt{N}/2\rfloor + 1)$. Plotting this on a km-coordinate plane, we can see this is the case because the next lattice point that $k^2 + m^2 = \alpha$ intersects which lies outside of the box $|k|, |m| \le \lfloor\sqrt{N}/2\rfloor$ is $(0, \lfloor\sqrt{N}/2\rfloor + 1)$. In Figure 4, we have that $N = 36$ so $\ell = 3$ and the next lattice point outside of the blue box $|k|, |m| \le 3$ is $(k, m) = (0, 4)$, corresponding to a decay rate of $\alpha = k^2 + m^2 = 16$.

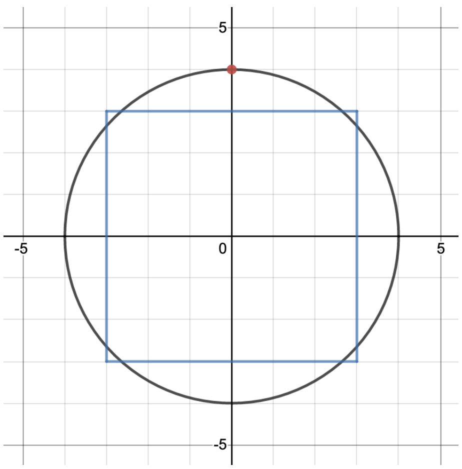

Figure 4: Decay rate level curves on S

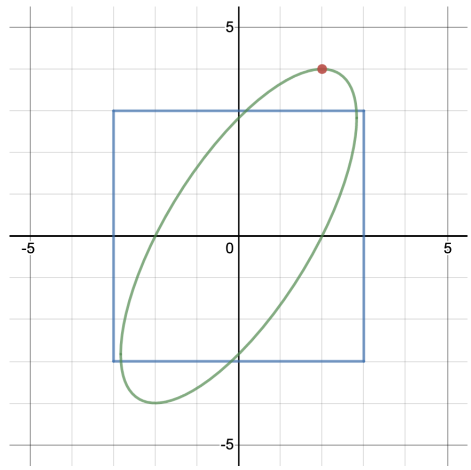

Figure 5: Decay rate level curves on P

On the other hand, consider a torus with associated matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The decay rate level curves now become ellipses (see Figure 5), so that when $N = 36$ and $\ell = 3$, the next lattice point outside of the blue box $|k|, |m| \le 3$ is $(k, m) = (2, 4)$. The slowest term in its Fourier series corresponds to a decay rate of $\alpha = k^2 + (m - k)^2 = 8$. So, since the decay rate of a solution on the square torus will not always match that of the corresponding solution on a parallelogram torus, we need to find new bounds on decay rate.

We now present our main theorem. For a torus generated by the parallelogram $P = A(S)$, consider the point set on P defined by $\{(x_n, y_n)_{n=1}^N\} = \{A(z_n, w_n)_{n=1}^N\}$ where $\{(z_n, w_n)_{n=1}^N\}$

is a solution set of the form in Eq. (1) on S.

Theorem 4. *For tori with associated matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that* $\max\left(\frac{a^2+c^2}{cd+ab}, \frac{b^2+d^2}{cd+ab}\right) \in \mathbb{Q}$ *and for every smooth heat distribution $\phi : \mathbb{T}^2 \to \mathbb{R}$, the initial heat distribution $u(0,x,y)$ given by*

$$u(0,x,y) = \sum_{n=1}^{N} \phi(x - x_n, y - y_n)$$

converges to equilibrium with speed at least $ce^{-\alpha t}$, where

$$\alpha = \min\left(\frac{1}{4(b^2+d^2)}, \frac{1}{4(a^2+c^2)}\right) N$$

and c is a constant independent of N and t.

To prove this theorem, we first introduce several lemmas on the geometry of elliptical decay curves.

Lemma 5. *All ellipses of the form $(dk - cm)^2 + (am - bk)^2 = \alpha(\det(A))^2$ for fixed a, b, c, d and varying α intersect the line $y = \ell + 1$ at points equidistant from $\left(\frac{cd+ab}{b^2+d^2}(\ell+1), \ell+1\right)$ and intersect the line $x = \ell + 1$ at points equidistant from $\left(\ell+1, \frac{cd+ab}{a^2+c^2}(\ell+1)\right)$.*

Proof. It is easy to check this computationally. Taking dx of both sides of the ellipse equation gives

$$\frac{dy}{dx}\left(2c^2y - 2cdx + 2a^2y - 2abx\right) + x(2d^2 + 2b^2) + y(-2cd - 2ab) = 0,$$

so

$$\frac{dy}{dx} = \frac{-x(b^2+d^2) + y(cd+ab)}{-x(cd+ab) + y(a^2+c^2)}.$$

The point at which an ellipse is tangent to the line $y = \ell + 1$ occurs when $-x(b^2+d^2) + y(cd+ab) = 0$ so $y = x\left(\frac{b^2+d^2}{cd+ab}\right)$. The point of tangency is $\left(\frac{cd+ab}{b^2+d^2}(\ell+1), \ell+1\right)$. For ease of notation, let $x' = \frac{cd+ab}{b^2+d^2}(\ell+1)$ so that our two equidistant points are $(x' - r, \ell+1)$ and

$(x' + r, \ell + 1)$ for $r \in \mathbb{R}$. Then,

$$\begin{aligned}\alpha(\det(A))^2 &= \left((d(x'-r) - c(\ell+1))^2 + (a(\ell+1) - b(x'-r))^2\right) \\ &= \left((d(x'+r) - c(\ell+1))^2 + (a(\ell+1) - b(x'+r))^2\right),\end{aligned}$$

which shows they must lie on the same decay rate level curve. An analogous computation can be done for the line $x = \ell + 1$. $\square$

The next lemma allows us to simplify the problem of optimizing lattice points on ellipses.

Lemma 6. *The point (k, m) on the ellipse $(dk - cm)^2 + (am - bk)^2 = \alpha(\det(A))^2$ that gives slowest decay rate outside of $|k|, |m| \leq \ell$ is exactly the point on $y = \ell + 1$ closest to the line $y = \frac{b^2+d^2}{cd+ab}x$ if $b^2 + d^2 > a^2 + c^2$ or the point on $x = \ell + 1$ closest to the line $x = \frac{a^2+c^2}{cd+ab}y$ if $a^2 + c^2 \geq b^2 + d^2$.*

Proof. This follows from Lemma 5 since the point of intersection between $y = \frac{b^2+d^2}{cd+ab}x$ and $y = \ell + 1$ is precisely $\left(\frac{cd+ab}{b^2+d^2}(\ell+1), \ell+1\right)$ and the point of intersection between $x = \frac{a^2+c^2}{cd+ab}y$ and $x = \ell + 1$ is precisely $(\ell + 1, \frac{a^2+c^2}{cd+ab}(\ell+1))$. The lattice point which gives slowest decay rate must then be the one closest to this point. $\square$

Our problem of optimizing lattice points on the ellipse now becomes equivalent to finding the point closest to $y = \frac{b^2+d^2}{cd+ab}x$ or $x = \frac{a^2+c^2}{cd+ab}y$ lying on each lattice line. Now we can finally prove our main result, Theorem 4.

Proof of Theorem 4. First, consider an optimal placement of heat sources $\{(z_n, w_n)_{n=1}^N\}$ on S of the form in Eq. (1). Then, the heat distribution of a single heat source can be expressed as a Fourier series:

$$\phi(z,w) = \sum_{(k,m)\in\mathbb{Z}} a_{k,m} e^{ikz} e^{imw}.$$

Pausinger-Steinerberger [1] state that the initial temperature is then

$$\begin{aligned} v(0,z,w) = \sum_{n=1}^{N} \phi(z-z_n, w-w_n) &= \sum_{n=1}^{N} \sum_{(k,m)\in\mathbb{Z}^2} a_{k,m} e^{ik(z-z_n)} e^{im(w-w_n)} \\ &= \sum_{n=1}^{N} \sum_{(k,m)\mathbb{Z}^2} a_{k,m} e^{-ikz_n} e^{-imw_n} e^{i(kz+mw)} \\ &= \sum_{(k,m)\in\mathbb{Z}^2} \underbrace{a_{k,m} \left(\sum_{n=1}^{N} e^{-ikz_n} e^{-imw_n} \right)}_{:=c_{k,m}} e^{i(kz+mw)}. \end{aligned}$$

The solution to the temperature function is

$$v(t,z,w) = \sum_{(k,m)\in\mathbb{Z}^2} c_{k,m} e^{-(k^2+m^2)t} e^{i(kz+mw)}.$$

We know that the coefficient $c_{k,m}$ vanishes for all $|k|, |m| \leq \lfloor \sqrt{N}/2 \rfloor$ and $(k,m) \neq (0,0)$ [1]. Theorem 3 tells us that the temperature function on P looks like

$$u(t,x,y) = \sum_{(k,m)\in\mathbb{Z}^2} c_{k,m} e^{i \frac{1}{\det(A)} [(dk-cm)x+(ak-bm)y]} e^{-\left(\frac{1}{\det(A)}\right)^2 [(dk-cm)^2+(am-bk)^2]t}$$

where, in particular, the $c_{k,m}$ coefficient is identical to that in $v(t,z,w)$. Our problem then becomes finding the pair (k,m) that gives the slowest decay rate outside of $|k|, |m| \leq \lfloor \sqrt{N}/2 \rfloor$ for various N, which by Theorem 6 is equivalent to finding the lattice point closest to $y = \frac{b^2+d^2}{cd+ab} x$ or $x = \frac{a^2+c^2}{cd+ab} y$ on the line $y = \ell + 1$ or $x = \ell + 1$ respectively.

Without loss of generality, assume $b^2 + d^2 > a^2 + c^2$. Then, the decay rate level curve ellipses always reach $y = \ell + 1$ first, so (k,m) points with slowest decay will lie on the line $y = \ell + 1$. Now, for ease of notation, define $p = \frac{b^2+d^2}{cd+ab}$. In the km-coordinate plane, the line $y = px$ for $p \in \mathbb{Q}$ goes through infinitely many lattice points. Take one such point $(\frac{1}{p}(\ell+1), \ell+1)$: then, consider the sequence of (k, m) points with slowest decay corresponding to increasing N.

The n^{th} point after our selected lattice point must be either $\left(\frac{\ell-n+1}{p} + \left\lfloor \frac{n}{p} \right\rfloor, \ell+1\right)$ or $\left(\frac{\ell-n+1}{p} + \left\lceil \frac{n}{p} \right\rceil, \ell+1\right)$. Thus, we know the decay rate must be the decay rate of either one of those two points. We can show that both points give linear decay rates in N with the same constant. Below, we perform the computation of the decay rate of $\left(\frac{\ell-n+1}{p} + \left\lfloor \frac{n}{p} \right\rfloor, \ell+1\right)$, and the other case is analogous by changing $\left\lfloor \frac{n}{p} \right\rfloor$ to $\left\lceil \frac{n}{p} \right\rceil$.

From the ellipse equation, we know

$$\alpha(\det(A))^2 = (b^2+d^2)k^2 - 2(cd+ab)km + (a^2+c^2)m^2.$$

Plugging in $(k,m) = \left(\frac{\ell-n+1}{p} + \left\lfloor \frac{n}{p} \right\rfloor, \ell+1\right)$ gives that the right hand expression is equal to:

$$(b^2+d^2)\left(\frac{\ell-n+1}{p} + \left\lfloor \frac{n}{p} \right\rfloor\right)^2 - 2(cd+ab)\left(\frac{\ell-n+1}{p} + \left\lfloor \frac{n}{p} \right\rfloor\right)(\ell+1) + (\ell+1)^2(a^2+c^2).$$

Expanding, plugging in $p = \frac{b^2+d^2}{cd+ab}$, completing the square and simplifying gives that the right hand expression is equal to:

$$(\ell+1)^2\left(a^2+c^2 - \frac{(cd+ab)^2}{b^2+d^2}\right) + \frac{1}{b^2+d^2}\left(n(cd+ab) - \left\lfloor \frac{n}{p} \right\rfloor (b^2+d^2)\right)^2.$$

We can now conclude that

$$\alpha(\det(A))^2 \geq (\ell+1)^2\left(a^2+c^2 - \frac{(cd+ab)^2}{b^2+d^2}\right),$$

which simplifies to

$$\alpha \geq \frac{1}{b^2+d^2}(\ell+1)^2 = \frac{1}{b^2+d^2}\left(\left\lfloor \frac{\sqrt{N}}{2} \right\rfloor + 1\right)^2 \geq \frac{N}{4(b^2+d^2)}.$$

As mentioned earlier, the case where $a^2+c^2 \geq b^2+d^2$ can be computed analogously, giving a decay rate of $\frac{N}{4(a^2+c^2)}$. Thus optimality of solutions is preserved by the matrix A, and we have that the decay rate of $\{A(z_n, w_n)_{n=1}^N\}$ is still linear in N. The overall decay follows that of the slowest term, which corresponds to taking the minimum of $\frac{N}{4(a^2+c^2)}$ and $\frac{N}{4(b^2+d^2)}$. $\square$

Remark 7. This theorem is true for all matrices $A \in SL(2, \mathbb{Z})$.

Remark 8. For tori with associated matrices A such that $p = \frac{cd+ab}{b^2+d^2} \in \mathbb{Z}$, this theorem can be checked even more explicitly. Let n be the integer such that $\ell \equiv n \pmod{p}$. Then each point (k, m) corresponds to either an x-coordinate of $\frac{\ell-n}{p}$ or $\frac{\ell-n}{p} + 1$. In the first case, the decay rate becomes

$$\left(\frac{1}{\det(A)}\right)^2 \left(a^2 + c^2 - \frac{(cd+ab)^2}{b^2+d^2}\right)(\ell+1)^2 + \frac{(cd+ab)^2}{b^2+d^2}(n+1)^2,$$

and plugging in $\ell = \left\lfloor \frac{\sqrt{N}}{2} \right\rfloor$ gives our result. In the second case, the decay rate is the same with an extra term of

$$\frac{((cd+ab)(n+1) - (b^2+d^2))^2}{b^2+d^2},$$

and plugging in $\ell = \left\lfloor \frac{\sqrt{N}}{2} \right\rfloor$ gives the same result.

Figure 6 is a comparison of $\max_{(x,y)\in\mathbb{T}^2} u(t, x, y)$ for different point sets on the torus associated with $A = \left(\begin{smallmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{smallmatrix}\right)$ for $N = 7$. The blue graph shows the convergence of our optimal point set which decays at a rate of $e^{-\frac{7}{4}t}$. The gold graph shows the convergence of a random point set which almost always decays at a rate of e^{-t}.

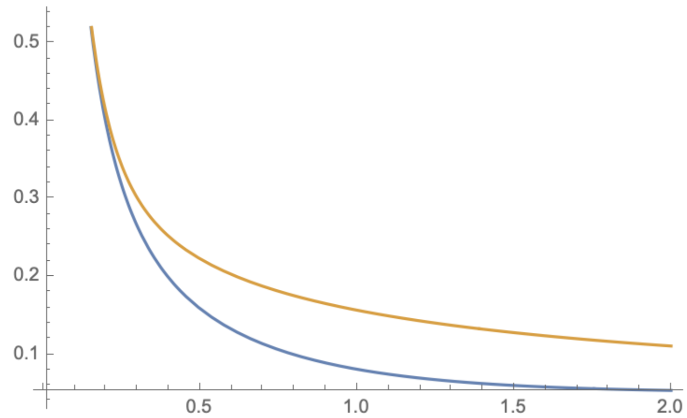

Figure 6: Decay rate of optimal point set vs random

4 Examples

4.1 Rectangular Torus

Consider the torus associated with the matrix $A = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$. Here, the parallelogram forming the fundamental domain of the torus is a rectangle of width a and height d. Even though $\frac{a^2+c^2}{cd+ab}, \frac{b^2+d^2}{cd+ab} \notin \mathbb{Q}$, our main theorem still holds true. That is, the point set $\{(x_n, y_n)_{n=1}^N\}$ such that

$$x_n = a\frac{n}{N} \quad \text{and} \quad y_n = d\frac{(pn+q) \mod N}{N}$$

for prime N, an integer p satisfying $\sqrt{N}/2 < p \leq \sqrt{N}$ and an arbitrary $q \in \mathbb{N}$ is still optimal.

Theorem 9. *For tori with associated matrices $A = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ and for every smooth heat distribution $\phi : \mathbb{T}^2 \to \mathbb{R}$, the initial heat distribution $u(0, x, y)$ given by*

$$u(0, x, y) = \sum_{n=1}^{N} \phi(x - x_n, y - y_n)$$

converges to equilibrium with speed at least

$$\max_{(x,y)\in\mathbb{T}^2} |u(t, x, y) - c_{0,0}| = ce^{-\alpha t}$$

where $c_{0,0}$ is the constant term of the Fourier series of $u(0,x,y)$, c is a constant independent of N and t, and

$$\alpha = \begin{cases} N/4d^2 & \textit{if } d > a \\ N/4a^2 & \textit{if } d \leq a. \end{cases}$$

Proof. The proof is very similar to that of Theorem 4. The decay rate level curves are now ellipses symmetric across the x and y axis with equation

$$\left(\frac{k}{a}\right)^2 + \left(\frac{m}{d}\right)^2 = \alpha.$$

Our $y = \frac{b^2+d^2}{cd+ab}x$ and $x = \frac{a^2+c^2}{cd+ab}y$ lines are just the x and y axes. Thus the point (k,m) with optimal decay rate is $(0,\ell+1)$ if $d > a$ and $(\ell+1,0)$ if $d \leq a$, corresponding to decay rates of

$$\alpha = \begin{cases} N/4d^2 & \text{if } d > a \\ N/4a^2 & \text{if } d \leq a. \end{cases}$$

□

4.2 Eisenstein Integer Lattice Torus

The Eisenstein integer lattice consists of all points of the form $a + b\omega$, where $a, b \in \mathbb{Z}$ and $\omega = e^{2\pi i/3}$. Consider the torus associated with $A = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$, so that the fundamental domain is a parallelogram P whose vertices lie on the Eisenstein integer lattice. In particular, the point x is equivalent to $x+1$ is equivalent to $x - \omega^2$. Then since $a^2 + c^2 = b^2 + d^2 = 1$, Theorem 4 tells us that solutions $\{A(z_n, w_n)_{n=1}^N\}$ will decay at a rate of $e^{-\frac{N}{4}t}$, which is actually equal to the decay rate of the square torus.

4.3 Square-Equivalent Torus

Consider the torus associated with the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which forms the square torus when folded up. It is interesting to see that the transformed point set $\{A(z_n, w_n)_{n=1}^N\}$ on P has a decay rate of $e^{-\frac{N}{8}t}$ as compared to the original point set on S with a decay rate of $e^{-\frac{N}{4}t}$.

5 Future Work

It would be interesting to explore the sequence of tori associated with iterations of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It seems that for the torus associated with A^j, the optimal point set decays at a rate of $e^{-\frac{1}{4(j^2+1)}t}$. This gives rise to the question of what would happen to the decay rate if we took the limit as j goes to infinity.

Moreover, since the square and the parallelogram obtained by transforming the square by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ exist on the same lattice grid, it is possible to shift points on the parallelogram back onto the square. This gives a new family of solutions on S for which it would be insightful to explore the optimality of.

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