On Generalizations of the Double Cap Conjecture

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Abstract

The problem of finding the largest $\frac{\pi}{2}$-avoiding spherical set first appeared in the AMS monthly journal in 1974. H. S. Witsenhausen asked readers to determine $\alpha(n) = \frac{|U|}{|S^{n-1}|}$, where U is the largest $\pi/2$-avoiding set in the n dimensional sphere S^{n-1}, that is, the largest set on the sphere in n dimensions containing no orthogonal vectors. We look at a variation of this problem: determine $\alpha(n,k) = \frac{|U|}{|S^{n-1}|}$, where U is the largest set on the n dimensional sphere S^{n-1} that contains no k mutually orthogonal vectors. We prove a lower bound for $\alpha(n,k)$ for any $n \geq 3$. We specialize to the case $n = k = 3$ by considering spherical sets that avoid three mutually orthogonal vectors in 3D in an attempt to determine $\alpha(3,3)$ which allows for more interesting configurations and visualization. We find the largest spherical subset U not containing three mutually orthogonal vectors in the following cases: (i) U is a large double cap; (ii) U is a centered band; or (iii) U is a wedge shape. We also find a good lower bound for the maximal area in the case that U is the union of two double caps and a centered band. The measures of these sets give a lower bound for $\alpha(3,3)$. Additional computational results suggest a configuration that gives an even better lower bound for $\alpha(3,3)$ as well as a possible relationship to the moving sofa problem.

Summary

The problem of determining the largest set in an n dimensional sphere that does not contain perpendicular vectors has been an open question since 1974. In this paper, we explore a variation of this problem. We find a lower bound for the largest spherical set in n dimensions that does not contain k vectors perpendicular to each other. To gain insight about more general cases, we study subsets of the usual unit sphere that does not contain three vectors perpendicular to each other. We pay special attention to configurations including: (i) large double caps, (ii) centered bands, (iii) wedge shapes, and (iv) combinations of two double caps and a centered band. We found maximums for the first three configuration, and a good lower bound for the maximal value attainable in the fourth configuration. The values from these computations give a lower bound for the largest set in the unit sphere that does not contain three mutually orthogonal vectors. Additional computational results suggest a candidate for the largest set in the unit sphere that does not have three vectors perpendicular to each other. This candidate reminds us of the moving sofa problem.

1 Introduction

In 1974, H. S. Witsenhausen posed the following problem in the American Mathematical Society's monthly journal [1]: Given a sphere in n dimensions, determine the largest subset of points in the sphere that contains no orthogonal vectors; that is, given any two points in the spherical set, the vectors drawn from the origin to the points do not form a right angle [1]. Gil Kalai proposed in 2009 that the largest configuration is the double cap of geodesic radius $\frac{\pi}{4}$ (Figure 1). This is called the double cap conjecture.

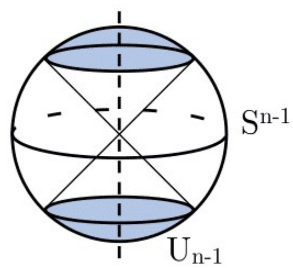


Figure 1: Double Cap of Geodesic Radius $\frac{\pi}{4}$

One approach to solving Witsenhausen's problem is to construct upper bounds for the largest surface area. The upper bound $\frac{1}{n}\,\mathrm{Area}(S^{n-1})$ was first shown by Witsenhausen using probabilistic methods [2]. Frankel and Wilson [3] improved this bound in 1981 using linear algebraic methods and the Ray-Chaudeuri-Wilson theorem, but this improvement was only a small step towards the conjectured value $\alpha(n) = (\sqrt{2} + o(1))^{-n}$ proposed by Kalai [4].

In this paper, we will begin by formalizing the double cap conjecture in Section 2. We then look at variations of the problem in Section 3. We start by modifying Witsenhausen's problem and adding restrictions to divide the problem into cases. We shall discuss the following variations: determine the largest set $U \subset S^{n-1}$ such that no k vectors in U are mutually orthogonal for $k \leq n$; in particular, determine the largest set $U \subset S^2$ such that no three vectors in U are mutually orthogonal. In Section 4, we show a lower bounds for k vectors in n dimensions based on a configuration that works in any dimension. In Sections 5–7, we compare this lower bound to cases in three dimensions, offering specific results as well as speculation backed by computations.

2 The double cap conjecture

Formalization of the conjecture

Let S^{n-1} denote the unit sphere in n dimensional space, that is $S^{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1^2 + ... + x_n^2 = 1\}$. We study subsets of S^{n-1} that have a $\frac{\pi}{2}$-avoiding property, meaning that for any two vectors drawn from the origin to points within the subset, the angle formed between these two vectors is not $\frac{\pi}{2}$. We call such a subset of S^{n-1} a $\frac{\pi}{2}$-avoiding subset, and denote by U a generic $\frac{\pi}{2}$-avoiding spherical subset. The $\frac{\pi}{2}$-avoiding property means that $\forall \vec{x}, \vec{y} \in U$ we have $\vec{x} \cdot \vec{y} \neq 0$. Note that $\vec{x} \perp \vec{y}$ if and only if $\vec{x} \perp -\vec{y}$. One such $\frac{\pi}{2}$-avoiding subset is in the open double cap which consists of a cap of geodesic radius $\frac{\pi}{4}$ and its antipodal copy; that is

$$U_{n-1} := (x_1, x_2, ...x_n) \in S^{n-1} : |x_n| > \frac{1}{\sqrt{2}}.$$

We define $\alpha(n)$ as

$$\alpha(n) := \sup_{U \subset S^{n-1}\ \frac{\pi}{2}\text{-avoiding}} \frac{|U|}{|S^{n-1}|}.$$

Proposition 2.1. *The double cap U_{n-1} of geodesic radius $\frac{\pi}{4}$ is $\frac{\pi}{2}$-avoiding.*

Proof. Let $\vec{x}, \vec{y} \in U_{n-1}$. If a vector is orthogonal to $\vec{p}$, it will be orthogonal to $-\vec{p}$ as well. Therefore, without loss of generality, we assume $x_n, y_n \geq 0$. For all $\vec{x}, \vec{y} \in U_{n-1}$, we show that the angle between them never reaches $\frac{\pi}{2}$, hence U_{n-1} is $\frac{\pi}{2}$-avoiding. Recall that the double cap U_{n-1} is open, and has a geodesic radius of $\frac{\pi}{4}$, thus the angle between the north pole $\vec{N} = (0, ..., 0, 1)$ and $\vec{x}$ is less than $\frac{\pi}{4}$. Similarly, the angle between $\vec{N}$ and $\vec{y}$ is also $< \frac{\pi}{4}$. Therefore, the angle between $\vec{x}$ and $\vec{y}$ is $< \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Hence, U_{n-1} is $\frac{\pi}{2}$-avoiding. $\square$

Kalai conjectured that the double cap U_{n-1} is the solution to Witenhausen's problem.

Conjecture 2.1 (Double Cap Conjecture [2]). *The largest measure of a Lebesgue measurable subset of the unit sphere of* $\mathbb{R}^n$ *containing no pair of orthogonal vectors is attained by* $U_{n-1} \in S^{n-1}$.

By Proposition 2.1, the Double Cap subset has the $\frac{\pi}{2}$-avoiding property, therefore we know that $\frac{|U_{n-1}|}{|S^{n-1}|} \leq \alpha(n)$, and as a restatement of the Double Cap conjecture we predict that $\frac{|U_{n-1}|}{|S^{n-1}|} = \alpha(n)$.

Existing Upper Bounds

Attempts to find the largest $\frac{\pi}{2}$-avoiding subset U led to finding various upper bounds. The bound proposed by Witsenhausen in 1974 used probabilistic methods of proof, which will be our approach as well (presented in detail in Section 6).

Proposition 2.2 (Witsenhausen's Upper Bound [1]). *Let* U *be any subset of* S^{n-1} *such that no two vectors in* U *are orthogonal. For a sphere in* n*-dimensional space,*

$$\alpha(n) = \frac{|U|}{|S^{n-1}|} \leq \frac{1}{n} \qquad\qquad \textit{for } n \geq 2.$$

Proof. For $n \geq 2$, choose a rotation ρ around the origin at random. The $\frac{\pi}{2}$-avoiding subset U must not contain orthogonal vectors, therefore the number of standard orthonormal vectors $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$ contained in $\rho(U)$ is at most 1. We define the indicator function $\chi_{\vec{e}_i}$ as

$$\chi_{\vec{e}_i} = \begin{cases} 0 & \vec{e}_i \notin \rho(U) \\ 1 & \vec{e}_i \in \rho(U). \end{cases}$$

We use expected value to derive the upper-bound. We know that

$$\mathbb{E}(\# \text{ of elements of } \{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\} \text{ in } \rho(U)) \leq 1.$$

Therefore for each $n \geq 2$

$$\begin{aligned} 1 \geq \mathbb{E}(\chi_{\vec{e}_1 \in \rho(U)} + ... + \chi_{\vec{e}_n \in \rho(U)}) =& \Pr(\vec{e}_1 \in \rho(U)) + ... + \Pr(\vec{e}_n \in \rho(U)) \\ =& \frac{|U|}{|S^{n-1}|} + \frac{|U|}{|S^{n-1}|} + ... + \frac{|U|}{|S^{n-1}|} = n\frac{|U|}{|S^{n-1}|}, \end{aligned}$$

or in other words $\frac{|U|}{|S^{n-1}|} \leq \frac{1}{n}$. $\square$

Proposition 2.3. *For $n = 2$, Witsenhausen's upper bound $\alpha \leq \frac{1}{n}$ is sharp, and the double cap conjecture is true.*

Proof. The measure of a curve created by an open geodesic radius is the same as the measure of the same curve created by a closed geodesic radius because single points have measure zero. Considering a unit circle, two caps each of $\frac{\pi}{2}$ radians have a cumulative arc length of π. The perimeter of the circle is 2π, therefore the two caps are $\frac{1}{2}$ the perimeter of the circle, and $\frac{1}{2} \leq \alpha(2)$. By Witsenhausen's Upper Bound, in two dimensions, $\alpha(2) \leq 1/2$, therefore, $\alpha(2) = \frac{1}{2}$. Therefore, for $n = 2$ the upper bound is sharp, and the double cap conjecture is true. $\square$

For $n \geq 3$ Witsenhausen's upper bound becomes less accurate. If the double cap subset is the largest subset, then we have that $\alpha(3) = 1 - \frac{1}{\sqrt{2}} \approx .2928$, while the upper-bound only states it must be less than 1/3 [2]. As n increases, $\frac{1}{n\ \alpha(n)}$ increases exponentially, assuming that the double cap is the largest configuration. Additionally, smaller bounds than the one supplied by Witsenhausen have also been found that are more accurate, notably that of Frankl and Wilson which states $\alpha(n) \leq (1 + o(1))(1.13)^{-n}$ [4]. This bound is of similar order to the conjectured value $\alpha(n) = (\sqrt{2} + o(1))^{-n}$ proposed by Kalai [2] and is a step towards proving the conjecture. For the remainder of the paper, we focus on a variation of the problem.

3 A Variant of Witsenhausen's problem

As mentioned in section 1, the double cap configuration is a proposed solution to the problem of determining the largest space on a n-dimensional sphere that has the property of containing no two orthogonal vectors. We wish to determine the largest space on a sphere that has the property of containing no k mutually orthogonal points for $k \leq n$. We define $\alpha(n,k)$ where $k \leq n$ to be the largest subset of points in the $(n-1)$-sphere in n dimensions which does not contain k mutually orthogonal vectors. For example, Witsenhausen's problem asks for the largest value for $\alpha(n,2)$. The upper bound for $\alpha(n,k)$ is $\frac{k-1}{n}$.

Remark. *The upper-bound $\alpha(n,k)$ is $\frac{k-1}{n}$ can be proven by probabilistic methods similar to the proof of Proposition 2.2.*

A specialization in three dimensions

The variation of the conjecture concerns all dimensions making the problem hard to visualize and find concrete examples to test. Therefore, we specialize to three dimensional spheres in order to determine the largest set that cannot contain three mutually orthogonal vectors, that is we wish to find $\alpha(3,3)$. Because we are only looking at three dimensional space, it is possible to visualize proposed configurations. Additionally because there are fewer constraints, Witsenhausen's original problem becomes simpler. To attempt to determine $\alpha(3,3)$, we impose restrictions to simplify the problem further and break it down into cases. Note that we are asking what the largest subset $U \subset S^2$ is, such that no three vectors in U are mutually orthogonal. We analyze these cases in Section 5-8, but first we will begin by finding a general lower bound that works in all dimensions.

4 A Lower Bound

In order to construct a lower bound, we must find a configuration that works in n dimensions. Building upon the double cap conjecture, we know that a single double cap cannot contain two orthogonal vectors, and so for a larger k, we can consider the addition of more double caps to the space. We consider $k-1$ of the original double caps U_2 on the sphere, rather than a single double cap, as shown in Figure 2 for three dimensions.

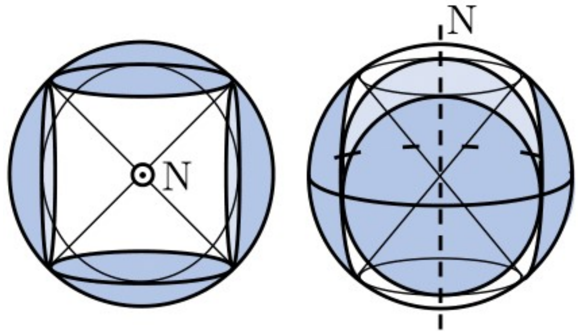


Figure 2: Two double caps in two dimensions

Proposition 4.1. *The disjoint union of $k-1$ double caps U_{n-1} cannot contain k mutually orthogonal vectors, that is,* $\alpha(k,n) \geq (k-1)\frac{|U_{n-1}|}{|S^{n-1}|}$

Proof. Assume to the contrary that the disjoint union of $k-1$ double caps U_{n-1} contains n mutually orthogonal vectors. By the pigeonhole principle, in order to fit k mutually orthogonal vectors within $k-1$ double caps, one of the double caps must contain two vectors orthogonal to each other. We know that U_{n-1} does not contain two vectors orthogonal to each other, which contradicts our assumptions. Therefore, we conclude that a given configuration of $n-1$ double caps in n dimensions fulfills the necessary conditions. If they were to have a larger geodesic radius, the combined geodesic diameters would add to over 2π, hence this is the largest area that the caps can have without beginning to overlap. With a geodesic radius of $\frac{\pi}{4}$ we know we can place the caps such that no two caps will intersect. Because we have an empty intersection,

$$\alpha_(k,n) \geq \frac{|\bigcup_{j=1}^{k-1} U_{n-1}^j|}{|S^{n-1}|} = \frac{\sum_{j=1}^{k-1} |U_{n-1}^j|}{|S^{n-1}|} = (k-1)\frac{|U_{n-1}|}{|S^{n-1}|}. \qquad \square$$

5 Large Double Cap and Complementary Band

To find lower bounds $\alpha(3,3)$, we start by imposing restrictions to simplify the problem. Note that we are asking what the largest subset $U \subset S^2$ is, such that no three vectors in U are mutually orthogonal. Recall that we define $\alpha(n,k)$ where $k \leq n$ to be the largest volume we can have in n dimensions such that it will never contain k mutually orthogonal vectors. We start by looking at two related cases with distinctive restrictions.

Large Double Cap

In the first case, we assume U_{dc} must be a double cap subset of some height h. Under these assumptions we find $\alpha(3,3)$. Note that this double cap is not equivalent to U_2, the double cap from Conjecture 2.1.

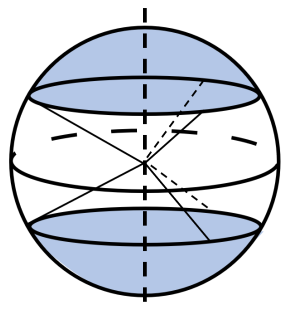


Figure 3: Large Double Cap

Definition 5.1. *Double caps* are sets of points of the form

$$U_{dc}(h) := \{(x,y,z) \in S^2 : |z| > 1-h\},$$

where h is the height of the cap.

Proposition 5.1. *The set $U_{dc}\left(1-\frac{1}{\sqrt{3}}\right) \subset S^2$ is the largest set of points forming a double cap that does not contain three mutually orthogonal points. Moreover, we have $\alpha(3,3) \geq 1-\frac{1}{\sqrt{3}} \approx 0.423$.*

Proof. Fix $\vec{v} = (a,b,c) \in U_{dc}\left(1-\frac{1}{\sqrt{3}}\right)$. Then, $|c| > \frac{1}{\sqrt{3}}$ and we assume $c > \frac{1}{\sqrt{3}}$ without loss of generality. We assume $a = 0$ without loss of generality because we can rotate S^2 around the z axis. Let A be the intersection of the plane perpendicular to $\vec{v}$ and the upper cap

$$A := \{(a_1,a_2,a_3) \in U_{dc}(1-1\sqrt{3}) : \vec{v}\cdot(a_1,a_2,a_3) < 0\}.$$

In order to show that no three points in the cap are mutually orthogonal, it suffices to show no two points in the arc A are orthogonal.

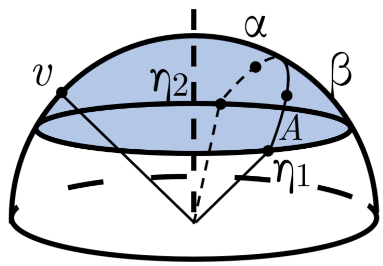


Figure 4: Visualization of points v, η_1, η_2, α, β and arc A

Let $\vec{\eta_1} = (x, y, 1/\sqrt{3})$ and $\vec{\eta_2} = (-x, y, 1/\sqrt{3})$ be points on the boundary of the cap orthogonal to $\vec{v}$. Let $\alpha, \beta \in A$ be arbitrary points, such that $\angle \alpha O \beta < \angle \eta_1 O \eta_2$ where O is the origin (Figure 4). We seek to find whether $\vec{\eta_1} \cdot \vec{\eta_2} > 0$ because if so, it implies that α and β are not orthogonal.

We solve the following system of equations for x and y

$$\begin{cases} \vec{v} \cdot \vec{\eta_1} = by + \frac{c}{\sqrt{3}} = 0 \\ x^2 + y^2 = \frac{2}{3}. \end{cases}$$

We get

$$\vec{\eta_1} \cdot \vec{\eta_2} = \frac{2c^2}{3b^2} - \frac{1}{3} = \frac{2}{3}\left(-1 + \frac{1}{1-c^2}\right) - \frac{1}{3},$$

which by the assumption $c > \frac{1}{\sqrt{3}}$ implies $\vec{\eta_1} \cdot \vec{\eta_2} > 0$. Therefore, because $\angle \alpha O \beta < \angle \eta_1 O \eta_2$, we determine that α and β cannot be orthogonal to each other, thus the cap cannot contain three mutually orthogonal vectors.

By setting $h = 1 - 1\sqrt{3}$, we find $\dfrac{|U_{dc}(1 - 1\sqrt{3})|}{|S^2|} = 1 - 1\sqrt{3} \approx 0.423$, and therefore $\alpha(3,3) \geq 1 - \frac{1}{\sqrt{3}}$. □

Centered Band

In the second case, we look at a centered band.

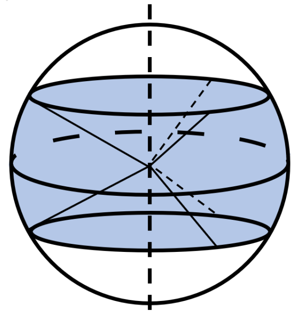


Figure 5: Centered Band

Definition 5.2. *Centered Bands* are sets of points of the form

$$U_b(h) := \{(x, y, z) \in S^2 : |z| < h\},$$

where $2h$ is the height of the band.

Proposition 5.2. *The set $U_{band}(1\sqrt{3}) \subset S^2$ is the largest set of points forming a centered band that does not contain three mutually orthogonal points. Moreover, we find that $\alpha(3,3) \geq \dfrac{1}{\sqrt{3}} \approx 0.577$.*

Proof. Take the threshold case where one vector is on the top boundary and two vectors are on the bottom boundary. Because mutually orthogonal vectors $(\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}})$ are in the band if $h > \frac{1}{\sqrt{3}}$, we know $h \leq \frac{1}{\sqrt{3}}$. Next, we wish to show that if $U_b(h)$ contains three mutually orthogonal vectors, then $h > 1/\sqrt{3}$. Fix three mutually orthogonal vectors $\vec{p}, \vec{q}, \vec{r}$ on the sphere. By taking $-\vec{p}$ instead of $\vec{p}$, $-\vec{q}$ instead of $\vec{q}$ and $-\vec{r}$ instead of $\vec{r}$ if necessary, we assume without loss of generality that $p_1, q_1, r_1 \geq 0$. We are then able to rotate the sphere such that two points are on the boundary of the band of height $z = -h$ and the third point is of on the plane of height $z = h$. If we solve for h, we find that $h = \frac{1}{\sqrt{3}}$. The band is open, therefore if $U_b(h)$ contains $\vec{p}, \vec{q}$ and $\vec{r}$, then $h > 1/\sqrt{3}$. Taking the contrapositive, we know that if $h \leq 1/\sqrt{3}$, then it cannot contain three mutually orthogonal vectors. $\square$

6 Wedges

In the next case, we assume U be a union of wedge shapes (Figure 6).

Definition 6.1. *Wedges* are sets of points of the form

$$U_\gamma = \{(x,y,z) \in \mathbb{R}^3 : \frac{1}{x^2+y^2}(x,y) \in \gamma\},$$

such that $\frac{|\gamma|}{S^1} = \frac{|U\gamma|}{S^2}$, where x and y not both 0 and $\gamma \subset S^1 \subset \mathbb{R}^2$.

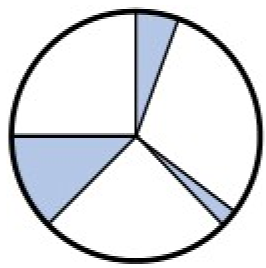


Figure 6: Arbitrary Wedges (top view)

It can be proven that the wedge shape will not contain three mutually orthogonal points, and that $\alpha(3,3)$ with these restrictions will never reach more than $\frac{1}{2}$.

An Upper Bound for the Wedges Case

Lemma 6.1 (Upper Bound for $\frac{|U\gamma|}{S^2}$)**.** *Given a wedge shaped set $U\gamma$, $\frac{|U\gamma|}{S^2} \leq \frac{1}{2}$.*

Proof. Suppose $U\gamma$ is any wedge shaped set not containing three mutually orthogonal vectors. We randomly and uniformly choose a rotation σ of the circle S^1 around the origin. Let

$$x = |\{(0,1),(1,0)\} \cap \sigma(\gamma)|,$$

then $\mathbb{E}(x) \leq 1$. Using methods similar to that of Witenhausen's proof of Proposition 2.2, we also have that

$$\begin{aligned} 1 \geq \mathbb{E}(x) =& \mathbb{E}(\chi_{(1,0)\in\gamma} + \chi_{(0,1)\in\gamma}) = \mathbb{E}(\chi_{(1,0)\in\gamma}) + \mathbb{E}(\chi_{(0,1)\in\gamma}) \\ =& \frac{|\gamma|}{|S^1|} + \frac{|\gamma|}{|S^1|} = 2\frac{|\gamma|}{|S^1|} \end{aligned}$$

or in other words, $\frac{|\gamma|}{|S^1|} \leq \frac{1}{2}$. Hence, from $\frac{|\gamma|}{|S^1|} = \frac{|U\gamma|}{|S^2|}$, we get that $\frac{|U\gamma|}{|S^2|} \leq \frac{1}{2}$. $\square$

An Exact Value for the Wedges Case

Lemma 6.2. *The upper bound for* $\frac{|U\gamma|}{S^2}$ *is attainable by a set* $Q \subset S^2$ *such that*

$$Q = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : p_1 \times p_2 > 0\}.$$

Proof. Assume to the contrary that Q contains three mutually orthogonal vectors, $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$, and $\vec{z} = (z_1, z_2, z_3)$. Because they are mutually orthogonal, we know

$$\begin{cases} x_1y_1 + x_2y_2 + x_3y_3 = 0 \quad (1) \\ x_1z_1 + x_2z_2 + x_3z_3 = 0 \quad (2) \\ y_1z_1 + y_2z_2 + y_3z_3 = 0 \quad (3). \end{cases}$$

Additionally, because p_1 and p_2 are always either both negative or both positive, we can assume without loss of generality that x_1, x_2, y_1, y_2, z_1, and z_2 are all positive. Therefore in order for the equations to be satisfied, x_3y_3, x_3z_3, and y_3z_3 must be negative. If x_3 is positive, then both y_3 and z_3 must be negative to satisfy Equations 1 and 2, but this leads to a contradiction. If both y_3 and z_3 are negative, then their product will be positive, leaving Equation 3 unsatisfied. Similarly, if x_3 is negative, a contradiction also follows. Therefore, Q cannot contain three mutually orthogonal points, and the configuration of Q is a valid configuration of U. □

7 A Union of Double Caps and a Band

Now that we have established several cases, we consider the union of two of these cases: two of the original double caps U_2 (see Section 4), and a band. Note, this band is a different height than the one found in Section 5).

Definition 7.1. The union of *Two double caps sets and a band* are the sets of points

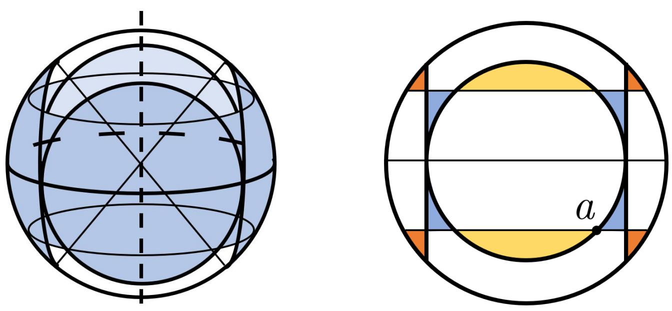
$U_{dcb}(h) = U_2^1 \cup U_2^2 \cup B(h)$ such that

$$U_2^1 := \left\{(x, y, z) \in S^2 : |x| > \frac{1}{\sqrt{2}}\right\}$$

$$U_2^2 := \left\{(x, y, z) \in S^2 : |y| > \frac{1}{\sqrt{2}}\right\}$$

$$B(h) := \left\{(x, y, z) \in S^2 : |z| < h\right\}.$$

We are looking for the largest configuration $U_{dbc}(h)$ that does not contain three mutually orthogonal vectors. We know $h < \frac{1}{2}$ because for $h > \frac{1}{2}$, $U_{dcb}(h)$ contains mutually orthogonal vectors $(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$. We know by calculation that $U_{dcb}(1/2)$ also contains three mutually orthogonal vectors, and while we don't know the optimal $B(h)$, we choose a lower bound $h \geq \sqrt{2}-1$ such that $U_{dcb}(\sqrt{2}-1)$ does not contain three mutually orthogonal vectors. To prove this, we reference the projection of the space onto the (y, z) plane as seen in Figure 7b.



(a) 3D Visualization (b) 2D Projection

Figure 7: The union of twodouble caps and a band. Yellow denotes $U_2^1 \setminus B$, Orange denotes $U_2^2 \setminus B$ and blue denotes $B \setminus (U_2^1 \cup U_2^2$.

Proposition 7.1. *The largest set of points $U_{dcb}(h) \subset S^2$ forming a union of two original double caps and a band that does not contain three mutually orthogonal points has a lower bound for the height of the band at $h > \sqrt{2}-1$. Moreover, we have $\alpha(3, 3) \gtrapprox 0.594$.*

Proof. Assume to the contrary that $U_{dcb}(\sqrt{2}-1)$ contains three mutually orthogonal vectors $\vec{p}, \vec{q}$ and, $\vec{r}$. By taking $-\vec{p}$ instead of $\vec{p}$, $-\vec{q}$ instead of $\vec{q}$ and $-\vec{r}$ instead of $\vec{r}$ if necessary, we assume without loss of generality that $p_1, q_1, r_1 \geq 0$. For the sake of brevity, we refer to $B(\sqrt{2}-1)$ as B. We know the band can contain at most two orthogonal vectors, therefore one vector must be in $(U_2^1 \cup U_2^2) \setminus B$. Symmetry allows us to assume without loss of generality that $\vec{p} \in U_2^1$ and $p_3 < 0$.

The double caps, $U_2^1 \cup U_2^2$, contain at most two orthogonal vectors. Hence, $B \setminus (U_2^1 \cup U_2^2)$ must contain at least one of the vectors $\vec{q}$ and $\vec{r}$. Without loss of generality we assume $\vec{q} \in B \setminus (U_2^1 \cup U_2^2)$. The plane perpendicular to $\vec{p}$ does not intersect $(B \setminus (U_2^1 \cup U_2^2)) \cap \{x, y, z \in S^2 : z < 0\}$, therefore $\vec{q}$ is in the upper half of the band.

The plane perpendicular to an arbitrary $\vec{p}$ will not intersect $(B \setminus (U_2^1 \cup U_2^2)) \cap \{x, y, z \in S^2 : z < 0\}$ twice. Thus, the final vector $\vec{r}$ must be in U_2^2. To determine the location of $\vec{r}$, we consider all vectors $\vec{p_\varepsilon} = \left(\frac{1}{\sqrt{2}}, \sqrt{\varepsilon\sqrt{2}-\varepsilon^2}, \varepsilon - \frac{1}{\sqrt{2}}\right)$ on the boundary of U_2^1 wtih $\varepsilon, 0 \leq \varepsilon \leq 1/\sqrt{2} - \sqrt{2} - 1$. If we take an arbitrary $\vec{p}$ in U_2^1 and the circle that contains all points orthogonal to $\vec{p}$ on the sphere, as we shift $\vec{p}$ to the right, the arc length between the intersects of the circle and the upper boundaries of B and U_2^2 decreases. Because we take $\vec{p_\varepsilon}$ to be the vector shifted as far right as possible for some vertical shift ε, we get the smallest arc length between the intersects of the circle and the upper boundaries of B and U_2^2. We assign the points of intersection to $\vec{q_\varepsilon} = (q_1, q_2, \sqrt{2}-1)$ and $\vec{r_\varepsilon} = (r_1, \frac{1}{\sqrt{2}}, r_3)$, respectively, where the values of q_1, q_2, r_1, r_3 are given in terms of ε by the system of equations

$$\begin{cases} \frac{1}{\sqrt{2}} q_1 + q_2\sqrt{\varepsilon\sqrt{2}-\varepsilon^2} + \left(\varepsilon - \frac{1}{\sqrt{2}}\right)\left(\sqrt{2}-1\right) = 0 \\ q_1^2 + q_2^2 + (\sqrt{2}-1)^2 = 1 \end{cases} \tag{1}$$

$$\begin{cases} \frac{1}{\sqrt{2}} r_1 + \frac{\sqrt{\varepsilon\sqrt{2}-\varepsilon^2}}{\sqrt{2}} + (\varepsilon - \frac{1}{\sqrt{2}}) r_3 = 0 \\ r_1^2 + r_3^2 = 1/2. \end{cases} \tag{2}$$

If the angle between $\vec{q_\varepsilon}$ and $\vec{r_\varepsilon}$ is obtuse, i.e. $\vec{q_\varepsilon} \cdot \vec{r_\varepsilon} < 0$, then in order for $\vec{r}$ to be perpendicular to $\vec{q}$, it must lie outside the set U_2^2, and we arrive at a contradiction.

We show $\vec{q_\varepsilon} \cdot \vec{r_\varepsilon} < 0$ by solving this system of equations for $\vec{q_\varepsilon}$ and $\vec{r_\varepsilon}$ using Wolfram

Mathematica to take their dot product in terms of ε (these values can be found in Appendix A). We graph $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon$ for $0 \le \varepsilon \le 1/\sqrt{2} - \sqrt{2} - 1$ (Figure 8).

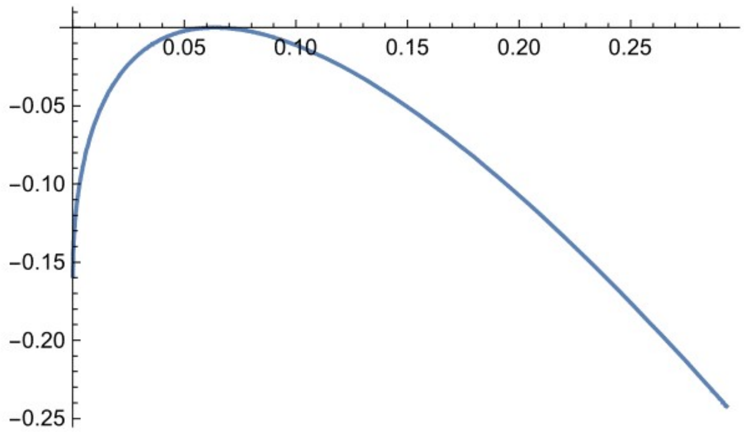


Figure 8: $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon$ as a function of ε

We find that for $0 \le \varepsilon \le 1/\sqrt{2} - \sqrt{2} - 1$, a maximum is attained at $\varepsilon \approx 0.0635$ where $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon = -2.78 \times 10^{-17} < 0$, and therefore, $\vec{r}$ must lie outside the set U_2^2. Hence, $U_{dcb}(\sqrt{2}-1)$ does not contain three mutually orthogonal vectors. Calculations to obtain $\alpha(3,3) \gtrapprox 0.594$ can be found in Appendix B. $\square$

Remark. *Such a small value for $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon$ suggests that $\sqrt{2}-1$ may be the value of h for the largest band possible for $U_{dcb}(h)$. Because Wolfram Mathematica cannot store all digits in every step, there may be a rounding error that returns a value for $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon$ of -2.78×10^{-17} rather than 0.*

8 Computational Results

In Section 7, we find results for $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon$ in terms of ε. Because $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon$ did not remain constant at 0, we can adjust the band in which $\vec{q}_\varepsilon$ lies, such that when on the boundaries of their respective areas, $\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon = 0$. To do this, we create a construction of the configuration on GeoGebra.

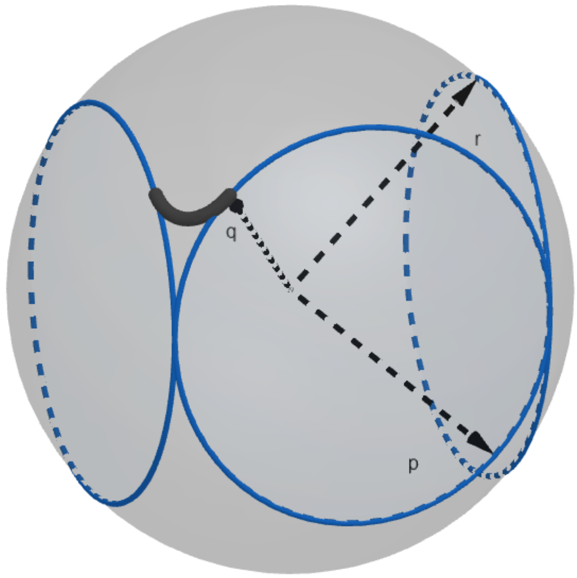


Figure 9: Combination of Curves

We find this curve by fixing $\vec{p}$ and $\vec{r}$ to the boundaries of U_2^1 and U_2^2 respectively, using a slider variable ε and vector $\vec{p}_\varepsilon = (\frac{1}{\sqrt{2}}, \sqrt{\sqrt{2}\varepsilon - \varepsilon^2}, \varepsilon - \frac{1}{\sqrt{2}})$ to calculate the other two vectors. By taking the intersection I_p of the plane perpendicular to $\vec{p}_\varepsilon$ and the sphere, we can find $\vec{r}_\varepsilon$ at the intersection of I and $y = \frac{1}{\sqrt{2}}$. This is equivalent to solving System 2 in Section 7. Then by taking the intersection I_r of the plane perpendicular to $\vec{r}_\varepsilon$ and the sphere, and taking the intersection of I_p and I_r, we can find $\vec{q}_\varepsilon$. This is equivalent to solving the system of equations

$$\begin{cases} \vec{p}_\varepsilon \cdot \vec{q}_\varepsilon = 0 \\ \vec{r}_\varepsilon \cdot \vec{q}_\varepsilon = 0 \\ \vec{q} \cdot \vec{q} = 1. \end{cases}$$

This system is very complicated to solve, therefore, the trace function in GeoGebra was used to draw the curve on the sphere for values of ε from 0 to $\frac{1}{\sqrt{2}} - \frac{1}{2}$, in increments of .01. The upper bound for the height of this curve is $h = 1/2$ and the lower bound can be approximated to be $h = \sqrt{2} - 1$, as seen in Section 7. Until the exact value of $\vec{q}_\varepsilon$ is found in terms of ε, we cannot compute the parametric equation for the curve on the sphere, and therefore, we cannot find the exact area of this configuration. No larger configurations on a sphere were found that avoid three mutually orthogonal vectors, and because of the way this configuration is constructed, this result may eventually suggest a value for $\alpha(3,3)$ once the equation of the curve is found.

Historical Remark

Combining multiple curves (Figure 9) to find the largest volume of points on a sphere that does not contain three mutually orthogonal vectors is similar to a solution posed to the ambidextrous moving sofa problem. In his solution, Dan Romik [5] combines 18 curves of different sizes (Figure 10) and proposes that the area bounded by these curves may be the largest solution to the ambidextrous moving sofa problem.

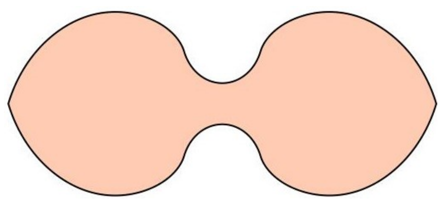
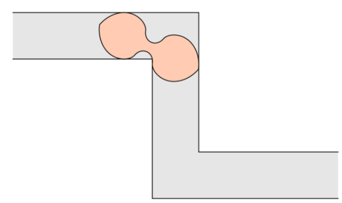



Figure 10: Romik's Ambidexterous Sofa [5]

9 Conclusion

In this paper at a variant of the Witsenhausen problem posed in the AMS Monthly Journal in 1974. Rather than discussing the largest $\frac{\pi}{2}$-avoiding set, we discuss the largest set in n dimensions that cannot contain k mutually orthogonal vectors, in particular, the largest set in three dimensions that does not contain three mutually orthogonal vectors. We find a lower bound for the largest set in n dimensions that cannot contain k mutually orthogonal vectors. In three dimensions, we consider subsets of the sphere in various configurations that give us further insight as to the largest set in three dimensions that does not contain three mutually orthogonal vectors. The configuration taking up the largest portion of the sphere that we are able to prove to not contain three mutually orthogonal vectors was a union of two double caps and a band, $B(\sqrt{2}-1)$ giving us $\alpha(3,3) \geq 0.594$. It is known that $\alpha(3,3) \leq \frac{2}{3}$. Computational results suggest a larger set of points with this property, but these remain a conjecture until the equation of the boundary is found and can be proven.

In further research, methods of construction for the various three dimensional cases may give insight as to large subsets in higher dimensions. Additionally, as we have not proved that the upper bound for the largest set in n dimensions that cannot contain k mutually orthogonal vectors is attainable, we wish to explore further configurations.

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A Computed vector values for $\vec{q_\epsilon}$, $\vec{r_\epsilon}$, and $\vec{q_\epsilon} \cdot \vec{r_\epsilon}$

$$
\begin{aligned}
\vec{q}_\varepsilon = &\left\{ \frac{\left(\sqrt{2}-2\right)e+\sqrt{2}\sqrt{e\left(e\left(2e\left(e-2\sqrt{2}\right)-4\sqrt{2}+9\right)-5\sqrt{2}+8\right)}+\sqrt{2}-1}{2\left(\sqrt{2}-e\right)e+1}, \right. \\
&\left. \frac{2\left(\sqrt{2}-1\right)e^3+3\left(\sqrt{2}-2\right)e^2+2\left(\sqrt{2}-1\right)e-\sqrt{e\left(e\left(2e\left(e-2\sqrt{2}\right)-4\sqrt{2}+9\right)-5\sqrt{2}+8\right)}}{\sqrt{\left(\sqrt{2}-e\right)e}\left(2\left(\sqrt{2}-e\right)e+1\right)}, \sqrt{2}-1 \right\}
\end{aligned}
$$

$$
\begin{aligned}
\vec{r}_\varepsilon = &\left\{ \frac{\left(\sqrt{2}-2\right)e+\sqrt{2}\sqrt{e\left(e\left(2e\left(e-2\sqrt{2}\right)-4\sqrt{2}+9\right)-5\sqrt{2}+8\right)}+\sqrt{2}-1}{2\left(\sqrt{2}-e\right)e+1}, \right. \\
&\left. \frac{2\left(\sqrt{2}-1\right)e^3+3\left(\sqrt{2}-2\right)e^2+2\left(\sqrt{2}-1\right)e-\sqrt{e\left(e\left(2e\left(e-2\sqrt{2}\right)-4\sqrt{2}+9\right)-5\sqrt{2}+8\right)}}{\sqrt{\left(\sqrt{2}-e\right)e}\left(2\left(\sqrt{2}-e\right)e+1\right)}, \sqrt{2}-1 \right\}
\end{aligned}
$$

$$
\begin{aligned}
\vec{q}_\varepsilon \cdot \vec{r}_\varepsilon = &\sqrt{2}-1\,\frac{\left(\sqrt{2}-1\right)\left(4\sqrt{\left(\sqrt{2}e-1\right)^2}-4\sqrt{\left(\sqrt{2}-e\right)e}\left(\sqrt{2}e-1\right)\right)}{2\left(4e\left(e-\sqrt{2}\right)+4\right)} \\
&+\frac{e\left(e\left(-2\left(\sqrt{2}-1\right)e-3\sqrt{2}+6\right)-2\sqrt{2}+2\right)+\sqrt{e\left(e\left(2e\left(e-2\sqrt{2}\right)-4\sqrt{2}+9\right)-5\sqrt{2}+8\right)}}{\sqrt{\left(\sqrt{2}-e\right)e}\left(2e\left(\sqrt{2}e-2\right)-\sqrt{2}\right)} \\
&+\frac{\left(\sqrt{\left(\sqrt{2}-2e\right)^2}e+\sqrt{\left(\sqrt{2}-e\right)e}-\sqrt{\left(\sqrt{2}e-1\right)^2}\right)\left(\left(\sqrt{2}-2\right)e+\sqrt{2}\sqrt{e\left(e\left(2e\left(e-2\sqrt{2}\right)-4\sqrt{2}+9\right)-5\sqrt{2}+8\right)}+\sqrt{2}-1\right)}{\left(2\left(\sqrt{2}-e\right)e-2\right)\left(2\left(\sqrt{2}-e\right)e+1\right)}
\end{aligned}
$$

B Lower Bound for $\alpha(3,3)$ from Proposition 7.1

We compute $|U_{dcb}|$ by taking the area of the double caps and the area of the band, then subtracting the intersection so that we do not over count.

$$
\begin{aligned}
\frac{|U_{dcb}|}{|S^2|} &= \frac{2|U_2| + |B| - |(U_2^1 \cup U_2^2) \cap B|}{|S^2|} \\
\frac{2|U_2| + |B|}{|S^2|} &= \frac{2\left(4\pi\left(1 - \frac{1}{\sqrt{2}}\right)\right) + 4\pi\left(2 - \sqrt{2}\right)}{4\pi} = 1 \\
|(U_2^1 \cup U_2^2) \cap B| &= 4 \times \int_{-\sqrt{2}+1}^{\sqrt{2}-1} \int_{-\sqrt{\frac{1}{2}-z^2}}^{\sqrt{\frac{1}{2}-z^2}} \frac{1}{1 - y^2 - z^2}\, dy\, dz \\
&= 4 \times \int_{-\sqrt{2}+1}^{\sqrt{2}-1} 2\arctan\left(\frac{\sqrt{\frac{1}{2} - z^2}}{\sqrt{1 - z^2 - (\frac{1}{2} - z^2)}}\right) dz \\
&= 4 \times \int_{-\sqrt{2}+1}^{\sqrt{2}-1} 2\arctan\left(\sqrt{1 - 2z^2}\right) dz \approx 5.105 \\
\frac{|(U_2^1 \cup U_2^2) \cap B|}{|S^2|} &\approx \frac{5.105}{4\pi} = 0.406 \\
\frac{|U_{dcb}|}{|S^2|} &\approx 0.594
\end{aligned}
$$