A Variation of the Erdös-Turán Conjecture and the Minimality of Multiplicative Bases

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Abstract

One problem that has been unsolved for nearly a century is the Erdös-Turán Conjecture, an important problem in additive number theory. It states that the efficiency of additive bases of order two of positive integers is always infinite. In this paper, we work towards a solution to the multiplicative analog of this problem. First, we prove that the Erdös-Turán Conjecture implies its multiplicative analog. Then, we introduce the density of a set and prove that finite efficiency is only possible if the basis has density zero. Furthermore, we provide examples of bases and calculate their densities, where one of our bases has density zero. Lastly, we consider a partially ordered set of the bases, and we consider which bases are minimal elements in this set. If a basis is not minimal, we attempt to discover the smallest subset which remains a basis along with its density.

Summary

In this paper, we explore a variation of a famous conjecture in additive number theory proposed by Erdös and Turán. We explore the relationship between the Erdös-Turán Conjecture and its multiplicative analog, the variation we consider. We define a basis to be a set of fundamental building blocks for the set of positive integers and its density to be the fraction of positive integers it contains. Then we prove a relation between the density and another property of a multiplicative basis. Lastly we introduce some examples of bases, calculate their densities, and determine whether or not they are minimal. A basis is minimal if we cannot remove building blocks while preserving all properties. If the basis is not minimal, we attempt to discover the smallest set of building blocks and calculate the new density.

1 Introduction

One interesting conjecture that has been extensively studied in the past century is the Erdös-Turán Conjecture, an additive number theory problem. This problem was motivated by Lagrange's Theorem, which states that all positive integers can be expressed as the sum of at most four perfect squares, and Vinogradov's Theorem, which states that all sufficiently large odd integers can be expressed as the sum of at most three primes. This leads to the question of whether these sets are efficient at representing positive integers as sums of their elements.

Let A be a subset of the positive integers. For each positive integer n, consider the number of times n can be written as the sum of two not necessarily distinct elements of A. The conjecture states that if this number is greater than 0 for all sufficiently large n, then it approaches infinity as n approaches infinity.

Any set A so that every sufficiently large integer may be written as a sum of at most k elements of A is called an additive basis of order k. Some examples include the set of odd numbers, which has order 2, the set of perfect squares, which has order 4 by Lagrange's Theorem, and the set of primes, which has order 3 if Goldbach's Conjecture is true.

We consider a variation of the Erdös-Turán Conjecture. Rather than use an additive basis of order 2, we use a multiplicative basis of order 2. This means that each positive integer except those in a finite set can be expressed as the product of two not necessarily distinct elements of a subset A. We conjecture that if A is a multiplicative basis of order 2, then the number of times a number n can be expressed as a product of two elements of A approaches infinity as n approaches infinity.

First, in Section 2, we define terms relating to the Erdös-Turán Conjecture and its multiplicative analog. In Section 3, we prove that the Erdös-Turán Conjecture implies its multiplicative analog. In Section 4, we first prove a major component of our conjecture relating to the densities of bases. Multiplicative bases have been studied much less rigorously than additive bases, so we then investigate a few properties. We construct some examples of bases and calculate their density. Lastly, in Section 5, we consider the set of all bases as a partially ordered set, and each set such that there is no set less than it is called a minimal set. We then determine whether our examples of multiplicative bases are minimal and if not, try to find a subset that is minimal.

2 Definitions and Background

First, we provide a mathematical definition for an additive basis as given by Erdös and Turán [1].

Definition 2.1. A subset $A \subseteq \mathbb{N}$ is an *additive base of order* k if all positive integers except a finite set can be written as a sum of k elements of A.

Next, we define the additive representation function and additive efficiency of a sequence.

Definition 2.2. The representation function of a subset $A \subset \mathbb{N}$ and $h \in \mathbb{N}$ is $R_{A,h}(n) = \#\{(a_1, a_2, \ldots, a_h) \in A^h | a_1 + a_2 + \cdots + a_h = n\}$.

The function $R_{A,h}(n)$ represents the number of times a positive integer n can be expressed as the sum of h not necessarily distinct elements of A.

Definition 2.3. The additive efficiency is given by

$$E(A) = \limsup_{n \to \infty} R_{A,h}(n).$$

The additive efficiency represents the smallest upper bound on the function $R_{A,h}(n)$ over all positive integers n. We now give the statement of the Erdös-Turán Conjecture.

Conjecture 2.1 (Erdös-Turán [2], 1941)**.** *The additive efficiency of any additive basis of order* 2 *is infinite.*

Now we introduce our variation of this conjecture. We define a multiplicative basis as given by Xiao [3].

Definition 2.4. A subset $A \subseteq \mathbb{N}$ is a *multiplicative base of order* k if all positive integers except a finite set can be written as a product of k not necessarily distinct elements of A.

Corollary 1. *A multiplicative basis A of order k must contain 1 and all sufficiently large primes.*

Proof. By the definition, we know all sufficiently large primes can be written as a product of k elements of A. A prime p can only be expressed in this way by $1^{k-1} \cdot p$, which means that both 1 and p must be elements of A. □

Similarly, we also define a multiplicative representation function and multiplicative efficiency for a given sequence.

Definition 2.5. Let the representation function of a set A and a number $h \in \mathbb{N}$ be $r_{A,h}(n) = \#\{(a_1, a_2, \ldots, a_h) \in A^h | a_1 a_2 \ldots a_h = n\}$.

The function $r_{A,h}(n)$ represents the number of times a positive integer n can be expressed as the product of h not necessarily distinct elements of A.

Definition 2.6. The multiplicative efficiency is given by

$$e(A) = \limsup_{n \to \infty} r_{A,h}(n)$$

The multiplicative efficiency represents the smallest upper bound on the function $r_{A,h}(n)$ over all positive integers n.

3 Variation of Erdös-Turán Conjecture

We introduce the main problem that is the focus of this paper.

Conjecture 3.1. *The multiplicative efficiency of any multiplicative basis of order 2 is infinite.*

3.1 Relation to Erdös-Turán Conjecture

Conjecture 3.1 is essentially the multiplicative case of Conjecture 2.1. Furthermore, the two conjectures are intrinsically related.

First, we introduce two operations, the sum-set and product-set on two sets A and B.

Definition 3.1. $A \oplus B = \{a + b | a \in A \cup \{0\}, b \in B \cup \{0\}\}$

The sum-set of two sets A and B is the set containing all elements of either A or B and positive integers that can be expressed as the sum of an element of A and an element of B.

Definition 3.2. $A \otimes B = \{ab | a \in A, b \in B\}$

The product-set of two sets A and B is the set containing all elements that can be expressed as the product of an element of A and an element of B. Now we proceed with the relationship between the two conjectures given by Dr. Khovanova [4].

Claim 3.2. *If the Erdös-Turán Conjecture is true, then Conjecture 3.1 is true.*

Proof. Consider any multiplicative basis A of order 2. Construct two sequences B and C such that $B = A \cap \{2^n | n \geq 0\}$ and $C = \log_2 B$. We know that $A \otimes A$ contains all positive integers except for a finite set. Powers of 2 can only be formed by the product of 2 powers of 2, which means that all pairs of elements of A whose product is a power of 2 are also elements of B. Hence, $B \otimes B$ contains all powers of 2 except for a finite set, and $C \oplus C$ contains all positive integers except a finite set. If the Erdös-Turán Conjecture is true, then the additive efficiency of C is infinite. This would mean that the multiplicative efficiency of B is infinite, and hence infinite for A as well. This means that the Erdös-Turán Conjecture implies Conjecture 3.1. $\square$

4 Efficiency with Positive Analytic Density

Given a subset A of $\mathbb{N}$, let f_A be the characteristic function of A.

Definition 4.1.

$$f_A(a) = \begin{cases} 1 & \text{if } a \in A \\ \\ 0 & \text{if } a \notin A \end{cases}.$$

Now we introduce the Dirichlet series of a sequence A.

Definition 4.2.

$$L_A(s) = \sum_{n=1}^{\infty} f_A(n) n^{-s} = \sum_{n \in A} n^{-s}.$$

Lastly, we define the analytic density of a sequence A.

Definition 4.3.

$$d(A) = \lim_{s \to 1} \frac{L_A(s)}{\zeta(s)} \text{ where } \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p \left(1 - p^{-s}\right)^{-1}.$$

Theorem 4.1. *Given a multiplicative basis A of order 2, its analytic density must be 0 if it has finite efficiency.*

Proof. We have

$$L_A(s)^2 = \sum_{n=1}^{\infty} \Big(\sum_{d|n} f_A(d) f_A\left(\frac{n}{d}\right) \Big) n^{-s}$$

Notice that because $f_A(d)f_A(\frac{n}{d}) = 1$ if and only if both d and $\frac{n}{d}$ are in A, we have

$$r_{A,2}(n) = \sum_{d|n} f_A(d) f_A\left(\frac{n}{d}\right).$$

Let $B = A \otimes A$. Because $r_{A,2}(n) > 0$ for all sufficiently large n, there exists a set C with a finite number of elements such that $B = \mathbb{N} \backslash C$. It follows that

$$L_B(s) = \sum_{n \in \mathbb{N} \backslash C} n^{-s}.$$

Suppose for the sake of contradiction that there exists a finite positive integer k such that $k \geq r_{A,2}(n)$ for all n. We then see that

$$\limsup_{n \to \infty} r_{A,2}(n) = k \implies k L_B(s) + k \sum_{n \in C} n^{-s} \geq L_A(s)^2.$$

Using the fact that the density of any sequence is at most 1, we have

$$k \geq k d(B) = \lim_{s \to 1} \frac{k L_B(s)}{\zeta(s)} \geq \lim_{s \to 1} \frac{L_A(s)^2 - k \sum_{n \in C} n^{-s}}{\zeta(s)} \geq 0.$$

However, C is a finite set, so the numerator is finite while the denominator is infinite,

$$\lim_{s\to 1}\frac{k\sum_{n\in C} n^{-s}}{\zeta(s)} = 0.$$

This implies that

$$k \geq \lim_{s\to 1}\frac{L_A(s)^2}{\zeta(s)} \geq 0 \implies \lim_{s\to 1}\frac{L_A(s)^2}{\zeta(s)^2} = 0 \implies d(A) = 0.$$

Therefore, for a subset to have finite multiplicative efficiency, it must have density 0. $\square$

Now we must determine whether or not there exists a multiplicative basis of density 0.

4.1 Examples of Multiplicative Bases

First, we introduce some examples of multiplicative bases. Let S_1 be the set containing 1 and all numbers $p_1^{e_1}\dots p_n^{e_n}$ such that $e_1+e_2+\dots+e_n$ is odd as given by Dr. Khovanova [4]. Let S_2 be the set containing 1 and all numbers $p_1^{e_1}\dots p_n^{e_n}$ such that each of $e_1, e_2, \dots, e_n$ is odd as given by Dr. Khovanova [4]. Let S_3 be the set containing 1 and all numbers $p_1^{e_1}\dots p_n^{e_n}$ such that n is odd. For our last example, we first split the sequence of all primes into subsequences P_i such that each prime is in exactly one subsequence, a subsequence contains consecutive primes, and

$$\prod_{p\in P_i}\left(1-\frac{1}{p}\right) \leq \frac{1}{2}.$$

For example, we have $P_1 = \{2\}$, $P_2 = \{3, 5, 7\}$, and so on. It is well known that

$$\prod_p (1-\frac{1}{p}) = 0,$$

which implies that there must be an infinite number of sets P_i. Let

$$A_p = \bigcup_{i=0}^{\infty} P_{2i+1} \text{ and } B_p = \bigcup_{i=0}^{\infty} P_{2i+2}$$

Now let A and B be the sets of all numbers which have all prime factors as elements of A_p and B_p respectively. Let $S_4 = A \cup B$ as given by [5].

We wish to calculate the analytic densities of these two sequences. First, we define another type of density, the asymptotic density.

Definition 4.4.

$$\partial(A, n) = \frac{|A \cap \{1, 2, \ldots, n\}|}{n} \text{ and } \partial(A) = \lim_{n \to \infty} \partial(A, n)$$

Given this new type of density, we relate it to the analytic density through the following result.

Theorem 4.2 (Dirichlet-Dedekind [6])**.** *If $\partial(A)$ exists, then $d(A) = \partial(A)$.*

This means that the asymptotic density and analytic density are equal as long as the asymptotic density exists.

4.1.1 Properties of S_1

First, we will show that S_1 is a multiplicative basis.

Claim 4.3. *S_1 is a multiplicative basis.*

Proof. Since $1 \in S_1$, all elements of S_1 are elements of $S_1 \otimes S_1$. Therefore, we now just need to show that all numbers greater than 1 where the sum of the exponents in its prime factorization is even will be in $S_1 \otimes S_1$. For any x of this form, we know $x > 1$, so there exists a prime $p|x$. We have $p \in S_1$ and $\frac{x}{p} \in S_1$, so $x \in S_1 \otimes S_1$. Therefore, since $S_1 \otimes S_1 = \mathbb{N}$, S_1 is a multiplicative basis. □

Now we solve for the density of S_1. First, we used a program to find the asymptotic density up to the first one million terms. Our results are shown in Table 1.

n	$\partial(S_1, n)$
100	0.52
1000	0.508
10000	0.5048
100000	0.50145
1000000	0.500266

Table 1: Density of the sequence S_1 over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.

As shown in Table 1, the asymptotic density of S_1 approaches 0.5 and is already very close with small values of n. We will now show analytically that this is indeed true using [7].

Claim 4.4. $d(S_1) = \frac{1}{2}$

Proof. For any positive integer x, we let the function $\Omega(x)$ be the sum of the exponents in its prime factorization. This means that

$$\frac{1-(-1)^{\Omega(x)}}{2} = f_{S_1}(x) \implies \partial(S_1) = \lim_{x\to\infty} \sum_{i=1}^{x} \frac{1-(-1)^{\Omega(i)}}{2x}.$$

It is well known that

$$\sum_{i=1}^{x} (-1)^{\Omega(i)} = o(x) \implies d(S_1) = \partial(S_1) = \frac{1}{2} - \lim_{x\to\infty} \sum_{i=1}^{x} \frac{(-1)^{\Omega(i)}}{2x} = \frac{1}{2}.$$

Hence, the density of S_1 is $\frac{1}{2}$. □

4.1.2 Properties of S_2

First, we will show that S_2 is a multiplicative basis.

Claim 4.5. *S_2 is a multiplicative basis.*

Proof. For any positive integer x we express it as $p_1^{e_1} \dots p_n^{e_n} q_1^{f_1} \dots q_n^{f_n}$ such that all e_i are even while all f_i are odd. We take

$$x_1 = \prod_{i=1}^{n} p_i \text{ and } x_2 = \prod_{i=1}^{n} p_i^{e_i - 1} \prod_{i=1}^{m} q_i^{f_i}.$$

Both x_1 and x_2 are elements of S_2. Therefore, since $S_2 \otimes S_2 = \mathbb{N}$, S_2 is a multiplicative basis. $\square$

n	$\partial(S_2, n)$
100	0.7
1000	0.707
10000	0.7055
100000	0.70457
1000000	0.704464

Table 2: Density of the sequence S_2 over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.

As shown in Table 2, the asymptotic density of S_2 approaches approximately 0.7044 and that it does not change much between small and large values of n. We will now show analytically that this is indeed true using [8]. First, we define a multiplicative sequence A.

Definition 4.5. A sequence A is multiplicative if for any two $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$, $f_A(a)f_A(b) = f_A(ab)$.

Now we define the Euler product expansion of a Dirichlet Series of a multiplicative sequence A.

Definition 4.6.

$$L_A(s) = \sum_{n=1}^{\infty} f_A(n)n^{-s} = \prod_p \left(\sum_{n=0}^{\infty} f_A(p^n)p^{-ns} \right).$$

We can use the Euler product expansion to calculate the density of S_2.

Claim 4.6. *$d(S_2) = \frac{K_1\pi^2}{6} \approx 0.7044422$ where K_1 is the carefree constant.*

Proof. Notice that S_2 is a multiplicative sequence, and $f_{S_2}(p^n) = 1$ if and only if n is odd or 0. The Euler product expansion for S_2 is

$$L_{S_2}(s) = \prod_p \left(1 + \frac{p^{-s}}{1 - p^{-2s}}\right).$$

Using the expression for $\zeta(s)$ given in Definition 4.3,

$$d(S_2) = \lim_{s\to 1} \frac{L_{S_2}(s)}{\zeta(s)} = \lim_{s\to 1} \prod_p \left(\frac{1 + p^{-s} - p^{-2s}}{1 + p^{-s}}\right) = \prod_p \left(\frac{1 - 2p^{-2} + p^{-3}}{1 - p^{-2}}\right)$$

$$= \prod_p \left(1 - \frac{2p-1}{p^3}\right) \prod_p \left(1 - p^{-2}\right)^{-1}.$$

The first product is known as the carefree constant which is approximately 0.4282495, while the second product is $\frac{1}{\zeta(2)} = \frac{\pi^2}{6}$. Hence, we obtain $d(A) = \frac{K_1 \pi^2}{6} \approx 0.7044422$. $\square$

4.1.3 Properties of S_3

Claim 4.7. *S_3 is a multiplicative basis.*

Proof. For any positive integer x we express it as $p_1^{e_1} \dots p_n^{e_n}$. If n is odd, then $x \in S_3 \implies x \in S_3 \otimes S_3$. If n is even, we take

$$x_1 = p_n^{e_n} \text{ and } x_2 = \prod_{i=1}^{n-1} p_i^{e_i}.$$

Both x_1 and x_2 are elements of S_3. Therefore, since $S_3 \otimes S_3 = \mathbb{N}$, S_3 is a multiplicative basis. $\square$

n	$\partial(S_3, n)$
100	0.44
1000	0.469
10000	0.5009
100000	0.50361
1000000	0.500955

Table 3: Density of the sequence S_3 over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.

As shown in Table 3, the asymptotic density of S_3 starts below 0.5 but then approaches 0.5 from above. It is not very close to 0.5 for small values of n, but approaches it when n is larger. We were unable to prove this, but it is supported by our numerical calculations.

4.1.4 Properties of S_4

Claim 4.8. *S_4 is a multiplicative basis.*

Proof. For any positive integer x we express it as $p_1^{e_1} \dots p_n^{e_n} q_1^{f_1} \dots q_n^{f_n}$ such that all $p_i \in A_p$ and all $q_i \in B_p$. Notice that $x_1 = p_1^{e_1} \dots p_n^{e_n} \in A \subset S_4$ and $x_2 = q_1^{f_1} \dots q_n^{f_n} \in B \subset S_4$. Both x_1 and x_2 are elements of S_4. Therefore, since $S_4 \otimes S_4 = \mathbb{N}$, S_4 is a multiplicative basis. $\square$

Claim 4.9. $d(S_4) = 0$.

Proof. Notice that A and B are both multiplicative sequences, and $f_A(p^n) = 1$ if and only if $p \in A_p$ and $f_B(p^n) = 1$ if and only if $p \in B_p$. The Euler product expansion for A is

$$L_A(s) = \prod_{p \in A_p} \left(1 - p^{-s}\right)^{-1}.$$

Furthermore, we know that A_p and B_p together contain all the primes without any overlap, so using the expression for $\zeta(s)$ given in Definition 4.3,

$$d(A) = \lim_{s \to 1} \frac{L_A(s)}{\zeta(s)} = \lim_{s \to 1} \prod_{p} \left(1 - p^{-s}\right) \prod_{p \in A_p} \left(1 - p^{-s}\right)^{-1} = \lim_{s \to 1} \prod_{p \in B_p} \left(1 - p^{-s}\right)$$

$$= \prod_{n=0}^{\infty} \prod_{p \in P_{2n+2}} \left(1 - p^{-1}\right) \leq \prod_{n=0}^{\infty} \frac{1}{2} = 0 \implies d(A) = 0.$$

Similarly, we may obtain that $d(B) = 0$. Since $S_4 = A \cup B$ and $A \cap B = \emptyset$,

$$L_{S_4}(s) = \sum_{n \in S_4} n^{-s} = \sum_{n \in A} n^{-s} + \sum_{n \in B} n^{-s} = L_A(s) + L_B(s)$$

$$\implies d(S) = d(A) + d(B) = 0.$$

Hence, the density of S_4 is 0.

$\square$

Corollary 2. *There exists a multiplicative basis with density* 0 *for any order greater than* 1.

Proof. We provide an example of a multiplicative basis with density 0 for order $k > 1$. We create the k sets

$$Q_{ip} = \bigcup_{n=0}^{\infty} P_{kn+i}.$$

These k sets are disjoint and cover all primes. Let Q_i be the set of all numbers which have all prime factors as elements of Q_{ip}. We obtain

$$d(Q_i) = \lim_{s\to 1} \prod_{p \notin Q_{ip}} \left(1 - p^{-s}\right) = \prod_{n \not\equiv i \pmod{k}} \prod_{p \in P_n} \left(1 - p^{-1}\right)$$
$$\leq \prod_{n \not\equiv i \pmod{k}} \frac{1}{2} = 0 \implies d(Q_i) = 0.$$

Notice that the set

$$Q = \bigcup_{n=1}^{k} Q_i$$

is a basis of order k and

$$d(Q) = \sum_{n=1}^{k} d(Q_i) = 0$$

We have constructed a basis of order k with density 0, so we are done. □

5 Minimality of Multiplicative Bases

Consider a partially ordered set (poset) containing all multiplicative bases. We consider a set A less than a set B if $A \subset B$ and there are an infinite number of positive integers that are elements of B and not A. We call a basis A *minimal* if there are no bases that are less than A. For this poset, there may be infinitely many minimal sets.

Previously, we found a basis of order 2 with density 0, so a natural question is to consider whether minimal bases are all of density 0 and if not, find other minimal bases based on the bases we found.

Theorem 5.1. *S_1 is a minimal set.*

Proof. Suppose we remove an element $k \in S_1$ and call the new set S_1'. We claim that $k \notin S_1' \otimes S_1'$. If $k = 1$, then this is clearly true since the only way to achieve 1 is through $1 \cdot 1$.

Now consider some other $k \in S_1$. Assume for the sake of contradiction that $k = xy$ where $x, y \in S_1'$. If $x, y \neq 1$, then the sum of the exponents in the prime factorization of xy is even. This means that $k \neq xy$. If $x = 1$, then $y = k \notin S_1'$, and similarly for $y = 1$. Hence, we have reached a contradiction, so $k \notin S_1' \otimes S_1'$.

Suppose there exists a basis S that is less than S_1. There must be an infinite number of elements of S_1 that are not in S, which means an infinite number of elements of S_1 are not in $S \otimes S$. However, this must imply that S is not a basis of order 2, which is a contradiction. Hence, S_1 is a minimal basis. □

This also shows that not all minimal bases have density 0.

Theorem 5.2. *S_2 is not a minimal set.*

Proof. We claim that we can remove all elements of the form $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$ such that n is even and the number of exponents which are 1 is less than $\frac{n}{2}$ to form a new basis S_2'.

We will show that any positive integer can be formed by the product of two not necessarily distinct elements of S_2'. We can express any positive integer as $k = p_1^{e_1} p_2^{e_2} \dots p_n^{e_n} q_1^{f_1} q_2^{f_2} \dots q_m^{f_m}$ such that $e_1, e_2, \dots e_n$ are even while $f_1, f_2, \dots, f_m$ are odd. We want to find two numbers x_1 and x_2 that multiply to k such that both x_1 and x_2 are in S_2'. This means that both x_1 and x_2 must be divisible by $p_1, p_2, \dots p_n$ while only one of the two is divisible by each of $q_1, q_2, \dots q_n$. We now split this proof into four different cases.

Case 1: $m + n$ is even and n is even

If $m = 0$, then we take

$$x_1 = \prod_{i=1}^{\frac{n}{2}} p_i \prod_{i=\frac{n}{2}+1}^{n} p_i^{e_i - 1} \text{ and } x_2 = \prod_{i=1}^{\frac{n}{2}} p_i^{e_i - 1} \prod_{i=\frac{n}{2}+1}^{n} p_i.$$

Both x_1 and x_2 have an even number of prime factors, but the number of exponents that are 1 in both numbers is at least $\frac{n}{2}$, so both x_1 and x_2 are elements of S_2'.

If $m > 0$, then we find odd positive integers a and b such that $a + b = m$. Take

$$x_1 = \prod_{i=1}^{n} p_i \prod_{i=1}^{a} q_i^{f_i} \text{ and } x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=a+1}^{m} q_i^{f_i}.$$

Both x_1 and x_2 have an odd number of prime factors, so they are both elements of S_2'.

Case 2: $m + n$ is even and n is odd

Take the two numbers

$$x_1 = q_1^{f_1} \prod_{i=1}^{n} p_i \text{ and } x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=2}^{m} q_i^{f_i}.$$

Notice that x_1 has an even number of prime factors but $n \geq 1$, so the number of exponents that are 1 is at least $\frac{n+1}{2}$, and x_2 has an odd number of prime factors, so they are both elements of S_2'.

Case 3: $m + n$ is odd and n is even

Take the two numbers

$$x_1 = \prod_{i=1}^{n} p_i \text{ and } x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=1}^{m} q_i^{f_i}.$$

Notice that x_1 has an even number of prime factors but $n \geq 0$, so the number of exponents that are 1 is at least $\frac{n}{2}$, and x_2 has an odd number of prime factors, so they are both elements of S_2'.

Case 4: $m + n$ is odd and n is odd

Take the two numbers

$$x_1 = \prod_{i=1}^{n} p_i \text{ and } x_2 = \prod_{i=1}^{n} p_i^{e_i-1} \prod_{i=1}^{m} q_i^{f_i}.$$

Both x_1 and x_2 have an odd number of prime factors, so they are both elements of S_2'.

We have now considered all possible parity cases for n and $m+n$, and every positive integer is able to be constructed from elements in S_2'. Therefore, S_2' is a multiplicative basis. $\square$

It is still possible to remove more elements, but we were unable to find a systematic way to do so. Using the new set, we numerically estimated the density.

n	$\partial(S_2', n)$
100	0.7
1000	0.704
10000	0.7045
100000	0.70423
1000000	0.70437

Table 4: Density of the sequence S_2' over the first 100, 1000, 10000, 100000, and 1000000 terms calculated with a computer.

As shown in Table 4, the asymptotic density of S_2' is barely different from S_2. Removing the elements we found unnecessary barely changed the densities. This suggests that the removed set has very few elements compared to the positive integers and has density 0.

Theorem 5.3. *S_3 is not a minimal set.*

Proof. We claim that we can remove all elements of the form $p_1^{e_1}p_2^{e_2}\dots p_n^{e_n}$ where n is not of the form 2^k-1 and $e_1, e_2, \dots, e_n > 1$ to form a new basis S_3'.

We will show that for all natural numbers $x = p_1^{e_1}p_2^{e_2}\dots p_n^{e_n}$ it is possible to express $x = x_1x_2$ such that $x_1, x_2 \in S_3'$. We now split this proof into three cases.

Case 1: $e_1, e_2, \dots, e_n > 1$

First we find the k such that $2^k - 1 \leq x < 2^{k+1} - 1$. Now let $y = x + 1 - 2^k$ and find the l such that $2^{l-1} - 1 \leq x + 1 - 2^k < 2^l - 1$. Let $z = 2^k + 2^l - x - 2$. We can express

$$x_1 = \prod_{i=1}^{z} p_i \prod_{i=z+1}^{2^k-1} p_i^{e_i} \text{ and } x_2 = \prod_{i=1}^{z} p_i^{e_i-1} \prod_{i=2^k}^{x} p_i^{e_i}.$$

Notice that x_1 has $2^k - 1$ prime factors and x_2 has $2^l - 1$ prime factors, so $x_1, x_2 \in S_3'$.

Case 2: $\min(e_1, e_2, \dots, e_n) = 1$ and n is odd

These elements themselves are already elements of S_3' since they were originally in S_3 and not removed.

Case 3: $\min(e_1, e_2, \ldots, e_n) = 1$ and n is even

Without loss of generality, suppose that $e_1 = 1$. Take

$$x_1 = \prod_{i=1}^{n-1} p_i^{e_i} \text{ and } x_2 = p_n^{e_n}.$$

We know $x_1 \in S_3'$ since it has an odd number of prime factors and at least one exponent is one. Hence, $x_1, x_2 \in S_3'$.

We have considered every possible case for the exponents in the prime factorization of x. Every positive integer is able to be expressed as the product of two elements of S_3', so therefore S_3' is a multiplicative basis. □

While it is possible to remove more elements, we were unable to find a systematic way to do so. We numerically estimated the density of the new set. We found that the values were the exact same as with S_3, so the set of numbers we removed does not contain any of the first million positive integers. This suggests that our removed set contains very few numbers compared to the set of all positive integers and has density 0.

Theorem 5.4. *S_4 is a minimal set.*

Proof. By our definition of S_4, we know that there is no element $x \in S_4$ such that it has prime factors in both A_p and B_p. Hence, the only way to express such a number $x = x_1 x_2$ such that $x_1, x_2 \in S_4$ is if $x_1 \in A$ and $x_2 \in B$ or vice versa.

Without loss of generality, suppose we remove an element $y > 1$ of A from S_4 to get S_4'. For any element $z > 1$ of B, the only way to form yz from S_4 is from y and z. However, $y \notin S_4'$, so it is impossible to form yz from two elements of S_4'. There are infinitely many elements of B, so there are infinitely many positive integers not in $S_4' \otimes S_4'$. Similarly, S_4' is not a basis if we remove an element of B.

Suppose there exists a basis S that is less than S_4. There must be elements in S_4 that are not in S, which means an infinite number of positive integers are not in $S \otimes S$. Therefore, S is not a basis of order 2, so we have reached a contradiction. Hence, S_4 is a minimal basis. □

6 Conclusion and Future Work

In summary, we have worked towards the multiplicative variation of the Erdös-Turán Conjecture. We proved that it is only possible for a basis to have finite efficiency if its density is 0. To explore this last case more, we considered some examples of multiplicative bases and calculated their density both numerically and analytically. However, we were not able to disprove the case of density 0 bases as we successfully found a construction for one. As a basis with density 0 existed, we investigated whether all bases had a subset of density 0 that was also a basis. For our bases, we proved whether they were minimal and tried to find a minimal subset of it if it was not minimal.

In the future, we plan on continuing our work on our conjecture. Although we were unable to prove the case with density 0, our conjecture is not disproved by the example. Furthermore, we will investigate more examples of multiplicative bases to understand more about them as well as the set of minimal bases.

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References

[1] T. Tao and V. Vu. *Additive Combinatorics*. New York: Cambridge University Press, 2006.

[2] P. Erdös and P. Turán. On a problem of Sidon in additive number theory, and on some related problems. 16(4):212–215, 1941.

[3] S. Y. Xiao. On the erdös-turán conjecture and related results. Master's thesis, University of Waterloo, 2011.

[4] T. Khovanova. personal communication, July 2019.

[5] Math Stack Exchange user coffeemath. Product of two sets with density zero has density zero? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1292144 (version: 2015-05-21). Online; accessed 24 July 2019.

[6] P. T. Bateman and H. G. Diamond. *Analytic Number Theory: An Introduction Course (Monographs in Number Theory)*. Wspc, 2004.

[7] MathOverflow user Peter Humphries. Density of the "multiplicative odd numbers". MathOverflow. URL:https://mathoverflow.net/q/116330 (version: 2012-12-13). Online; accessed 11 July 2019.

[8] MathOverflow user François Brunault. Numbers whose prime factors all have odd exponent. MathOverflow. URL:https://mathoverflow.net/q/312045 (version: 2018-10-05). Online; accessed 11 July 2019.