Cyclic Koszul algebras and oriented graphs

Chavdar Lalov

under the direction of

Guangyi Yue
Department of Mathematics
Massachusetts Institute of Technology

Research Science Institute
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Abstract

Koszul algebras are one of the most studied types of quadratic algebras due to their numerous applications in other fields of mathematics. We generalise the notion of a Koszul algebra through the use of oriented graphs. The defining properties of Koszul algebras can be attached to a special type of oriented graphs: a line with vertices, where all edges are oriented towards the right. Reversing this combinatorial correspondence, we introduce a new type of algebras, Cyclic Koszul algebras, associated with oriented cycles. Firstly, we show that the new structure recovers some of the fundamental properties of classical Koszul algebras; for instance, if an algebra is Cyclic Koszul, then so is its dual algebra. Conversely, when $\dim V = 2$, we prove that the symmetric algebra $S(V)$ is not Cyclic Koszul, contrasting the fact that it is Koszul. However, we partially prove a conjecture that if we quantize the symmetric algebra $S(V)$, it will be Cyclic Koszul in most cases.

Summary

We introduce a generalisation of the so-called quadratic Koszul algebras. Imagine we had an alphabet of any size. Using it we can construct words and combine them together to form sentences. However, because addition is commutative, interchanging words produces the same sentence. To a big extent, this “language” resembles the so-called tensor algebra. We can eliminate part of the language by forbidding some sentences of two-letter words. The new language resembles a quadratic algebra. Quadratic algebras are quite difficult to be described mathematically. However, when we forbid two-letter combinations in a more specific way, we understand the new language quite better. The equivalent, well-understood quadratic algebras are called Koszul algebras. The defining properties of Koszul algebras are best illustrated by relating them to a specific oriented graph: a line with vertices, where each edge is oriented towards the right. Reversing this combinatorial correspondence, we introduce a new type of algebras, Cyclic Koszul algebras, associated with oriented cycles. We investigate how different is the behaviour of classical Koszul algebras in comparison to Cyclic Koszul algebras. At first glance, both algebras share some common fundamental properties. However, on a deeper level, Cyclic Koszul algebras prove to have interesting properties which are not seen in Koszul algebras.

1 Introduction

The tensor algebra $T(V)$ of a vector space V is one of the most fundamental objects in mathematics. Many other important algebras, such as the symmetric algebra $S(V)$ and the exterior algebra $\bigwedge(V)$, can be obtained as a quotient of the tensor algebra. The simplest nontrivial way to take a quotient is by an ideal generated by quadratic relations. This gives us the notion of a quadratic algebra. The motivation for introducing quadratic algebras also comes naturally from the study of quantum groups [1]. More precisely, as said in Polishchuk and Positselski [2], quadratic algebras provide a convenient framework for "noncommutative spaces" on which quantum groups act. That implies the need to control the size of quadratic algebras as measured by their Hilbert series. However, the latter proves to be quite hard to do for general quadratic algebras and so the notion of a Koszul algebra, a special type of quadratic algebra, was defined. An example of the nice properties of Koszul algebras is the identity $h_A(t)h_{A^!}(-t) = 1$, where $h_A(t)$ is the Hilbert series of a Koszul algebra A and $h_{A^!}(t)$ is the Hilbert series of the dual algebra $A^!$ of A. This identity does not hold for a general quadratic algebra A.

Surprisingly, Koszul algebras have many applications in different areas of mathematics such as representation theory, algebraic geometry, topology, number theory and others (a list of references is given in [2, p. viii]). For example, as said in Polishchuk and Positselski [2], in representation theory certain subcategories of the category $\mathcal{O}$ for a semisimple complex Lie algebra are governed by Koszul algebras.

In this project we study a generalisation of Koszul algebras arising from graphs. To a Koszul algebra we can associate a simple type of oriented graphs: a line with arrows to the right (see Figure 1.) This connection will be explained in detail in Section 3.1. Reversing the combinatorial correspondence, in this paper we study the algebraic counterpart of oriented cycles, which give rise to Cyclic Koszul algebras.

Figure 1: Oriented graph of Koszul algebras.

We succeed in proving that Cyclic Koszulness is preserved upon dualization. Moreover, we present a way in which one can construct high-dimensional Cyclic Koszul algebras from low-dimensional Cyclic Koszul algebras.

In the second section, we go through the main definitions and theorems needed. In the third section, we present our generalisation of the Koszul algebras and derive some standard properties about the new structure.

In the fourth section, a special type of quadratic algebras, which behave in a quite intriguing way, is examined. More precisely, we prove that the symmetric algebra $S(V) = T(V)/\langle y \otimes x - x \otimes y \rangle$, where $\dim V = 2$ and $\{x, y\}$ is a basis of V, is not Cyclic Koszul, which contrasts to the fact that it is Koszul. However, we *conjecture* that if we quantize $S(V)$, i.e. consider taking the quotient over the ideal generated by $y \otimes x - ax \otimes y$, where $a \in \mathbb{C}$ is not a root of unity and is different from 0, then the algebra $T(V)/\langle y \otimes x - ax \otimes y \rangle$ is Cyclic Koszul. We partially prove the conjecture although there is much more to be done. The studied cases strongly suggest the validity of the conjecture in the general case.

2 Preliminaries

In the first subsection we provide some basic definitions and theorems and in the second we introduce the notion of chain complexes of vector spaces. Throughout the paper we work with finite-dimensional vector spaces over the field of complex numbers.

2.1 Basic definitions and theorems

Quadratic algebras are constructed from the tensor algebra.

Definition 2.1. The *tensor algebra* of a vector space V is the direct sum of the tensors of all ranks on V

$$T(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots.$$

Definition 2.2. A *quadratic algebra* A is the data of a vector space V and a subspace of relations $I \subseteq V \otimes V$

$$A = T(V)/\langle I \rangle,$$

where $\langle I \rangle$ is the ideal generated by I.

Distributive lattices in which the elements are vector spaces are necessary for the definition of Koszul algebras.

Definition 2.3. Let $(L, \vee, \wedge)$ be an algebraic structure consisting of a set L and two binary operations $\vee$ and $\wedge$ on L. Then $(L, \vee, \wedge)$ is a *lattice* if $\vee$ and $\wedge$ are commutative and associative, and if the following identities called the absorption laws hold

$$a \vee (a \wedge b) = a,$$

$$b \wedge (a \vee b) = b,$$

where $a, b \in L$. If also the identities $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ are true, then the lattice is *distributive*.

For a quadratic algebra to be Koszul we impose the following restrictions.

Definition 2.4. Fix a subspace $I \subseteq V \otimes V$. For each $n \geq 2$ consider the $n-1$ subspaces $V^{\otimes i} \otimes I \otimes V^{\otimes n-2-i}$ where $i \in \{0, ..., n-2\}$. If for a given $n \geq 2$ these subspaces form a distributive lattice, where the $\vee$ is defined to be vector sum and the $\wedge$ is the usual intersection considering the vector spaces as sets, the algebra $T(V)/\langle I \rangle$ is said to be n-*Koszul*. If for each $n \geq 2$, the algebra $T(V)/\langle I \rangle$ is n-Koszul we call it a *Koszul algebra*.

Example 1. The simplest type of Koszul algebras are quadratic algebras where the quotient is an ideal generated by a monomial, i.e. $A = T(V)/\langle x^2 \rangle$. Let V be two-dimensional with basis $\{x, y\}$. We show that A is 4-Koszul, although one can show it for every $n \geq 2$. We prove, as an example, that

$(V \otimes V \otimes I + I \otimes V \otimes V) \cap V \otimes I \otimes V = V \otimes V \otimes I \cap V \otimes I \otimes V + I \otimes V \otimes V \cap V \otimes I \otimes V$

The basis for $V \otimes I \otimes V$ is $\{x^4, x^3y, yx^3, yx^2y\}$ where we have ignored the symbol $\otimes$ for simplicity. All elements of the basis except yx^2y are contained in either $V \otimes V \otimes I$ or $I \otimes V \otimes V$. However, since yx^2y is a basis element of $V^{\otimes 4}$ it cannot be obtained from the other basis elements, nor from the sum $V \otimes V \otimes I + I \otimes V \otimes V$.

The most famous example of a Koszul algebra is the symmetric algebra $T(V)/\langle yx - xy \rangle$. Nevertheless, to prove Koszulness one needs to use free linear resolutions, see Polishchuk and Positselski [2, p. 20].

There are several equivalent definitions of Koszul algebras which can be found in Polishchuk and Positselski [2] and Ufnarovskii [3]. A condition for $A = T(V)/\langle I \rangle$ to be Koszul is given by the following theorem.

Theorem 2.1 (Polishchuk and Positselski [2, p. 15])**.** *Let W be a vector space and $X_1, \ldots, X_n \subseteq W$ be a collection of its subspaces. Then the following conditions are equivalent:*

1. *the collection $X_1, \ldots, X_n$ forms a distributive lattice;*
2. *there exists a basis B of W such that each of the subspaces X_i is the linear span of a subset of B.*

Since $V^{\otimes i} \otimes I \otimes V^{\otimes n-2-i}$ are subspaces of $V^{\otimes n}$, then to prove that $T(V)/\langle I \rangle$ is Koszul, we use Theorem 2.1 with an appropriate basis of $V^{\otimes n}$. We note, however, that finding such a basis from scratch is usually quite complicated. There are other methods shown in Polishchuk and Positselski [2].

Koszulness is preserved when taking the dual algebra.

Theorem 2.2 (Polishchuk and Positselski [2, p. 27])**.** *Let V^* be the dual of a vector space V and let $I^\perp \in V^* \otimes V^*$ be the orthogonal complement of $I \in V \otimes V$. An algebra $A = T(V)/\langle I \rangle$ is n-Koszul if and only if the dual algebra $A^! = T(V^*)/\langle I^\perp \rangle$ is n-Koszul. More generally, A is Koszul if and only if $A^!$ is Koszul.*

Example 2. It is well-known that dual algebra of the symmetric algebra is the exterior algebra $T(V)/\langle x \otimes y + y \otimes x \rangle$. Thus, by Theorem 2.2 the exterior algebra is also Koszul.

2.2 Complexes of vector spaces

A collection of vector spaces forming a distributive lattice is equivalent to certain properties related to complexes of those vector spaces.

Definition 2.5. Let $X_1, X_2 \dots$ be vector spaces and let $\phi_1, \phi_2 \dots$ be linear maps, such that ϕ_i maps X_i to X_{i-1} for $i > 2$ and ϕ_1 is just the zero map from X_1 to 0. Note that the number of vector spaces can be finite as well as infinite. A *chain complex* $C_\bullet(X_1, X_2, \dots)$ is a sequence

$$C_\bullet : \quad \cdots \xrightarrow{\phi_3} X_2 \xrightarrow{\phi_2} X_1 \xrightarrow{\phi_1} 0,$$

where $\phi_i \circ \phi_{i+1} = 0$ for all $i \geq 1$

Among chain complexes, of particular interest are the exact ones.

Definition 2.6. Let $C_\bullet(X_1, X_2, \dots)$ be a complex of vector spaces. If $\operatorname{Im} \phi_{k+1} = \ker \phi_k$ for some fixed k, then we say the complex is *exact* at X_k. If the complex is exact at all X_k, we call the complex *exact*.

With this definition in mind we present a wonderful proposition which connects the distributivity of vector spaces to exactness of complexes. The notation $\{x_1, \dots, \hat{x_k}, \dots, x_n\}$ means that we take all elements from x_1 to x_n without the element x_k.

Proposition 2.3 (Polishchuk and Positselski [2, p. 16]). *Let W be a vector space and let $X_1, \dots, X_n \subset W$ be a collection of subspaces such that any proper subset $X_1, \dots, \hat{X}_k, \dots, X_n$ is distributive. Then the following conditions are equivalent*

1. *The collection $X_1, \dots, X_n$ form a distributive lattice;*

2. *the complex of vector spaces $B_\bullet(W; X_1, \dots, X_n)$*

$$W \to \bigoplus_t W/X_t \to \dots \to \bigoplus_{t_1<\dots<t_{n-i}} W/(\sum_{s=1}^{n-i} X_{ts}) \to \dots \to W/\sum_s X_s \to 0, \qquad (1)$$

   where $s, t, t_1, t_2, \dots \in \{1, \dots, n\}$, is exact everywhere except for the leftmost term;

3. *the complex of vector spaces $B^\bullet(W; X_1, \dots, X_n)$*

$$0 \to \bigcap_s X_s \to \cdots \to \bigoplus_{t_1<\dots<t_{n-i}} \bigcap_{s=1}^{n-i} X_{t_s} \to \cdots \to \bigoplus_t X_t \to W, \qquad (2)$$

   where $s, t, t_1, t_2, \dots \in \{1, \dots, n\}$, is exact everywhere except for the rightmost term;

The map from $\bigoplus_{p_1<\dots<p_{n-i}} \bigcap_{s=1}^{n-i} X_{p_s}$ to $\bigoplus_{q_1<\dots<q_{n-i-1}} \bigcap_{s=1}^{n-i-1} X_{q_s}$ is defined by $(\alpha_1, \dots, \alpha_{\binom{n}{n-i}}) \rightsquigarrow (a_1, \dots, a_{\binom{n}{n-i-1}})$, where $a_j = (-1)^{sign(j_1,j)}\alpha_{j_1} + \dots + (-1)^{sign(j_{i+1},)}\alpha_{j_{i+1}}$. The numbers $j_1, \dots, j_{i+1}$ and the function $sign(,)$ are defined in the following way: Let us denote by I_j the j^{th} addend of the direct sum $\bigoplus_{p_1<\dots<p_{n-i}} \bigcap_{s=1}^{n-i} X_{p_s}$ with respect to lexicographical order. In the same way we define Y_j to be j^{th} addend in $\bigoplus_{q_1<\dots<q_{n-i-1}} \bigcap_{s=1}^{n-i-1} X_{q_s}$. There are exactly $i+1$ intersections $Y_{j_1}, \dots, Y_{j_{i+1}}$ which contain I_j. Let X_w be the space which is being intersected in I_j, but not in Y_{j_k}. Then the function $sign(j_k, j)$ depends on the position of the index w in the string $p_1, \dots, p_{n-i}$. If the position has an odd index, then $sign(j_k, j) = 0$, otherwise $sign(j_k, j) = 1$.

Exactness of a complex is preserved when taking the duals of the vector spaces and maps in the complex.

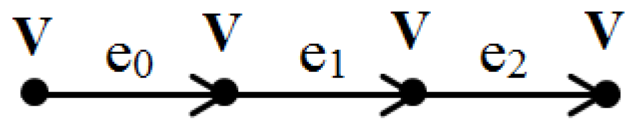

Figure 2: Visualising $V^{\otimes i} \otimes I \otimes V^{\otimes n-2-i}$ for $n = 4$.

Theorem 2.4. *If the complex $C_\bullet(X_1, X_2, \dots)$*

$$C_\bullet : \quad \cdots \xrightarrow{\phi_3} X_2 \xrightarrow{\phi_2} X_1 \xrightarrow{\phi_1} 0$$

is exact at X_k, then the complex $C^\bullet({X_1}^*, {X_2}^*, \dots)$

$$C^\bullet : \quad \cdots \xleftarrow{{\phi_3}^*} {X_2}^* \xleftarrow{{\phi_2}^*} {X_1}^* \xleftarrow{{\phi_1}^*} 0,$$

where ${\phi_1}^*, {\phi_2}^*, \dots$ are the dual maps of $\phi_1, \phi_2, \dots$ respectively, is exact at ${X_k}^*$.

3 Cyclic Koszulness and Fundamental Theorems

We study a generalisation of Koszul algebras by examining whether more complicated collections of subspaces of $V^{\otimes n}$ form a distributive lattice. To illustrate and define this explicitly, we use oriented graphs.

3.1 A generalisation of Koszulness and oriented graphs

The collection of subspaces $V^{\otimes i} \otimes I \otimes V^{\otimes n-2-i}$ can be represented through an oriented graph (Figure 2) with n vertices where each vertex is marked with a number from 0 to $n-1$ and an edge between two vertices is drawn if and only if their indices are consecutive numbers. Moreover, the edge is always oriented towards the vertex with the bigger index.

We also denote by e_i the edge connecting the vertices with indices i and $i+1$. We place a copy of a given vector space V at every vertex. We define e_i to correspond to the vector space $V^{\otimes i} \otimes I \otimes V^{\otimes n-2-i}$. The indices of the vertices connected by e_i imply that I will occupy the $(i+1)^{st}$ and $(i+2)^{nd}$ position of the tensor product.

Our objective is to consider subspaces of $V^{\otimes n}$ not just of the form $V^{\otimes i} \otimes I \otimes V^{\otimes n-2-i}$, but others where I is represented by two nonconsecutive positions in the tensor product.

Example 3. Let $S(V)$ be the symmetric algebra where V has a basis $\{x, y\}$ and let $n = 3$. We can consider the subspace W spanned by the sum

$$a_1(\mathbf{y}x\mathbf{x} - \mathbf{x}x\mathbf{y}) + a_2(\mathbf{y}y\mathbf{x} - \mathbf{x}y\mathbf{y})$$

and ask whether it creates a distributive lattice together with $V \otimes I$ and $I \otimes V$. Note that W can also be defined by taking all vectors of the form $\mathbf{y}v\mathbf{x} - \mathbf{x}v\mathbf{y}$ where v can be any vector in V. Notice that the product $yx - xy$, which spans I, is "split" in first and third position of the tensor product.

We use oriented graphs to explicitly define our generalisation of Koszul algebras. For a given oriented graph $G([n], E)$ where n is the number of vertices and E is the set of edges, one can again place copies of V at each vertex. Then again each edge represents a subspace of $V^{\otimes n}$, however now a given edge may connect vertices with nonconsecutive indexes, for example m and $m + p$ where $p \geq 2$.

These new vector spaces have a more complex structure as in Example 3. Let I have a basis $\alpha_1, \ldots, \alpha_s$. Therefore, each α_q ($q \in \{1, \ldots, s\}$) can be represented as a sum $\sum_{j_1, j_2 \in [1,\ldots,k]} v_{j_1}^{(q)} \otimes v_{j_2}^{(q)}$ where $v_{j_1}^{(k)}$, $v_{j_2}^{(k)} \in V$ and k can be any positive integer smaller than or equal to the square of the dimension of V. The restriction for k comes from the fact that $(\dim V)^2 = \dim V \otimes V$. If we, for example, accept that the edge between the vertices with indices m and $m + p$ (Recall that our first vertex had index 0.) is oriented towards the vertex with index m we can define the vector space of that edge to contain all vectors which are spanned by the sum

$$\sum_{i=1}^{s} \sum_{j_1, j_2 \in \{1,\ldots,k\}} v_{(i,0)} \otimes \ldots \otimes v_{(i,m-1)} \otimes v_{j_2}^{(i)} \otimes v_{(i,m+1)} \otimes \ldots \otimes v_{(i,m+p-1)} \otimes v_{j_1}^{(i)} \otimes v_{(i,m+p+1)} \otimes \ldots \otimes v_{(i,n-1)}, \tag{3}$$

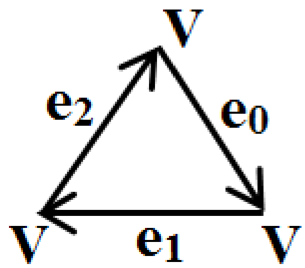

Figure 3: Visualising $n = 3$ for cyclic graphs.

where $v_{(i,y)}$ can be any vector in V for $y \in \{0, 1, \dots, \widehat{m}, m+1 \dots, \widehat{m+p}, \dots, n-1\}$. If the edge is oriented towards the vertex with index $m+p$ then the vectors $v_{j_1}^{(i)}$ and $v_{j_2}^{(i)}$ just interchange their positions in the tensor product. Note that when $p = 1$, we get the vector space $V^{\otimes m} \otimes I \otimes V^{\otimes n-m-2}$ and thus recover the original construction when our graph has the form from Figure 2.

What are the properties, which the subspaces, corresponding to edges, satisfy for different graphs? Do they still form a distributive lattice and for which I does that happen? Are all theorems from the usual case still true?

3.2 Cyclic Koszulness and basic properties

The object of study in this paper is the case when the graph G is a cycle (see Figure 3). In other words we have one more subspace of $V^{\otimes n}$, where I is represented by the first and last position of the tensor product. More precisely, the vectors $v_{j_2}^{(i)}$ and $v_{j_1}^{(i)}$ are in the first and last position of the tensor products in (3), respectively. For instance, this happened in Example 3.

The edge between the vertices with indices 0 and $n-1$ is denoted by e_{n-1}. We denote the vector space represented by the edge e_i by E_i ($i \in \{0, \dots, n-1\}$), so that no confusion arises.

Definition 3.1. For each $n \geq 2$, let $e_0, \dots, e_{n-1}$ be the edges of a cycle where all edges, except e_{n-1}, are oriented towards the vertex with the bigger index. An algebra $A = T(V)/\langle I \rangle$ is *Cyclic n-Koszul* if the vector spaces $E_0, \dots, E_{n-1}$ form a distributive lattice. The algebra A is *Cyclic Koszul* if it is cyclic n-Koszul for all $n \geq 2$.

We continue with our first result.

Theorem 3.1. *An algebra $A = T(V)/\langle I \rangle$ is Cyclic n-Koszul if and only if the dual algebra $A^! = T(V^*)/\langle I^\perp \rangle$ is Cyclic n-Koszul. More generally, an algebra A is Cyclic Koszul if and only if the dual algebra $A^!$ is Cyclic Koszul.*

Before proceeding with the proof we need a few lemmas.

Lemma 3.2. *If $E_0, \dots, E_{n-2}$ form a distributive lattice, then the collection $E_0, \dots, \hat{E}_k, \dots E_{n-1}$ also forms a distributive lattice .*

Proof. By Theorem 2.1 there is a basis B of $V^{\otimes n}$ which contains bases for $E_0, \dots, E_{n-2}$. Every vector in B has the form $\sum_{i=1}^{m^n} \bigotimes_{j=1}^{n} v_{i_j}$ where each $v_{i_j} \in V$ and $\dim V = m$. Let us change the vectors v_{i_j} by mapping them to $v_{i_{j+k+1}}$ with the rule that if $j + k + 1 > n$ then we just take the value modulo n. Because of the bilinearity of the tensor product the transformed vectors are still a basis of $V^{\otimes n}$. Moreover, for the same reason, if in the beginning a collection of vector were a basis for E_j, they are now a basis for E_{j+k+1}. Because $n - 1 + k + 1 \equiv k \pmod{n}$, no basis is given to E_k. Because each of the other vector spaces gets one basis transported to it, we can conclude that the collection of vector spaces $E_0, \dots, \widehat{E_k}, \dots E_{n-1}$ forms a distributive lattice. $\square$

The next lemma is standard and relates the notions of a dual and direct sum.

Lemma 3.3. *Let us have two finite-dimensional vector spaces V and W. Then if $\oplus$ is the exterior direct sum, the vector spaces $(V \oplus W)^*$ and $V^* \oplus W^*$ are isomorphic.*

The next lemma connects the notions of quotient, duality and orthogonal complement.

Lemma 3.4 (e.g. Conrad [4, p. 7])**.** *Let V be a finite dimensional vector space. If W is a subspace of V, then the vector spaces $(V/W)^*$ and $W^\perp$ are isomorphic.*

The next lemma is again widely used.

Lemma 3.5 (Sharipov [5, p. 97])**.** *Let V be a vector space and X_1 and X_2 be subspaces. Then for ${X_1}^\perp,\ {X_1}^\perp \in V^*$ we have ${X_1}^\perp \cap {X_2}^\perp = (X_1 + X_2)^\perp$*

Proof of Theorem 3.1. We need to prove that if the collection of vector spaces $E_0, \ldots, E_{n-1}$ forms a distributive lattice then so do $E_0^{'}, \ldots, E_{n-1}^{'}$, where $E_i^{'}$ is the same as E_i, however, instead of V we use V^* and in the place of $I \subseteq V \otimes V$, we use $I^\perp \subseteq V^* \otimes V^*$. We will prove two things. Firstly, that any proper subset of $E_0^{'}, \ldots, E_{n-1}^{'}$ forms a distributive lattice and secondly, that the complex $B_\bullet(V^{*\otimes n}; E_0^{'}, \ldots, E_{n-1}^{'})$ (see (1)) is exact everywhere except for the leftmost term.

Because $E_0, \ldots, E_{n-1}$ form a distributive lattice, so does the collection $\{E_0, \ldots, E_{n-2}\}$. Notice that this subset is visualised not by a cycle but by the same graph which visualises the normal Koszul algebras (Figure 2). We just do not consider the edge that connects the first with the last vertex. Therefore, by Theorem 2.2 we have that the collection $E_0^{'}, \ldots, E_{n-2}^{'}$ forms a distributive lattice. Thus, from Lemma 3.2 we obtain that each of the collections $\{E_0^{'}, \ldots, \widehat{E_k^{'}}, \ldots, E_{n-1}^{'}\}$ form a distributive lattice. This finishes the first part of the proof.

From Proposition 2.3 it follows that the complex $B^\bullet(V^{\otimes n}; E_0, \ldots, E_{n-1})$ is exact everywhere except for its rightmost term. We prove that this is equivalent to $B_\bullet(V^{*\otimes n}; E_0^{'}, \ldots, E_{n-1}^{'})$ being exact everywhere except at its leftmost term.

Let us take the dual of any of the elements in $B_\bullet(V^{*\otimes n}; E_0^{'}, \ldots, E_{n-1}^{'})$ apart from $V^{*\otimes n}$ and 0. From Lemma 3.3 and Lemma 3.4 it follows that

$$\left(\bigoplus_{t_1<\cdots<t_{n-i}} V^{*\otimes n}/(\sum_{s=1}^{n-i} E_{i_s}^{'})\right)^{*} \cong \bigoplus_{t_1<\cdots<t_{n-i}} \left(V^{\otimes n}/(\sum_{s=1}^{n-i} E_{i_s}^{'})\right)^{*} \cong \left(\sum_{s=1}^{n-i} E_{i_s}^{'}\right)^{\perp},$$

By Lemma 3.5 we have that

$$\left(\sum_{s=1}^{n-i} E_{i_s}^{'}\right)^{\perp} \cong \bigcap_{s=1}^{n-i} (E_{i_s}^{'})^{\perp},$$

Finally, we prove that $(E_k^{'})^{\perp} \cong E_k$, where $k \in \{0, \dots, n-1\}$. For $k \in \{0, \dots, n-2\}$

$$E_k = V^{\otimes k} \otimes I \otimes V^{\otimes n-2-k},$$
$$E_{i_s}^{'} = V^{*\otimes k} \otimes I^{\perp} \otimes V^{*\otimes n-2-k}.$$

Note that if $\psi \in V^*$, $v \in V$, $w \in W$, $\psi \in W^*$, for some vector spaces V and W, then $(\phi \otimes \psi)(v \otimes w) = \phi(v) \times \psi(w)$. Since I and $I^{\perp}$ both occupy the $k+1^{th}$ and $k+2^{nd}$ position in their respective tensor products E_k and $E_{i_s}^{'}$, it follows that

$$V^{\otimes k} \otimes I \otimes V^{\otimes n-2-k} \subseteq (V^{*\otimes k} \otimes I^{\perp} \otimes V^{*\otimes n-2-k})^{\perp}.$$

When we look at E_{n-1} and $E_{n-1}^{'}$ the same argument still holds because in the sum (3) the vector subspaces I and $I^{\perp}$ are placed in the first and last position of all tensor products. We dot not prove the inverse inclusion, instead we prove that E_k and $E_k^{'}$ have the same dimension. Let $\dim I = i$. Then $\dim V^{\otimes k} \otimes I \otimes V^{\otimes n-2-k} = m^{n-2} i$ (recall that m is the dimension of V). Moreover, $\dim V^{*\otimes k} \otimes I^{\perp} \otimes V^{*\otimes n-2-k} = m^{n-2}(m^2 - i)$. Therefore,

$$\dim(V^{*\otimes k} \otimes I^{\perp} \otimes V^{*\otimes n-2-k})^{\perp} = m^n - m^{n-2}(m^2 - i) = m^{n-2} i.$$

Thus, since the dual of 0 is 0, we can directly apply Theorem 2.4 to prove the exactness of $B_{\bullet}(V^{*\otimes n}; E_0^{'}, \dots, E_{n-1}^{'})$ at every term except for the leftmost one. This finishes the second part of our proof. □

Theorem 3.6. *Let $A = T(V)/\langle I \rangle$ and $B = T(W)/\langle J \rangle$ be two Cyclic n-Koszul algebras. Then the algebra $C = T(V \otimes W)/\langle I \otimes J \rangle$ is also a Cyclic n-Koszul algebra. More generally, if A and B are Cyclic Koszul algebras, then C is also a Cyclic Koszul algebra.*

Proof. Let $E_{(0,a)}, \dots, E_{(n-1,a)}$; $E_{(0,b)}, \dots, E_{(n-1,b)}$ and $E_{(0,c)}, \dots, E_{(n-1,c)}$ be the vector spaces created from the cyclic graphs when at each vertex we place V, W or $V \otimes W$, respectively. From Theorem 2.1 it follows that there is a basis B_1 of $V^{\otimes n}$ which contains bases $B_{(0,a)}, \ldots, B_{(n-1,a)}$ for $E_{(0,a)}, \ldots, E_{(n-1,a)}$, respectively. Similarly, there is a basis B_2 of $W^{\otimes n}$ which contains bases $B_{(0,b)}, \ldots, B_{(n-1,b)}$ for $E_{(0,b)}, \ldots, E_{(n-1,b)}$, respectively. Let us consider the vector space $R = V^{\otimes n} \otimes W^{\otimes n}$. It has a basis $B_1 \otimes B_2$. Each of the subspaces $E_{(i,a)} \otimes E_{(i,b)}$ has a basis $B_{(i,a)} \otimes B_{(i,b)}$. Let us consider the canonical isomorphism between $V^{\otimes n} \otimes W^{\otimes n}$ and $(V \otimes W)^{\otimes n}$. Then $E_{(i,a)} \otimes E_{(i,b)}$ is mapped to $E_{(i,c)}$. Respectively, $B_{(i,a)} \otimes B_{(i,b)}$ is mapped to a basis of $E_{(i,c)}$. $\square$

This theorem will allow one to construct high-dimensional Cyclic Koszul algebras with complicated quotients if a few low-dimensional Cyclic Koszul algebras are found.

4 Cyclic Koszulness of $S(V)$ and quantization

We use the notation from the previous section.

One of the most well studied algebras obtained as a quotient of the tensor algebra is the symmetric algebra $S(V) = T(V)/\langle I \rangle$, where $\langle I \rangle$ is generated by the differences of products $v \otimes w - w \otimes v$ for all $v, w \in V$. It is well-known that $S(V)$ is Koszul for all possible dimensions of V, see e.g. Polishchuk and Positselski [2, p. 20].

Later E. You investigated for which I is $T(V)/\langle I \rangle$ Koszul when $\dim V = 2$. He obtained the following result.

Theorem 4.1 (You [6, p. 12]). *We assume that $A = \bigoplus_{i=0}^{\infty} A_i = T(V)/\langle I \rangle$ is a quadratic algebra, where $V = A_1$, $\dim V = 2$, and I is is a set of all quadratic relations of A.*

(1) When $\dim I = 0, 1, 3, 4$, this A is Koszul;

(2) When $\dim I = 2$, only A, which has $\dim A_3 = 2$, is Koszul.

In this chapter we ask whether similar theorems hold for Cyclic Koszulness. A surprisingly interesting structure arises.

Conjecture 1. *Let V be a vector space with a basis x, y and let $a \in \mathbb{C}$ be a nonzero number. The algebra $A = T(V)/\langle y \otimes x - ax \otimes y \rangle$ is Cyclic 2-Koszul. For each $n \geq 3$, A is Cyclic n-Koszul if and only if a is not an n^{th} root of unity. More generally, the algebra A is Cyclic Koszul if and only if a is not a root of unity.*

In other words, the symmetric algebra is not Cyclic Koszul, but when we quantize it, it is often Cyclic Koszul. For comparison, the symmetric algebra, as well as its quantized version are Koszul which follows directly from Theorem 4.1. We partially prove the above conjecture and the studied cases strongly suggest it is true.

4.1 The $a = 1$ and $n \geq 2$ case

When $a = 1$, the algebra A becomes the symmetric algebra $S(V)$, because $\dim V = 2$. From Theorem 4.1 we know that $S(V)$ is Koszul. Thus, we can investigate its Cyclic n-Koszulness through the use of Proposition 2.3 and Lemma 3.2. More precisely, for $S(V)$ to be Cyclic n-Koszul, we need to prove that the complex $B^\bullet(V^{\otimes n}; E_0, \ldots, E_{n-1})$

$$0 \to \bigcap_s E_s \to \cdots \to \bigoplus_{t_1 < \cdots < t_{n-i}} \bigcap_{s=1}^{n-i} E_{t_s} \to \cdots \to \bigoplus_t E_t \to V^{\otimes n}, \tag{4}$$

where $s, t, t_1, \ldots, t_s \in \{0, \ldots, n-1\}$, is exact everywhere except for its rightmost term.

Case 1. Let $n = 2$. Then we end up with the complex

$$0 \xrightarrow{\phi_3} I \xrightarrow{\phi_2} I \otimes I \xrightarrow{\phi_1} V^{\otimes 2}.$$

Note that the second term of the complex is $I \cap I = I$. The map $\phi_2 : a \rightsquigarrow (a, -a)$ is injective which confirms the exactness at I. The kernel of ϕ_1 is also one dimensional since $\phi_1(\alpha_1, \alpha_2) = 0$ if and only if $\alpha_1 + \alpha_2 = 0$. Thus, the complex is exact at $I \otimes I$ as well which proves that $S(V)$ is Cyclic 2-Koszul.

Case 2. Let $n \geq 3$. Let us look only at the exactness at $\bigoplus_t E_t$

$$\cdots \to \bigoplus_{t_1<t_2} E_{t_1} \cap E_{t_2} \xrightarrow{\phi_2} \bigoplus_t E_t \xrightarrow{\phi_1} V^{\otimes n}.$$

We will show that there is an element in the $\ker \phi_1$ which is not an element in $\operatorname{Im} \phi_2$ by constructing an example. When multiplying x and y with tensor product we ignore the symbol $\otimes$ in order for the formulas to be clearer. Consider the element

$$\left((yx - xy)x^{n-2}, x(yx - xy)x^{n-3}, \ldots, x^{n-2}(yx - xy), yx^{n-1} - x^{n-1}y\right).$$

When we sum all n of its parts we get zero which implies that it is in the kernel of ϕ_1. However, the intersections $E_{t_1} \cap E_{t_2}$ are zero if t_1 and t_2 are consecutive numbers or if they are the numbers 0 and $n-1$. The latter can be checked directly by showing that $(I \otimes V) \cap (V \otimes I) = \{0\}$. If t_1 and t_2 are not consecutive then we have $E_{t_1} \cap E_{t_2} = V^{\otimes t_1} \otimes I \otimes V^{\otimes t_2 - t_1 - 1} \otimes I \otimes V^{\otimes n - t_2 - 2}$. Because I is the span of $yx - xy$, a linear sum of elements in $E_0 \cap E_2, \ldots, E_0 \cap E_{n-2}$ can never give us the element $(yx - xy)x^{n-2}$, where we have $n-2$ consecutive x's. Therefore, the case $a = 1$ is solved.

4.2 The cases $n = 2, 3, 4, 5$ and a not a root of unity

Firstly we note that for $n = 2$, one can prove that for any nonzero a, the algebra $A = T(V)/\langle yx - axy\rangle$ is Cyclic 2-Koszul by using the same method as in section 4.1, because the value of a is not used in the proof.

We show that if $n = 3, 4, 5$ and if a is not a $3^{rd}, 4^{th}, 5^{th}$ root of unity, respectively, then the algebra A is Cyclic n-Koszul. We solve the case $n = 4$, but the other two are analogous.

We have the complex $B^\bullet(V^{\otimes n}; E_0, \ldots, E_{n-1})$

$$0 \to \bigcap_{s=0}^{3} E_s \to \bigoplus_{t_1<t_2<t_3} \bigcap_{s=1}^{3} E_{t_s} \to \bigoplus_{t_1<t_2} \bigcap_{s=1}^{2} E_{t_s} \to \bigoplus_{t=0}^{3} E_t \to V^{\otimes 4}.$$

The relation $(V \otimes I) \cap (I \otimes V) = 0$ simplifies it to

$$0 \to E_0 \cap E_2 \oplus E_1 \cap E_3 \xrightarrow{\phi_2} E_0 \oplus E_1 \oplus E_2 \oplus E_3 \xrightarrow{\phi_1} V^{\otimes 4}.$$

The map ϕ_2 sending a pair (α_1, α_2) to $(-\alpha_1, -\alpha_2, \alpha_1, \alpha_2)$ is injective which proves the exactness at $E_0 \cap E_2 \oplus E_1 \cap E_3$. The rank of ϕ_2 is two, so to prove exactness at $\bigoplus_{t=0}^{3} E_t$ we need to show that the kernel of ϕ_1 is two-dimensional. Note that the dimension of $\bigoplus_{t=0}^{3} E_t$ is 16. Therefore, our problem is equivalent to showing that the matrix representing ϕ_1 with respect to some bases has rank 14. For the standard bases the matrix was found and the rank was calculated to be 14 with a computer program (see Appendix A). When $a = 1$, for example, the rank was 12, not implying exactness as stated by our conjecture.

5 Conclusion and Future development

We introduced a new structure - Cyclic Koszul algebras, and proved fundamental properties for it. We plan to continue the project by searching for properties of Cyclic Koszul algebras which are not generally present in Koszul algebras. An example would be to look at the Hilbert series of Cyclic Koszul algebras and try to prove some kind of identities.

We believe that Conjecture 1 is implied by two theorems from noncommutative geometry, although this observation is not rigorous. The first theorem by Berest, Felder and Ramadoss in [7], says that the homology of the quantized symmetric algebra $A = T(V)/\langle yx - axy \rangle$ vanishes when a is not a root of unity. The second theorem by Feigin and Tsygan in [8], states that the cyclic homology of A is isomorphic to its homology, so it also vanishes. A rigorous proof to our conjecture, with the apparatus in this paper can be of certain interest.

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A Matrices

The basis of V is $\{x, y\}$ and the basis of I is $(yx - axy)$. Then we can naturally define the basis of E_i for all i and from there the bases for $E_0 \oplus E_1 \oplus E_2 \oplus E_3$ and $V^{\otimes 4}$ are also defined. The matrix representing ϕ_1 is

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 1 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$