The Lusztig-Vogan Bijection in the Case of the Trivial Representation

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Abstract

Let G be a connected complex reductive algebraic group, and let $\mathfrak{g}$ be its Lie algebra. Let Λ^+ be the set of dominant weights of G, and let Ω be the set of pairs of the form $(\mathcal{O}, \mathcal{E})$, where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$ and $\mathcal{E}$ is an irreducible representation of the centralizer G^e for $e \in \mathcal{O}$. Then there exists a natural bijection between Λ^+ and Ω, which has come to be known as the Lusztig-Vogan bijection. For certain groups G, both Λ^+ and Ω can be indexed with relatively simple combinatorial structures. Using these descriptions, algorithms computing the Lusztig-Vogan bijection $\gamma : \Omega \to \Lambda^+$ when G is the complex general linear group $GL_n(\mathbb{C})$ were previously described by Achar and Rush. Here we study the algorithm given by Rush and give a closed form for γ in the case where $\mathcal{E}$ is the trivial representation.

Summary

In 2003, Bezrukavnikov proved a conjecture independently formulated by Lusztig and Vogan which described a correspondence, or bijection, between two particular sets associated with a reductive group G. This particular bijection came to be known as the Lusztig-Vogan bijection. A bijection exists between two sets if each element of one set can be paired with exactly one element of the other set, and vice versa.

Here we focus on the case where G is the complex general linear group, often denoted $GL_n(\mathbb{C})$, where n is some positive integer. In this case, many of the objects involved in the bijection give way to simple combinatorial descriptions, which Achar and Rush used to describe elementary algorithms computing the bijection for $G = GL_n(\mathbb{C})$. We study the algorithm given by Rush and then derive a closed-form expression that explicitly computes the bijection for a particular subcase referred to as the trivial representation.

1 Introduction

Representation theory seeks to better understand a particular algebraic structure by associating each element of that structure with a linear transformation on some vector space. In this way, techniques from linear algebra may be applied to better understand the algebraic structure. In particular, one can better understand a group through its group representations. Group representations arise frequently in a variety of situations, with applications both inside and outside of mathematics, such as in number theory and physics.

Starting in the late 1800s, several mathematicians, including Maurer, Chevalley, Borel, and Kolchin, began studying an important class of groups called algebraic groups. This theory was developed using methods from algebraic geometry and Lie theory, with numerous applications in representation theory and group theory. In particular, one of the more recent discoveries in the representation theory of algebraic groups is the Lusztig-Vogan bijection.

The Lusztig-Vogan bijection holds an important place in the representation theory of reductive groups, a subclass of algebraic groups. Investigating the bijection in important special cases would further reveal some of the properties of the bijection. Here, we calculate a closed form for the bijection in the case of the complex general linear group for the trivial representation, one of the most fundamental special cases.

To describe the bijection, we first take G to be a connected complex reductive algebraic group with Lie algebra $\mathfrak{g}$, and take $\mathcal{N}$ to be the nilpotent cone in $\mathfrak{g}$. Let Λ^+ be the set of dominant weights of G and Ω be the set of pairs, up to isomorphism, of the form $(\mathcal{O}, \mathcal{E})$, where $\mathcal{O} \subseteq \mathcal{N}$ is a nilpotent orbit, with respect to the conjugation action, and $\mathcal{E}$ is an irreducible representation of G^e, the centralizer of $e \in \mathcal{O}$. In 1989, while studying cells in the affine Weyl groups, Lusztig conjectured the existence of a natural bijection between Λ^+ and Ω [3]. Vogan independently reached the same conclusion in 1991 while studying associated varieties [7]. In 2003, Bezrukavnikov demonstrated the existence of such a bijection by considering $\mathcal{D}$, the bounded derived category of G-equivariant coherent sheaves on $\mathcal{N}$, showing the following [2]:

Theorem 1 (Bezrukavnikov)**.** *The Grothendieck group $K_0(\mathcal{D})$ is a free abelian group for which the two sets Λ^+ and Ω index bases. Moreover, the transition matrix between these two bases is upper-triangular.*

However, Bezrukavnikov's proof was nonconstructive. Achar has described algorithms to compute this bijection in the case of $G = GL_n(\mathbb{C})$ in [1], which Rush has improved [4]. We follow the algorithm in [4] to compute the bijection in the case that $\mathcal{E}$ is the trivial representation.

In Section 2, we recall the theory of representations of GL_n and dominant weights, and we calculate the dominant weights for the semisimple Lie algebra $\mathfrak{sl}_{n+1}$. In Section 3, we describe the algorithm given in [4] and provide example calculations. Section 4 contains an analysis of the algorithm when $\mathcal{E}$ is the trivial representation of G^e. Section 5 contains some directions for future research. For the rest of this paper, we take all ground fields to be the field of complex numbers.

2 Preliminaries

2.1 Representations of $GL_n(\mathbb{C})$

A representation of the group GL_n is a group homomorphism $\rho : GL_n \to GL(V)$ for some vector space V. To simplify, for all $g \in GL_n$ and $v \in V$, we write $g \cdot v$ to mean $\rho(g)(v)$. When ρ is clear from the context, we also refer to V itself as the representation.

The *trivial representation* is where ρ sends all $g \in GL_n$ to the identity map on V; i.e., for all $g \in GL_n$ and $v \in V$, we have $g \cdot v = v$. In addition, for the rest of this paper, we assume for simplicity that V is 1-dimensional when ρ is the trivial representation.

We say a representation ρ is *irreducible* if the only subspaces of V that are invariant under the action of GL_n are $\{0\}$ and V itself. For instance, the trivial representation is irreducible, assuming V is 1-dimensional.

The *torus* T of GL_n is the set of invertible diagonal $n \times n$ matrices. For all (rational) group homomorphisms $\lambda : T \to \mathbb{C}^*$, we write $V_\lambda := \{v \in V \mid t \cdot v = \lambda(t)v,\ \forall t \in T\}$. We say λ is a *weight* of the representation V if $V_\lambda \neq \{0\}$, and V_λ is its *weight space*.

Because we can identify T with $(\mathbb{C}^*)^n$, or the set of the n-tuples $(t_1, \ldots, t_n)$, all such λ can be written in the form $(t_1, \ldots, t_n) \mapsto t_1^{m_1} \ldots t_n^{m_n}$, for some $m_1, \ldots, m_n \in \mathbb{Z}$. This means each weight corresponds to an n-tuple of integers $(m_1, \ldots, m_n)$. We also have the weight space decomposition $V = \bigoplus_{m_1,\ldots,m_n \in \mathbb{Z}} V_{(m_1,\ldots,m_n)}$. By the definition of the trivial representation, the only weight of the trivial representation of GL_n is the n-tuple $(0, \ldots, 0)$, and $V = V_{(0,\ldots,0)}$.

We can define a partial order on the set of weights, saying $\lambda \geq \mu$ for two weights λ and μ if and only if we can write $\lambda - \mu = a_1\alpha_1 + \cdots + a_{n-1}\alpha_{n-1}$, where $a_i \geq 0$ for all i and $\alpha_1 = (1, -1, 0, \ldots)$, $\alpha_2 = (0, 1, -1, \ldots)$, and so on, until $\alpha_{n-1} = (0, 0, \ldots, 1, -1)$.

This partial order allows us to use the idea of *highest weights*. We say V is a *highest weight representation with weight* λ if $V_\lambda \neq \{0\}$ and $\mu \leq \lambda$ for all μ where $V_\mu \neq \{0\}$. If we let $\mathbb{Z}^n_+$ be the set of all n-tuples of weakly decreasing integers, then for all $\lambda \in \mathbb{Z}^n_+$, there is an irreducible representation $V(\lambda)$ of highest weight λ, with a one-dimensional highest weight space. Moreover, $V(\lambda)$ is unique up to isomorphism. In fact, the reverse is also true; from Proposition 2.1 of [6], we know that $\{V(\lambda) \mid \lambda \in \mathbb{Z}^n_+\}$ is actually a complete collection of irreducible representations of GL_n. So the weights in the set $\mathbb{Z}^n_+$ index irreducible representations of GL_n. These weights are called the *dominant weights* of GL_n.

2.2 Complex Semisimple Lie Algebras

Every algebraic group has an associated Lie algebra, which can be constructed from the tangent space at the identity of the group. It turns out that many properties of the Lie algebra correspond to the algebraic group, and vice versa.

Recall preliminaries on Lie algebras: let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then there exists a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is both abelian and self-normalizing (self-normalizing means if for some $x \in \mathfrak{g}$ we have $[x, \mathfrak{h}] \subseteq \mathfrak{h}$, then $x \in \mathfrak{h}$). We call $\mathfrak{h}$, which is unique up to conjugation, a *Cartan subalgebra*.

The *adjoint action* of an element $x \in \mathfrak{g}$ on $\mathfrak{g}$ is defined as $\mathrm{ad}_x := [x, -]$. The *Killing form* is defined as $\kappa_{\mathfrak{g}}(x, y) := \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$ for $x, y \in \mathfrak{g}$. Its restriction to $\mathfrak{h}$ is nondegenerate, so we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ under the isomorphism that takes x to the map $\kappa_{\mathfrak{g}}(x, -)$.

Now $\forall \alpha \in \mathfrak{h}^* \backslash \{0\}$, let

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h} : [h, x] = \alpha(h)x\}.$$

If $\mathfrak{g}_\alpha \neq \{0\}$, then we call α a *root* and $\mathfrak{g}_\alpha$ its *root space*. Let R be the set of all roots of $\mathfrak{g}$. When $\mathfrak{g}$ is complex and semisimple, there exists a *root space decomposition*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \mathfrak{h}^* \\ \alpha \neq 0}} \mathfrak{g}_\alpha.$$

R satisfies the following properties:

1. R spans $\mathfrak{h}^*$.

2. If $\alpha \in R$ and $\lambda \in \mathbb{C}$, then $\lambda\alpha \in R$ if and only if $\lambda = \pm 1$.

3. For all $\alpha, u \in \mathfrak{h}^*$, let $\langle\langle \alpha, u \rangle\rangle := 2\frac{(u,\alpha)}{(\alpha,\alpha)}$, where $(-,-)$ is the Killing form on $\mathfrak{h}^*$ inherited from the isomorphism with $\mathfrak{h}$. Then for all $\alpha, u \in R$, $\langle\langle \alpha, u \rangle\rangle \in \mathbb{Z}$.

4. There exists a subset $\Pi = \{\alpha_1, \ldots, \alpha_s\} \subseteq R$ such that Π forms a basis for $\mathfrak{h}^*$, and for all $\alpha \in R$, $\alpha = a_1\alpha_1 + \cdots + a_s\alpha_s$, where the a_i are either all nonnegative or all nonpositive. We refer to the α_i as the *simple roots*, and we call α a *positive root* if the a_i are nonnegative, and a *negative root* otherwise. We write R_+ as the set of all positive roots, and R_- as the set of all negative roots, so that R_+ and R_- partition R.

The *root lattice* Q of $\mathfrak{g}$ is defined as the lattice in $\mathfrak{h}^*$ generated by the root system R (or $\mathbb{Z}R$, the $\mathbb{Z}$-span of R). The set of *weights* of $\mathfrak{g}$, or the *weight lattice*, is defined as the set $P := \{\beta \in \mathfrak{h}^* \mid \langle\langle \alpha, \beta \rangle\rangle \in \mathbb{Z},\ \forall \alpha \in Q\}$. The set of *dominant weights* is defined as $P^+ := \{\lambda \in P \mid \langle\langle \alpha, \lambda \rangle\rangle \geq 0,\ \forall \alpha \in R_+\}$. This definition of dominant weights for semisimple Lie algebras corresponds to the definition in Section 2.1, which was given for dominant weights of GL_n.

2.3 Calculations of Dominant Weights

To study irreducible representations of GL_n, we first calculate the dominant weights of $\mathfrak{sl}_{n+1}$, the set of $(n+1) \times (n+1)$ matrices with complex entries and trace 0, using the commutator as the Lie bracket. In this case, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the set of all diagonal matrices with trace 0.

For all $1 \leq i, j \leq n+1$ such that $i \neq j$, let E_{ij} be the $(n+1) \times (n+1)$ matrix with all zero entries, except for a 1 entry in the (i, j) position. Additionally, for

$1 \leq i \leq n+1$, define $\epsilon_i \in \mathfrak{h}^*$ such that $\epsilon_i(h)$ equals the (i,i) entry of the matrix h for all $h \in \mathfrak{h}$.

Now for all $h \in \mathfrak{h}$ and $i \neq j$, we have that $[h, E_{ij}] = (\epsilon_i - \epsilon_j)(h)$. Thus the roots of $\mathfrak{g}$ are of the form $\epsilon_i - \epsilon_j$. The set Π of simple roots is $\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_n - \epsilon_{n+1}\}$. To simplify, we write $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

For all $\lambda \in \mathfrak{h}^*$, let $\lambda = a_1\alpha_1 + \cdots + a_n\alpha_n = \lambda_1\epsilon_1 + \cdots + \lambda_{n+1}\epsilon_{n+1}$. Then

$$\lambda = a_1\alpha_1 + \cdots + a_n\alpha_n = a_1\epsilon_1 + (a_2 - a_1)\epsilon_2 + \cdots + (a_n - a_{n-1})\epsilon_n - a_n\epsilon_{n+1}.$$

So we write $\lambda_1 = a_1, \lambda_2 = a_2 - a_1, \ldots, \lambda_n = a_n - a_{n-1}, \lambda_{n+1} = -a_n$.

We now compute the dominant weights of $\mathfrak{g}$, using the fact that λ is a dominant weight if and only if $\langle\langle \alpha, \lambda \rangle\rangle \in \mathbb{Z}_{\geq 0}$ for all simple roots $\alpha \in \Pi$.

So for $2 \leq i \leq n-1$, we have $\langle\langle \alpha_i, \lambda \rangle\rangle = -a_{i-1} + 2a_i - a_{i+1} = \lambda_i - \lambda_{i+1}$. Additionally, $\langle\langle \alpha_1, \lambda \rangle\rangle = 2a_1 - a_2 = \lambda_1 - \lambda_2$ and $\langle\langle \alpha_n, \lambda \rangle\rangle = 2a_n - a_{n-1} = \lambda_n - \lambda_{n+1}$.

Thus $\lambda \in P^+$ if and only if $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ is a nonincreasing sequence, where each difference $\lambda_{i+1} - \lambda_i$ is a nonnegative integer. Hence P^+ can be identified with the set of n-tuples of weakly decreasing sequences of nonnegative integers.

This type of reasoning can be used to deduce the fact that the dominant weights of GL_n correspond to n-tuples of weakly decreasing integers, as stated in Section 2.1. And similar reasoning can also be applied to the other classical Lie algebras, $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1}$, and $\mathfrak{sp}_{2n}$, and their corresponding Lie groups. For more details on Lie algebras, refer to [5].

3 An Algorithm for Computing the Lusztig-Vogan Bijection for $G = GL_n(\mathbb{C})$

In [1], Achar explicitly described the Lusztig-Vogan bijection for the case $G = GL_n$, which was improved and expanded upon by Rush in [4]. While Achar's algorithm featured structures called weight diagrams, the focus of Rush's paper was an algorithm using integer sequences.

In the case $G = GL_n(\mathbb{C})$, we identify both Λ^+ and Ω with combinatorial structures. First, every element of Λ^+ corresponds to a weakly decreasing sequence of n integers. As for Ω, let $X \in \mathfrak{g}$ be nilpotent, let its orbit, under conjugation, be $\mathcal{O}_X$, and let G_X be the stabilizer of X in G. The Jordan normal form of $\mathcal{O}$ corresponds to a partition $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_l] = [k_1^{a_1}, \ldots, k_m^{a_m}]$ of n, where the α_i are the sizes of the l Jordan blocks.

Now every irreducible representation of G_X can be identified with an irreducible representation of the reductive quotient $G_X^{\text{red}} \cong GL_{a_1} \times \cdots \times GL_{a_m}$. Using the concept of highest weight representations as given in Section 2.1, irreducible representations of $GL_{a_1} \times \cdots \times GL_{a_m}$ are indexed by m-tuples $(\mu_1, \ldots, \mu_m)$, where each μ_i is a dominant weight of GL_{a_i}. Say that an integer sequence $\nu = [\nu_1, \ldots, \nu_l]$ is *dominant* with respect to $\alpha = [\alpha_1, \ldots, \alpha_l]$ if $\alpha_i = \alpha_{i+1}$ implies $\nu_i \geq \nu_{i+1}$, and let Ω_α be the set of sequences which are dominant with respect to α. Thus Ω can be viewed as ordered pairs (α, ν), where α is a partition of n and $\nu \in \Omega_\alpha$.

Let $\alpha^* = [\alpha_1^*, \ldots, \alpha_s^*]$ be the transpose partition of α, where α_k^* equals the number of parts of α that are at least k, for all $1 \leq k \leq s := \alpha_1$. Let $L_\alpha \cong GL_{\alpha_1^*} \times \cdots \times GL_{\alpha_s^*}$ be the Levi factor of the parabolic subgroup associated with X, and Λ_α^+ be the set of dominant weights of L_α with respect to the Borel subgroup B_α. Each weight in Λ_α^+ can be identified with the concatenation of a dominant weight each of $GL_{\alpha_i^*}$ for all

$1 \leq i \leq s$.

3.1 The Algorithm

We now give a description of the integer-sequences algorithm in [4] to compute the Lusztig-Vogan bijection $\gamma : \Omega \to \Lambda^+$. Briefly, given a partition $\alpha \vdash n$ and $\nu \in \Omega_\alpha$, the algorithm $\mathfrak{A}$ outputs $\mathfrak{A}(\alpha, \nu) \in \Lambda^+_\alpha$. Then Rush [4] showed that:

Theorem 2. $\gamma(\alpha, \nu) = \mathrm{dom}(\mathfrak{A}(\alpha, \nu) + 2\rho_\alpha)$.

Here, for an integer sequence μ, $\mathrm{dom}(\mu)$ finds the unique element of Λ^+ in the W-orbit of μ, where W is the Weyl group of G. This definition coincides wih rearranging the terms in μ into nondecreasing order. In addition, ρ_α is defined as the half-sum of the positive roots of L_α.

We first define a few functions: let $\alpha = [\alpha_1, \ldots, \alpha_l]$ be a sequence of l positive integers, and $\nu = [\nu_1, \ldots, \nu_l]$ be a sequence of l integers. Let i be an integer where $1 \leq i \leq l$, and let the (possibly empty) sets I_a, I_b partition the set $\{1, \ldots, l\} \backslash \{i\}$. We define

$$\mathcal{C}(\alpha, \nu, i, I_a, I_b) := \left\lceil \frac{\nu_i - \sum_{j \in I_a} \min\{\alpha_i, \alpha_j\} + \sum_{j \in I_b} \min\{\alpha_i, \alpha_j\}}{\alpha_i} \right\rceil,$$

We now define the function $\mathcal{R}(\alpha, \nu)$. First, let $\mathfrak{S}_l$ be the set of permutations on the set $\{1, \ldots, l\}$. Then $\mathcal{R}$ is a function $\mathbb{N}^l \times \mathbb{Z}^l \to \mathfrak{S}_l$, and $\mathcal{R}$ is computed iteratively over l steps, where each step determines a new value of the permutation.

Specifically, let $\mathcal{R}(\alpha, \nu) = \sigma$. The i^{th} step of the computation finds $\sigma^{-1}(i)$, where σ^{-1} is the inverse permutation of σ. We write $J_i := \sigma^{-1}(\{1, \ldots, i-1\})$, the image of $\{1, \ldots, i-1\}$ under σ^{-1}, and $J'_i := \{1, \ldots, l\} \backslash J_i$.

Then $\sigma^{-1}(i)$ is chosen to be the value of $j \in J'_i$ such that the tuple

$$\left(\mathcal{C}\left(\alpha, \nu, j, J_i, J'_i \backslash \{j\}\right), \alpha_j, \nu_j, -j\right)$$

is lexicographically maximal.

We now define the algorithm $\mathcal{U}$ as a function $\mathbb{N}^l \times \mathbb{Z}^l \times \mathfrak{S}_l \to \mathbb{Z}^l$. In particular, $\mathcal{U}$ outputs a weakly decreasing sequence. The algorithm $\mathcal{U}$ is also computed over l steps.

If we let $\mathcal{U}(\alpha, \nu, \sigma) = [\mu_1, \ldots, \mu_l]$, then the i^{th} step determines μ_i. On the i^{th} step, as in $\mathcal{R}$, let $J_i := \sigma^{-1}(\{1, \ldots, i-1\})$ and $J'_i := \{1, \ldots, l\} \backslash J_i$. Then μ_i is computed as

$$\min\left\{\mu_{i-1}, \mathcal{C}\left(\alpha, \nu, \sigma^{-1}(i), J_i, J'_i \backslash \{\sigma^{-1}(i)\}\right) - l + 2i - 1\right\},$$

unless $i = 1$, in which case μ_i is just $\mathcal{C}\left(\alpha, \nu, \sigma^{-1}(i), J_i, J'_i \backslash \{\sigma^{-1}(i)\}\right) - l + 2i - 1$.

Now using the functions above, we can describe $\mathfrak{A}$ itself. We set $\mathfrak{A}(\alpha, \nu)$ to be a function $\mathfrak{A} : \mathbb{Y}_{n,l} \times \mathbb{Z}^l \to \mathbb{Z}^n$, where $\mathbb{Y}_{n,l}$ denotes the set of partitions of n into l parts.

The algorithm $\mathfrak{A}$ is computed iteratively. To compute $\mathfrak{A}(\alpha, \nu)$, it first sets, in order:

$$\sigma := \mathcal{R}(\alpha, \nu),$$

$$\mu := \mathcal{U}(\alpha, \nu, \sigma),$$

$$\alpha' := [\alpha^*_2, \ldots, \alpha^*_s]^*,$$

$$\nu' := \left[\nu_1 - \mu_{\sigma(1)}, \ldots, \nu_{\alpha^*_2} - \mu_{\sigma(\alpha^*_2)}\right].$$

Here, $\alpha^* = [\alpha^*_1, \ldots, \alpha^*_s]$ is the transpose partition of α as defined above, and μ_i is the i^{th} integer in the sequence μ for all $1 \leq i \leq l = \alpha^*_1$. In particular, note that both α'

and ν' have α_2^* parts.

Finally, $\mathfrak{A}(\alpha, \nu)$ is set to be the result when $\mathfrak{A}(\alpha', \nu')$ is appended to the end of μ.

Now to compute $\gamma(\alpha, \nu)$ as stated in Theorem 2, we first add $2\rho_\alpha$ to $\mathfrak{A}(\alpha, \nu)$ componentwise, and then order all n integers in weakly decreasing order to obtain a dominant weight of GL_n.

In this case, ρ_α is just the concatenation of s sequences, one per part in α^*, starting with α_1^* and ending with α_s^*. The sequence corresponding to the k^{th} part α_k^* is $\left[\frac{\alpha_k^*-1}{2}, \frac{\alpha_k^*-3}{2}, \ldots, \frac{-(\alpha_k^*-1)}{2}\right]$. For instance, if $\alpha = [3,2]$, then $\alpha^* = [2,2,1]$ and $\rho_\alpha = \left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0\right]$.

3.2 An Example Calculation

To show the algorithm in action, we compute it in the case of $n = 4$, $\alpha = [4]$. Here $\alpha^* = [1,1,1,1]$, $l = 1$, and $s = 4$. As $\Omega_{[4]} = \{[\nu_1] \mid \nu_1 \in \mathbb{Z}\}$, we have $\nu := [\nu_1]$ for some integer ν_1. To clarify the notation, we write α^k, ν^k, σ^k, and μ^k to denote the values of α, ν, σ, and μ, respectively, on the k^{th} iteration on the algorithm.

We must compute $\mathfrak{A}(\alpha, \nu)$. First, $\sigma^1 = \mathcal{R}([4], [\nu_1])$ is a permutation on $l = 1$ element, and so it is the identity permutation 1. Now we compute $\mu^1 = \mathcal{U}([4], [\nu_1], 1)$, which is a 1-term sequence. There is only one step, and we find

$$\begin{aligned}
\mu_1^1 &= \mathcal{C}([4], [\nu_1], 1, \emptyset, \{1\} \backslash \{1\}) - 1 + 2 - 1 \\
&= \left\lceil \frac{\nu_1 - 0 + 0}{4} \right\rceil \\
&= \left\lfloor \frac{\nu_1 + 3}{4} \right\rfloor .
\end{aligned}$$

So $\mu^1 = \left[\left\lfloor \frac{\nu_1+3}{4} \right\rfloor\right]$.

For the next iteration, we have $\alpha^2 = [1,1,1]^* = [3]$ and $\nu^2 = \left[\nu_1 - \left\lfloor \frac{\nu_1+3}{4} \right\rfloor\right] =$

$\left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor\right]$. So we must calculate $\mathfrak{A}\left([3], \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor\right]\right)$. Again, $\sigma^2 = \mathcal{R}\left([3], \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor\right]\right)$ is only a permutation on 1 element, and so it is the identity permutation 1. To find $\mu^2 = \mathcal{U}\left([3], \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor\right], 1\right)$, again a 1-term sequence, we compute

$$\begin{aligned}
\mu_1^2 &= \mathcal{C}\left([3], \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor\right], 1, \emptyset, \{1\}\backslash\{1\}\right) - 1 + 2 - 1 \\
&= \left\lceil \frac{\left\lfloor \frac{3\nu_1}{4} \right\rfloor - 0 + 0}{3} \right\rceil \\
&= \left\lfloor \frac{\nu_1 + 2}{4} \right\rfloor .
\end{aligned}$$

So $\mu^2 = \left[\left\lfloor \frac{\nu_1+2}{4} \right\rfloor\right]$.

For the third iteration, we have $\alpha^3 = [1,1]^* = [2]$ and $\nu^3 = \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor - \left\lfloor \frac{\nu_1+2}{4} \right\rfloor\right]$. We must find $\mathfrak{A}([2], \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor - \left\lfloor \frac{\nu_1+2}{4} \right\rfloor\right])$. Then $\sigma^3 = 1$, the identity permutation, and to find $\mu^3 = \mathcal{U}\left([2], \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor - \left\lfloor \frac{\nu_1+2}{4} \right\rfloor\right], 1\right)$, we find

$$\begin{aligned}
\mu_1^3 &= \mathcal{C}\left([2], \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor - \left\lfloor \frac{\nu_1+2}{4} \right\rfloor\right], 1, \emptyset, \{1\}\backslash\{1\}\right) - 1 + 2 - 1 \\
&= \left\lceil \frac{\left(\left\lfloor \frac{3\nu_1}{4} \right\rfloor - \left\lfloor \frac{\nu_1+2}{4} \right\rfloor\right) - 0 + 0}{2} \right\rceil \\
&= \left\lfloor \frac{\nu_1 + 1}{4} \right\rfloor .
\end{aligned}$$

Thus $\mu^3 = \left[\left\lfloor \frac{\nu_1+1}{4} \right\rfloor\right]$.

Finally, on the fourth iteration, we have $\alpha^4 = [1]^* = [1]$ and $\nu^4 = \left[\left\lfloor \frac{3\nu_1}{4} \right\rfloor - \left\lfloor \frac{\nu_1+2}{4} \right\rfloor - \left\lfloor \frac{\nu_1+1}{4} \right\rfloor\right] = \left[\left\lfloor \frac{\nu_1}{4} \right\rfloor\right]$. To find $\mathfrak{A}([1], \left[\left\lfloor \frac{\nu_1}{4} \right\rfloor\right])$, we first have again $\sigma^4 = 1$, and then we find $\mu^4 =$

$\mathcal{U}\left([1], \left[\left\lfloor \frac{\nu_1}{4} \right\rfloor\right], 1\right)$:

$$\begin{aligned}
\mu_1^4 &= \mathcal{C}\left([1], \left[\left\lfloor \frac{\nu_1}{4} \right\rfloor\right], 1, \emptyset, \{1\}\backslash\{1\}\right) - 1 + 2 - 1 \\
&= \left\lceil \frac{\left\lfloor \frac{\nu_1}{4} \right\rfloor - 0 + 0}{1} \right\rceil \\
&= \left\lfloor \frac{\nu_1}{4} \right\rfloor .
\end{aligned}$$

Thus $\mu_1^4 = \left[\left\lfloor \frac{\nu_1}{4} \right\rfloor\right]$, and the algorithm stops. To find $\mathfrak{A}([4], [\nu_1])$, we concatenate the four sequences $\mu^1, \mu^2, \mu^3, \mu^4$ to obtain $\left[\left\lfloor \frac{\nu_1+3}{4} \right\rfloor, \left\lfloor \frac{\nu_1+2}{4} \right\rfloor, \left\lfloor \frac{\nu_1+1}{4} \right\rfloor, \left\lfloor \frac{\nu_1}{4} \right\rfloor\right]$.

To find $\gamma([4], [\nu_1])$, we must first find $2\rho_\alpha$. Because $\alpha^* = [1, 1, 1, 1]$, $\rho_\alpha = [0, 0, 0, 0]$. So

$$\begin{aligned}
\gamma([4], [\nu_1]) &= \operatorname{dom}\left(\mathfrak{A}([4], [\nu_1]) + 2[0, 0, 0, 0]\right) \\
&= \operatorname{dom}\left(\left[\left\lfloor \frac{\nu_1+3}{4} \right\rfloor, \left\lfloor \frac{\nu_1+2}{4} \right\rfloor, \left\lfloor \frac{\nu_1+1}{4} \right\rfloor, \left\lfloor \frac{\nu_1}{4} \right\rfloor\right] + 2[0, 0, 0, 0]\right) \\
&= \left[\left\lfloor \frac{\nu_1+3}{4} \right\rfloor, \left\lfloor \frac{\nu_1+2}{4} \right\rfloor, \left\lfloor \frac{\nu_1+1}{4} \right\rfloor, \left\lfloor \frac{\nu_1}{4} \right\rfloor\right].
\end{aligned}$$

4 The Case of the Trivial Representation

The algorithm can be computed more easily when $\nu = [0^l]$, which corresponds to $\mathcal{E}$ being the trivial representation. We claim that:

Theorem 3. *If $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_l]$ is a partition of n, then $\mathfrak{A}(\alpha, [0^l]) = [0^n]$.*

Proof. As defined above, let $\alpha^* = [\alpha_1^*, \ldots, \alpha_s^*]$ be the transpose partition of α. We induct on s.

For both the base case $s = 1$ and the inductive step, it suffices to show that for any α with l parts, the function $\mathcal{U}\left(\alpha, [0^l], \sigma\right)$ outputs $[0^l]$, where $\sigma = \mathcal{R}\left(\alpha, [0^l]\right)$. Indeed, suppose this is true. Then for the base case $s = 1$, the algorithm has only one iteration and $\mathfrak{A}(\alpha, [0^l]) = \mathcal{U}\left(\alpha, [0^l], \sigma\right) = [0^l] = [0^n]$. As for the inductive step, the iteration following the calculation of $\mathcal{U}\left(\alpha, [0^l], \sigma\right)$ uses the values $(\alpha')^* = [\alpha_2^*, \ldots, \alpha_s^*]$ and $\nu' = [0^l] - [0^l] = [0^l]$. Because the value of s decreases by 1, we know $\mathfrak{A}(\alpha', \nu') = \left[0^{n-l}\right]$, as the partition α' has parts summing to $n - \alpha_1^* = n - l$. Thus appending the $[0^{n-l}]$ to the end of $\mu = \mathcal{U}\left(\alpha, [0^l], \sigma\right) = [0^l]$ results in $[0^n]$, as desired.

We now proceed to show that $\mathcal{U}\left(\alpha, [0^l], \sigma\right) = [0^l]$. For all $1 \leq i \leq l$, define J_i and J_i' in the same way as in the description of the algorithm, namely $J_i = \sigma^{-1}(\{1, \ldots, i-1\})$ and $J_i' = \{1, \ldots, l\} \backslash J_i$. We will induct on i to show that $\mu_i = 0$ for all i.

For the base case $i = 1$, by the definitions of $\mathfrak{A}$, $\mathcal{R}$, and $\mathcal{U}$, it suffices to show that the expression $\mathcal{C}\left(\alpha, [0^l], j, J_i, J_i' \backslash \{j\}\right) = \mathcal{C}\left(\alpha, [0^l], j, \emptyset, \{1, \ldots, l\} \backslash \{j\}\right)$ attains a maximum of exactly $l - 2i + 1 = l - 1$ when it is taken over all $j \in J_i' = \{1, \ldots, l\}$. We compute:

$$
\begin{aligned}
\mathcal{C}\left(\alpha, [0^l], j, \emptyset, \{1, \ldots, l\} \backslash \{j\}\right) &= \left\lceil \frac{0 - 0 + \displaystyle\sum_{k \in \{1, \ldots, l\} \backslash \{j\}} \min\{\alpha_j, \alpha_k\}}{\alpha_j} \right\rceil \\
&= \left\lceil \frac{\alpha_j(j-1) + (\alpha_{j+1} + \cdots + \alpha_l)}{\alpha_j} \right\rceil \\
&= j - 1 + \left\lceil \frac{\alpha_{j+1} + \cdots + \alpha_l}{\alpha_j} \right\rceil \\
&\leq j - 1 + \left\lceil \frac{\alpha_j(l-j)}{\alpha_j} \right\rceil \\
&= l - 1.
\end{aligned}
$$

Equality holds at $j = l$, so the maximum over all j is exactly $l - 1$.

For the inductive step, where $i \geq 2$, it suffices to show that the expression $\mathcal{C}\left(\alpha, [0^l], j, J_i, J_i' \backslash \{j\}\right)$ attains a maximum of at least $l - 2i + 1$ when it is taken over all $j \in J_i'$.

In fact, such a j can be easily constructed as the maximum value of r such that $\alpha_r \in J_i'$. Let m be this maximum value of r. Again, we just compute:

$$\begin{aligned}
\mathcal{C}\left(\alpha, [0^l], m, J_i, J_i'\backslash\{m\}\right) &= \left\lceil \frac{0 - (\alpha_m(i-1-(l-m)) + (\alpha_{m+1} + \cdots + \alpha_l)) + (l-i)(\alpha_m)}{\alpha_m} \right\rceil \\
&= \left\lceil \frac{\alpha_m(2l-2i-m+1) - (\alpha_{m+1} + \cdots + \alpha_l)}{\alpha_m} \right\rceil \\
&= \left\lceil 2l-2i-m+1 - \frac{\alpha_{m+1} + \cdots + \alpha_l}{\alpha_m} \right\rceil \\
&\geq 2l-2i-m+1 + \left\lceil -\frac{(l-m)\alpha_m}{\alpha_m} \right\rceil \\
&= 2l-2i-m+1-(l-m) \\
&= l-2i+1.
\end{aligned}$$

Thus $\mathcal{C}\left(\alpha, [0^l], j, J_i, J_i'\backslash\{j\}\right)$ does indeed attain a maximum of at least $l-2i+1$.

After inducting on i, we have that $\mu_i = 0$ for each $1 \leq i \leq l$, so $\mathcal{U}(\alpha, [0^l], \sigma) = [0^l]$. And after inducting on s, this implies $\mathfrak{A}(\alpha, [0^l]) = [0^n]$ for any α. □

Corollary 3.1. *For all partitions α with l parts, $\gamma(\alpha, [0^l]) = \mathrm{dom}(2\rho_\alpha)$.*

This result follows directly from Theorem 3 and Theorem 2, as $\gamma(\alpha, [0^l]) = \mathrm{dom}(\mathfrak{A}(\alpha, [0^l]) + 2\rho_\alpha) = \mathrm{dom}(2\rho_\alpha)$. This provides a direct calculation for $\gamma(\alpha, [0^l])$ which avoids the intermediary $\mathfrak{A}$.

Corollary 3.2. *For all positive integers n, $\gamma([n], [0]) = [0^n]$ and $\gamma([1^n], [0^n]) = [n-1, n-3, \ldots, -(n-1)]$.*

This follows directly from Corollary 3.1 and the explicit formulas for ρ_α in these two cases given in Section 3.1.

For example, using Corollary 3.1, $\gamma([3,2],[0^2]) = \mathrm{dom}([1,-1,1,-1,0]) = [1,1,0,-1,-1]$. Using Corollary 3.2, $\gamma([3],[0]) = [0,0,0]$ and $\gamma([1^3],[0^3]) = [2,0,-2]$.

5 Future Work

The relative recentness of work done on the Lusztig-Vogan bijection opens several possible avenues for future research. An obvious direction would be to extend Theorem 3 and Corollary 3.1 by investigating other cases of the algorithm for GL_n. Another direction is to continue the work in [4] on the Lusztig-Vogan bijection for other classical groups, such as the symplectic group $Sp_n(\mathbb{C})$ and the special orthogonal group $SO_n(\mathbb{C})$. It may be possible to develop similar algorithms to compute the bijection in these cases. Another alternative would be to use Lusztig's work on cells in affine Weyl groups to study the bijection from a different point of view.

6 Conclusion

We discussed group representations and dominant weights, focusing on the complex general linear group GL_n. Then we studied an algorithm computing the Lusztig-Vogan bijection for this reductive group, focusing in particular on the case of the trivial representation for G^e. Using the algorithm, we found an explicit description for the dominant weight corresponding to the trivial representation for any nilpotent orbit.

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