Bounding the Deviation of the Valuation Property of Quermassintegrals for Non-convex Sets

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Abstract

The first quermassintegral of a set $C \subset \mathbb{R}^d$ is a characteristic of C closely related to its surface area. Quermassintegrals have traditionally been viewed in the context of convex geometry, and their definition relies on a classic theorem in the field, Steiner's Formula for Parallel Bodies. Nonetheless, the concept can apply to any set, and in this paper we look at how the properties of quermassintegrals of non-convex sets differ from those of convex sets. Specifically, the first quermassintegral, denoted by W_1, satisfies the so-called valuation property, $W_1(C \cup D) = W_1(C) + W_1(D) - W_1(C \cap D)$, when C, D, and their union are convex. Here we determine how far non-convex sets deviate from the valuation property of quermassintegrals by placing bounds on the value $\eta(C,D) = W_1(C) + W_1(D) - W_1(C \cup D) - W_1(C \cap D)$, for $C \cup D$ non-convex. Using both set-theoretic approaches and fundamental results of integral geometry, we show that in $\mathbb{R}^2$, $0 \leq \eta(C,D) \leq W_1(\text{conv}(C \cup D) \setminus (C \cup D))$, provided C and D are convex and their union is connected. Furthermore, we prove that the lower bound holds for all $C, D \subset \mathbb{R}^d$.

Summary

Convex geometry is the study of the properties of convex bodies – figures with surfaces that curve outwards, such as circles, cubes or spheres. Beyond the familiar characteristics of these sets of points in space, like volume and surface area, convex geometry also studies other characteristics, including the so-called *quermassintegral* of a set. The quermassintegral, which is related to the surface area, has traditionally been studied only in the context of convex sets, but can equally be applied to non-convex ones (crescents, stars, or tubes). In this paper we study an equation which is known to be satisfied by the quermassintegrals of convex sets but not all non-convex sets. We place upper and lower limits on how far the quermassintegrals of non-convex sets can deviate from satisfying this equation. We employ two different methods to establish these limits, one taking advantage of the connection between the quermassintegral and the surface area and the other using an existing formula for the quermassintegral of a set. We prove that sets in any dimension comply with these limits.

1 Introduction

Convex geometry is a branch of Euclidean geometry that developed as a distinct field of study only in the early 20th century. Modern convex geometry has its origins in the work of Hermann Minkowski and Hermann Brunn, whose research into the properties of convex sets was motivated by applications in number theory. More recently the field of convex geometry has found extensive use in optimization, maximizing or minimizing values on convex sets in parameter space, but it has also been applied to other problems in geometry, in particular isoperimetric problems [1]. Brunn and Minkowski, building on the work of Jakob Steiner, were responsible for some of the most fundamental results in the field, including Minkowski's theorem on mixed volumes. This theorem generalized Steiner's classic formula for parallel bodies. While Brunn and Minkowski's work focused on sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, later research by Fenchel, Aleksandrov, and Bonnesen extended the area of study to d dimensions [2].

Fundamental to all of these results is the concept of the Minkowski sum, which defines a notion of addition for sets in $\mathbb{R}^d$. The Minkowski sum of n sets of vectors is the set of vectors produced by summing one element from each set. One of the simplest, and most studied, Minkowski sums is the sum of an arbitrary two-dimensional set A with a two-dimensional disc B. To visualize the resulting set, imagine dipping the disc B in paint and dragging its center along the boundary and through the interior of the set A. The painted area, A surrounded by a strip of width equal to the radius of B, constitutes $A+B$ (Figure 1), where the Minkowski sum is denoted by the conventional plus sign.

The concept of Minkowski addition has wide-ranging practical applications, from engineering to computer science, including facilitating the calculation of the routes of drones through crowded spaces [3]. In mathematics, however, one of the most studied properties of Minkowski sums of sets is their volume.

Consider a convex body (compact convex set) C in $\mathbb{R}^d$ and a d-dimensional ball of radius t,

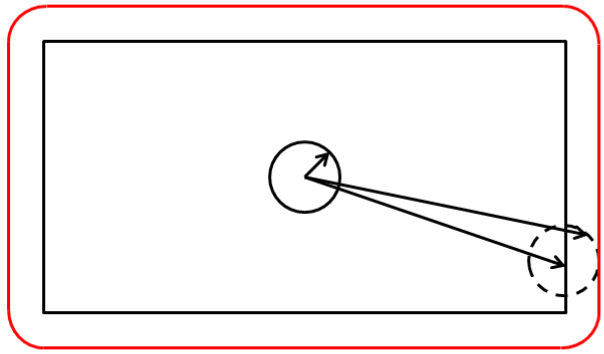

Figure 1: The Minkowski sum of a rectangle and a disc

denoted by tB^d. Steiner's formula, a classic result of convex geometry, states that the volume of the sum of these two sets is a polynomial in t, as follows [4]:

$$\mathrm{Vol}(C + tB^d) = W_0(C) + \binom{d}{1} W_1(C)t + \cdots + \binom{d}{d} W_d(C)t^d,$$

where $W_i(C)$ is the so-called i^{th} *quermassintegral* of C. These quermassintegrals have a number of properties, one of which, the *valuation property*, is the focus of this paper. The valuation property of the first quermassintegral [4] states that, provided C, D, and $C \cup D$ are convex bodies,

$$W_1(C \cup D) = W_1(C) + W_1(D) - W_1(C \cap D).$$

This is directly analogous to the more intuitive valuation property of volumes,

$$\mathrm{Vol}(C \cup D) = \mathrm{Vol}(C) + \mathrm{Vol}(D) - \mathrm{Vol}(C \cap D).$$

We see by the valuation property that for convex bodies C, D, and $C\cup D$, the value

$$\eta(C,D) := W_1(C)+W_1(D)-W_1(C\cup D)-W_1(C\cap D)$$

is equal to 0. In this paper we analyze $\eta(C,D)$ in the general case where $C\cup D$ need not be convex.

Section 2 gives further background on quermassintegrals and presents a definition of them that can be applied to non-convex sets. Section 3 then lays out an integral-geometric proof of upper and lower bounds on $\eta(C,D)$ based on the connection between the first quermassintegral and the surface area of a set. Because of the use of the Cauchy-Crofton formula in these proofs, they only show that the bounds apply for convex bodies with a connected union in $\mathbb{R}^2$, so in Section 4 we present an alternative proof of the bounds that extends them to any compact sets in $\mathbb{R}^d$. Finally, we calculate $\eta(C,D)$ for some simple examples and compare it to our bounds, to illustrate the concepts discussed in the previous sections.

2 Defining Quermassintegrals

Quermassintegrals are members of a broader family of operators known as *mixed volumes*. Traditionally, the mixed volume of a collection of convex bodies is defined using their Minkowski sum. The Minkowski sum of two sets A and B scaled by constants t_1 and t_2 is the set of all sums of vectors in A scaled by t_1 and vectors in B scaled by t_2, or

$$t_1A+t_2B = \{t_1x+t_2y : x\in A, y\in B\}.$$

Minkowski's theorem describes the volume of this sum. Let $A_1,\ldots,A_n$ be convex bodies in $\mathbb{R}^d$ and $t_1,\ldots,t_n\in\mathbb{R}$.

Theorem (Minkowski's theorem on mixed volumes [4]). *The volume* $\mathrm{Vol}(\sum_{k=1}^{n} t_kA_k)$ *is a polyno-*

mial of degree d in $t_1, \ldots, t_n$.

When every A_k is convex, the mixed volume $V(A_{x_1}, \ldots, A_{x_d})$, for indices $x_1, \ldots, x_d \in \{1, \ldots, n\}$, is defined as the coefficient of $t_{x_1} t_{x_2} \cdots t_{x_d}$ in the polynomial expression of $\mathrm{Vol}(\sum_{k=1}^{n} t_n A_n)$. From this definition and from Steiner's formula for parallel bodies, a special case of Minkowski's theorem, comes the notion of the *quermassintegrals* of a set. Steiner's formula addresses the specific case of only two convex bodies, where one is the d-dimensional ball of radius 1, B^d, stating that

$$\mathrm{Vol}(C + tB^d) = W_0(C) + \binom{d}{1} W_1(C)t + \cdots + \binom{d}{d} W_d(C)t^d,$$

where $W_i(C)$ is called the i^{th} quermassintegral of C. The quermassintegrals are mixed volumes, defined as

$$W_i(C) = V(\underbrace{C, \ldots, C}_{d-i}, \underbrace{B^d, \ldots, B^d}_{i}),\ i = 0, \ldots, d.$$

Hence $V(C, B^d, \ldots, B^d)$ is the first quermassintegral of C, and can be alternatively written as $W_1(C)$. It is this first quermassintegral with which we concern ourselves for the remainder of the paper.

However, the definitions of quermassintegrals and mixed volumes presented above apply only to convex bodies. Minkowski's theorem and Steiner's formula do not require that the volume of the Minkowski sum of non-convex sets be a polynomial, so definitions that depend on the coefficients of such a polynomial cannot be extended to non-convex sets. To define the quermassintegrals of non-convex sets we turn instead to Kubota's formula for quermassintegrals.

Let C be a set in $\mathbb{R}^d$, κ_{d-1} the $(d-1)$-dimensional volume of B^{d-1}, and w_i for $i = 0, \ldots, d-1$ the i^{th} $(d-1)$-dimensional quermassintegral. Additionally let $u \in S^{d-1}$, $d\sigma(u)$ the surface area element of S^{d-1}, and $C|u^{\perp}$ the orthogonal projection of C onto the space $u^{\perp} = \{x : u \cdot x = 0\}$ (Figure 2).

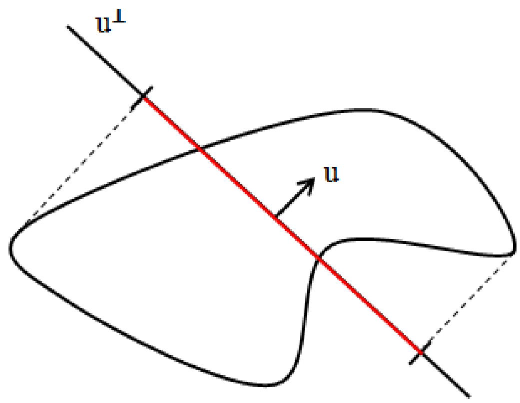

Figure 2: The setup for Kubota's formula in $\mathbb{R}^2$

Theorem (Kubota's formula for quermassintegrals [4])**.** *The i^{th} quermassintegral is given by*

$$W_i(C) = \frac{1}{d\kappa_{d-1}} \int_{S^{d-1}} w_{i-1}(C|u^{\perp}) d\sigma(u)$$

for $i = 1, \ldots, d$.

The base case, $W_0(C) = \mathrm{Vol}(C)$ for any d, can be found by letting $t = 0$ in Steiner's formula above.

Throughout the rest of this paper, we will use Kubota's formula as the fundamental definition of the quermassintegral.

3 An Integral-Geometric Approach

We first present a proof of the bounds on $\eta(C,D)$ that depends on the close link between the first quermassintegral and the surface area of a set. We begin by proving a lemma that establishes this relation, based on a formula for the surface area of convex bodies.

Let C be a convex body in $\mathbb{R}^d$ and v denote $(d-1)$-dimensional volume, with u, $C|u^{\perp}$, and

$d\sigma(u)$ defined as above.

Theorem (Cauchy's surface area formula [4]). *The surface area of C is given by*

$$S(C) = \frac{1}{\kappa_{d-1}} \int_{S^{d-1}} v(C|u^\perp) d\sigma(u).$$

Comparing this expression with Kubota's formula above, we see that for a convex body C,

$$W_1(C) = \frac{1}{d} \cdot S(C).$$

The following lemma extends this property to non-convex sets.

Lemma 1. *For all compact, connected $C \subset \mathbb{R}^2$,*

$$W_1(C) = \frac{1}{2} \cdot S(\mathrm{conv}(C)),$$

where $\mathrm{conv}(C)$ *denotes the convex hull of C.*

Proof. It is well-known that for some $C \subset \mathbb{R}^d$ and some $(d-1)$-dimensional hyperplane $u^\perp$, $\mathrm{conv}(C)|u^\perp = \mathrm{conv}(C|u^\perp)$. In $\mathbb{R}^2$, $\mathrm{conv}(C|u^\perp) = C|u^\perp$ if C is connected. Hence

$$\begin{aligned} W_1(C) &= \frac{1}{2\kappa_1} \int_{S^1} w_0(C|u^\perp) d\sigma(u) \\ &= \frac{1}{4} \int_{S^1} w_0(\mathrm{conv}(C)|u^\perp) d\sigma(u) \\ &= \frac{1}{2} \cdot S(\mathrm{conv}(C)). \end{aligned}$$

□

Before applying this property and our extended definition of quermassintegrals to sets in $\mathbb{R}^2$, it is helpful to introduce another theorem relating to surface area, specifically in two dimensions.

Let α be a rectifiable curve in the plane, and ℓ a line in the plane. Parameterizing lines by the angle θ of their slope and their closest distance to the origin, p, we denote the number of times α and ℓ intersect by $n_\alpha(\ell) = n_\alpha(\theta, p)$.

Theorem (The Cauchy-Crofton formula [5])**.** *The length of α is given by*

$$L(\alpha) = \frac{1}{4} \iint_{\mathbb{R}^2} n_\alpha(\theta, p)\, d\theta\, dp.$$

When applied to a closed curve, the Cauchy-Crofton formula provides an alternative method of calculating the perimeter of a set in $\mathbb{R}^2$.

Theorem 1. *Let C and D be convex bodies in $\mathbb{R}^2$ and $C \cup D$ be connected. Then*

$$0 \leq \eta(C,D) \leq W_1(\operatorname{conv}(C \cup D) \setminus (C \cup D)).$$

Proof. By Lemma 1 above, we see that

$$\eta(C,D) = \frac{1}{2} \cdot S(\operatorname{conv}(C)) + \frac{1}{2} \cdot S(\operatorname{conv}(D)) - \frac{1}{2} \cdot S(\operatorname{conv}(C \cup D)) - \frac{1}{2} \cdot S(\operatorname{conv}(C \cap D)),$$

or

$$\eta(C,D) = \frac{1}{2} \left[S(C) + S(D) - S(\operatorname{conv}(C \cup D)) - S(C \cap D) \right],$$

since C and D are convex. The surface area, unlike the first quermassintegral, is a valuation for all compact, connected sets, so

$$S(C) + S(D) = S(C \cup D) + S(C \cap D).$$

Hence we must compare $S(C \cup D)$ with $S(\operatorname{conv}(C \cup D))$.

Let $E := \operatorname{conv}(C \cup D) \setminus (C \cup D)$, and ℓ be the line given by an arbitrary (θ, p). Only three cases for ℓ are possible (Figure 3):

1. ℓ does not intersect $\partial\,\mathrm{conv}(C \cup D)$

2. ℓ intersects $\partial\,\mathrm{conv}(C \cup D)$ but not E

3. ℓ intersects $\partial\,\mathrm{conv}(C \cup D)$ and E

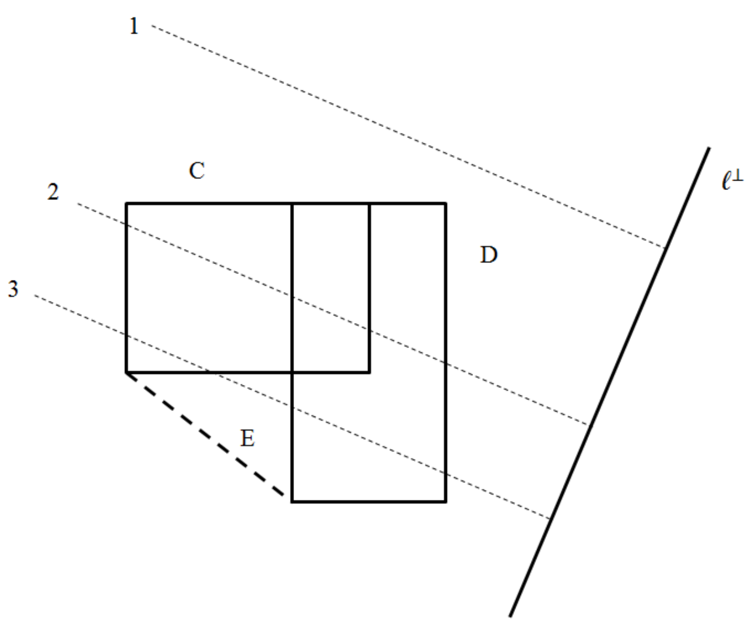

Figure 3: The three cases

We disregard cases that occur only a finite number of times for a given θ, since they have measure 0 and therefore will not factor into Kubota's formula. Thus if ℓ intersects the convex hull of $C \cup D$, then $n_{\partial\,\mathrm{conv}(C \cup D)} = 2$, otherwise $n_{\partial\,\mathrm{conv}(C \cup D)} = 0$.

Lemma 2. *A line ℓ intersects the boundary of a compact, connected set C if and only if it intersects the boundary of* $\mathrm{conv}(C)$.

Proof. If ℓ does not pass through $\mathrm{conv}(C)$, it cannot pass through C, since $C \subset \mathrm{conv}(C)$. Hence if ℓ intersects ∂C it must also intersect $\partial\,\mathrm{conv}(C)$.

Now suppose ℓ does not intersect ∂C, so ℓ does not pass through C. Then either there are points in C on both sides of ℓ in the plane, in which case C cannot be connected, or there are points in C on only on side of ℓ, in which case there must be a convex set containing C but not intersecting ℓ, so the line cannot intersect $\partial\,\mathrm{conv}(C)$. In each case there is a contradiction, so ℓ must intersect ∂C. □

By Lemma 2, we have that in case 1, $n_{\partial(C\cup D)} = n_{\partial\,\mathrm{conv}(C\cup D)} = 0$. No line can intersect the boundary of a convex set more than twice, so in case 2, $n_{\partial\,\mathrm{conv}(C\cup D)} = 2$. Since ℓ does not intersect E, it must intersect $\partial\,\mathrm{conv}(C\cup D)$ and $\partial(C\cup D)$ at the same points, so $n_{\partial(C\cup D)} = 2$. In case 3, the intersections with the boundary and the boundary of the convex hull do not coincide, and the greatest possible $n_{\partial(C\cup D)}$ in this case is 4, since ℓ may intersect at most twice with ∂C and twice with ∂D. The least possible $n_{\partial(C\cup D)}$ is 2, since by Lemma 2, ℓ must intersect $\partial(C\cup D)$. In the three cases, therefore, we have

1. $n_{\partial(C\cup D)} - n_{\partial\,\mathrm{conv}(C\cup D)} = 0$
2. $n_{\partial(C\cup D)} - n_{\partial\,\mathrm{conv}(C\cup D)} = 0$
3. $0 \leq n_{\partial(C\cup D)} - n_{\partial\,\mathrm{conv}(C\cup D)} \leq 2$

For a given θ, let $\ell^{\perp}$ be the line through the origin perpendicular to all lines with angle θ. Then $n_{\partial(C\cup D)} - n_{\partial\,\mathrm{conv}(C\cup D)} = 0$ for all p not in $E|\ell^{\perp}$, and $0 \leq n_{\partial(C\cup D)} - n_{\partial\,\mathrm{conv}(C\cup D)} \leq 2$ for p in $E|\ell^{\perp}$. Hence

$$0 \leq \int_{\mathbb{R}} n_{\partial(C\cup D)}\,dp - \int_{\mathbb{R}} n_{\partial\,\mathrm{conv}(C\cup D)}\,dp \leq \int_{E|\ell^{\perp}} 2\,dp.$$

Integrating over all angles θ and scaling by $\frac{1}{4}$, we have

$$0 \leq \frac{1}{4}\iint_{\mathbb{R}^2} n_{\partial(C\cup D)}\,dp\,d\theta - \frac{1}{4}\iint_{\mathbb{R}^2} n_{\partial\,\mathrm{conv}(C\cup D)}\,dp\,d\theta \leq \frac{1}{2}\int_0^{2\pi} w_0(E|\ell^{\perp})\,d\theta.$$

Applying the Cauchy-Crofton formula and Kubota's formula, we get that

$$0 \leq S(C \cup D) - S(\text{conv}(C \cup D)) \leq 2W_1(E).$$

Since for any C and D,

$$S(C) + S(D) - S(C \cup D) - S(C \cap D) = 0,$$

we obtain

$$0 \leq \frac{1}{2}\left[S(C) + S(D) - S(\text{conv}(C \cup D)) - S(C \cap D)\right] \leq W_1(E),$$

or

$$0 \leq \eta(C,D) \leq W_1(\text{conv}(C \cup D) \setminus (C \cup D)).$$

□

4 A Set-Theoretic Approach

The proof in the preceding section provides a visual intuition for working with quermassintegrals, but it applies only to convex bodies with a connected union, and only in $\mathbb{R}^2$. This section presents a pair of set-theoretic proofs showing that the same bounds apply more generally to all compact sets in $\mathbb{R}^d$.

4.1 The Lower Bound

Theorem 2. *Let C and D be compact sets in $\mathbb{R}^d$. Then $\eta(C,D) \geq 0$.*

Proof. Let $u \in S^{d-1}$ be an arbitrary unit vector and $u^{\perp}$ the hyperplane $\{x : u \cdot x = 0\}$. Let x be a point in the projection $(C \cup D)|u^{\perp}$. Then

$$\begin{aligned} x \in (C \cup D)|u^{\perp} &\iff \exists y \in C \text{ or } D \quad \text{such that} \quad y|u^{\perp} = x \\ &\iff x \in C|u^{\perp} \text{ or } D|u^{\perp} \\ &\iff x \in (C|u^{\perp} \cup D|u^{\perp}), \end{aligned}$$

which implies that

$$(C|u^{\perp}) \cup (D|u^{\perp}) = (C \cup D)|u^{\perp}.$$

Similarly,

$$\begin{aligned} x \in (C \cap D)|u^{\perp} &\Rightarrow \exists y \in C \text{ and } D \quad \text{such that} \quad y|u^{\perp} = x \\ &\Rightarrow x \in C|u^{\perp} \text{ and } D|u^{\perp} \\ &\Rightarrow x \in (C|u^{\perp} \cap D|u^{\perp}), \end{aligned}$$

so

$$(C|u^{\perp}) \cap (D|u^{\perp}) \supseteq (C \cap D)|u^{\perp}.$$

Letting v denote $(d-1)$-dimensional volume, we have that

$$v((C|u^{\perp}) \cup (D|u^{\perp})) = v((C \cup D)|u^{\perp})$$

and

$$v((C|u^{\perp}) \cap (D|u^{\perp})) \geq v((C \cap D)|u^{\perp}).$$

Since volume is a valuation for all sets C and D, we also have that

$$v((C|u^{\perp}) \cup (D|u^{\perp})) + v((C|u^{\perp}) \cap (D|u^{\perp})) = v(C|u^{\perp}) + v(D|u^{\perp}).$$

Substituting, we get

$$v((C \cup D)|u^{\perp}) + v((C \cap D)|u^{\perp}) \leq v(C|u^{\perp}) + v(D|u^{\perp}).$$

Integrating over all u and noting that v is equal to w_0,

$$\begin{aligned}\int_{S^{d-1}} w_0((C \cup D)|u^{\perp}) d\sigma(u) + &\int_{S^{d-1}} w_0((C \cap D)|u^{\perp}) d\sigma(u) \\ &\leq \int_{S^{d-1}} w_0(C|u^{\perp}) d\sigma(u) + \int_{S^{d-1}} w_0(D|u^{\perp}) d\sigma(u),\end{aligned}$$

so

$$W_1(C \cup D) + W_1(C \cap D) \leq W_1(C) + W_1(D).$$

Therefore $\eta(C,D) \geq 0$. □

4.2 The Upper Bound

Theorem 3. *For compact sets C and D in* $\mathbb{R}^d$,

$$\eta(C,D) \leq W_1(\text{conv}(C \cup D) \setminus (C \cup D)).$$

Proof. As above, let $u \in S^{d-1}$ be an arbitrary unit vector and $u^{\perp}$ the hyperplane $\{x : u \cdot x = 0\}$. We have shown in the previous section that

$$(C|u^{\perp}) \cup (D|u^{\perp}) = (C \cup D)|u^{\perp}$$

and

$$(C|u^{\perp}) \cap (D|u^{\perp}) \supseteq (C \cap D)|u^{\perp}.$$

Now consider a point p in $((C|u^{\perp}) \cap (D|u^{\perp})) \setminus ((C \cap D)|u^{\perp})$. This point is the projection onto

$u^\perp$ of a point c in C and a point d in D but not of any point in $C \cap D$, so the line ℓ perpendicular to $u^\perp$ that passes through p intersects C and D but not $C \cap D$. Now suppose every point on ℓ between c and d is either in C or in D. Then, since C is closed, C must have a boundary point b in this line segment. Because it is a boundary point, every $\frac{1}{n}$-neighborhood of b contains a point b_n not in C, which by assumption is in D. The sequence $\{b_n\}$ converges to b, so $b \in C$ is the limit of a sequence in D. This is a contradiction since D is closed, so there must exist a point on ℓ between c and d which is not in $C \cup D$. By the definition of convexity, the point is in the convex hull of $C \cup D$, so p is the projection of a point in E. Therefore we have

$$p \in ((C|u^\perp) \cap (D|u^\perp)) \setminus ((C \cap D)|u^\perp) \Rightarrow p \in E|u^\perp,$$

so

$$((C|u^\perp) \cap (D|u^\perp)) \setminus ((C \cap D)|u^\perp) \subseteq E|u^\perp,$$

which implies that

$$v((C|u^\perp) \cap (D|u^\perp)) - v((C \cap D)|u^\perp) \leq v(E|u^\perp).$$

As in the proof of the lower bound, the valuation property of volumes gives

$$v((C|u^\perp) \cup (D|u^\perp)) + v((C|u^\perp) \cap (D|u^\perp)) = v(C|u^\perp) + v(D|u^\perp).$$

Substituting, we get

$$v((C \cup D)|u^\perp) + v((C \cap D)|u^\perp) + v(E|u^\perp) \geq v(C|u^\perp) + v(D|u^\perp).$$

Integrating over all u and substituting v with w_0,

$$\int_{S^{d-1}} w_0((C \cup D)|u^{\perp}) d\sigma(u) + \int_{S^{d-1}} w_0((C \cap D)|u^{\perp}) d\sigma(u) + \int_{S^{d-1}} w_0((E)|u^{\perp}) d\sigma(u)$$
$$\geq \int_{S^{d-1}} w_0(C|u^{\perp}) d\sigma(u) + \int_{S^{d-1}} w_0(D|u^{\perp}) d\sigma(u),$$

so

$$W_1(C \cup D) + W_1(C \cap D) + W_1(E) \geq W_1(C) + W_1(D).$$

Therefore $\eta(C,D) \leq W_1(\text{conv}(C \cup D) \setminus (C \cup D))$. □

5 Visualizing the Bounds

To aid the reader in visualizing the quermassintegrals and the bounds on $\eta(C,D)$, we conclude by providing two simple examples of the calculation of $\eta(C,D)$ and comparing the values obtained to the bounds.

We consider one example in which C and D are line segments of length 1, and one in which C and D are right triangles with side lengths 1, 1, and $\sqrt{2}$ (Figure 4).

(a) (b)

Figure 4: Simple Connected Examples in $\mathbb{R}^2$

In Figure 4a, the intersection of the two lines C and D is a point, so

$$W_1(C \cap D) = \frac{1}{2} \cdot S(\text{conv}(C \cap D)) = \frac{1}{2} \cdot S(C \cap D) = 0.$$

$W_1(C) = W_1(D) = 1$, since the perimeter of a line of length 1 is counted as 2 (it is easy to see why by imagining the line as the limit of a rectangle with sidelengths 1 and x as x approaches 0). The remaining term in $\eta(C,D)$ is $W_1(C \cup D) = \frac{1}{2} \cdot S(\mathrm{conv}(C \cup D))$, which can be calculated with simple geometry (the convex hull of $C \cup D$ is marked with a dotted line in Figure 4). Following the analogous procedure for the second example, we get the following values for $\eta(C,D)$ in each case:

Example 4a: $\eta(C,D) = 1 + 1 - \sqrt{2} - 0 = 2 - \sqrt{2}$

Example 4b: $\eta(C,D) = 2 + \sqrt{2} - (\frac{\sqrt{2}}{2} + \frac{1}{2}) - 2 = \frac{-1}{2} + \frac{\sqrt{2}}{2}$

Both of these values are positive, as predicted by our lower bound. We can now check the upper bound for each of these examples: In Figure 4a, we see that $\mathrm{conv}(C \cup D) \setminus (C \cup D)$ has a convex hull equal to $\mathrm{conv}(C \cup D)$, namely the entire square shown in the diagram. Hence for Example 4a

$$W_1(\mathrm{conv}(C \cup D) \setminus (C \cup D)) = \sqrt{2} \geq 2 - \sqrt{2} = \eta(C,D).$$

Again, we repeat the procedure for Figure 4b, and we obtain

$$W_1(\mathrm{conv}(C \cup D) \setminus (C \cup D)) = \frac{1}{2} + \frac{\sqrt{2}}{2} \geq \frac{-1}{2} + \frac{\sqrt{2}}{2} = \eta(C,D).$$

6 Future Work

The natural line of further investigation for any problem involving bounding a value is to refine the bounds. The lower bound on the deviation $\eta(C,D)$ in this problem is optimal, since convex sets have $\eta(C,D) = 0$, but the upper bound could be refined by means of the integral-geometric approach presented in Section 3. Another interesting question to study would be the following: assuming that a non-convex body is expressible as the union of two convex bodies, decompose it into two bodies, such that the bounds on the deviation of the quermassintegrals from the valuation property are as tight as possible. This process of "optimally" decomposing a set has the potential, along with the results in S. Alesker's "Theory of valuations on manifolds" [6], to non-trivially estimate the Lipschitz-Killing measure of some Riemannian manifolds with corners. Since this measure is an intrinsic property of a manifold, while the bounds depend on convexity properties of its isometric embeddings, this estimate could provide a link between intrinsic Riemannian geometry and the extrinsic geometry of isometric embeddings.

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