Growth of Module Dimensions in Auslander-Reiten Quivers of $\tilde{D}_n$-type and $\tilde{E}_{6,7,8}$-type Quivers

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Abstract

Representations of quivers are frequently used to classify algebras and describe their structure, and so they have a wide range of applications across mathematics and theoretical science. A quiver is a set of vertices connected by arrows, similar to a directed graph, and a representation of a quiver assigns a vector space to each vertex and a map to each arrow. For a quiver Q, the Auslander-Reiten quiver of Q is a quiver with each vertex corresponding to a unique indecomposable module of the path algebra of Q. We study the dimensions of the indecomposable modules assigned to each vertex of the infinite Auslander-Reiten quivers of $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ type quivers. We prove that the dimensions are bounded linearly for both $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ type quivers.

Summary

A quiver is a set of points and a set of arrows, with every arrow pointing from one point to another. Quivers have diverse applications in mathematics and particle physics, and are especially useful in proving special properties in abstract algebra. In this project, we study a family of special quivers frequently used to represent abstract algebraic structures and obtain information about their structures. In quivers of this family, each point is assigned an algebraic structure that has an integer dimension. Our goal is to determine how those dimensions increase as we move down each quiver, and we show that the dimensions are bounded linearly for certain infinite cases.

1 Introduction

The term *quiver* was first used in mathematics by the French mathematician Peter Gabriel [1] in a 1972 article on irreducible representations. He described a set of points and a set of arrows, with each arrow having a start and an end in the set of points. Instead of calling this a directed graph, he suggested the name quiver, to distinguish it from other concepts attached to the term graph.

Since then, quivers have become a distinct concept expanding far beyond the conventional applications of graphs in mathematics. Representations of quivers in particular have applications across a diverse range of fields; beyond representation theory and linear algebra, they have been used to describe moduli spaces in algebraic geometry [2] and interactions in particle physics [3], and are often studied in association with cluster algebras [4].

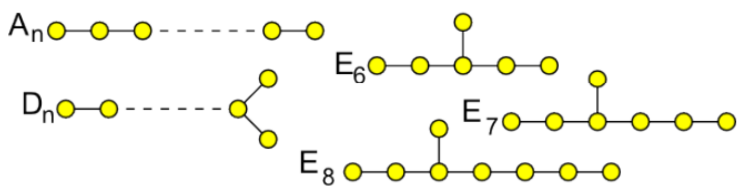

Figure 1: Dynkin diagrams types [5]

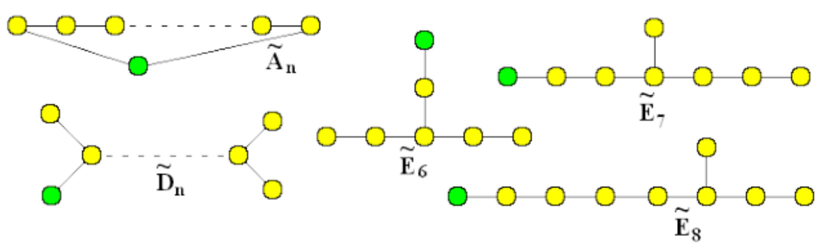

Figure 2: Euclidean graph types [6]

Quivers may be classified by the structures of their underlying graphs as either Dynkin, Euclidean, or wild. The relevant Dynkin and Euclidean types of graphs are shown in Figures 1 and 2. Dynkin diagrams are a family of graphs that represent root systems of Lie algebras, and Euclidean graphs similarly represent affine Lie algebras. Euclidean graphs are categorized as $\tilde{A}_n$, $\tilde{D}_n$, or $\tilde{E}_{6,7,8}$ (Figure 2). We study quivers with underlying graphs of type $\tilde{D}_n$ or $\tilde{E}_{6,7,8}$.

The Auslander-Reiten quiver of the quiver Q has indecomposable modules of the path algebra kQ of Q at its vertices. Gabriel's theorem [1] tells us that the Auslander-Reiten quivers of Dynkin type quivers are finite and well-defined. However, the Auslander-Reiten quivers of Euclidean type quivers are infinite and much more complex, and less is known about their structures. We study the dimensions of the indecomposable kQ-modules in such Auslander-Reiten quivers and their relationship with the locations of the modules in the quiver. We find that the dimensions are bounded linearly in Auslander-Reiten quivers of $\tilde{D}_n$ or $\tilde{E}_{6,7,8}$ type quivers.

In Section 2, we provide the necessary definitions and notations and describe the knitting algorithm, which we use frequently in the proofs of the lemmas and main results. In Section 3, we introduce the $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ type quivers and present several lemmas important to the main theorems. In Section 4, we prove our first main theorem, that the dimensions of the modules in the Auslander-Reiten quivers of $\tilde{D}_n$ quivers are linearly bounded. In Section 5, we similarly prove that dimensions in the Auslander-Reiten quivers of $\tilde{E}_{6,7,8}$ quivers are linearly bounded.

2 Definitions and Notations

We begin with the definition of a quiver.

Definition 2.1. A *quiver* Q is composed of a set Q_0 of vertices $\{1, 2, \ldots, n\}$ and a set Q_1 of arrows that connect pairs of vertices.

An example of a quiver can be seen in Figure 3.

Given a quiver Q, a *path* is a sequence of arrows $\rho_1\rho_2\ldots\rho_m$ in Q such that the start $s(\rho_i)$ of each arrow is the tail $t(\rho_{i+1})$ of the next. The product of two paths xy is their composition

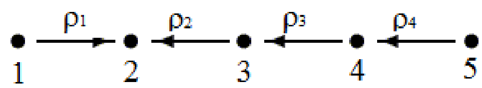

Figure 3: A quiver with five vertices

if the start of x is the tail of y, and is 0 otherwise. For example, for the quiver in Figure 3 if $x = \rho_2\rho_3$ and $y = \rho_4$, then $xy = \rho_2\rho_3\rho_4$ and $yx = 0$.

The paths of Q generate an associative algebra.

Definition 2.2. Given a quiver Q and a field k, the *path algebra* kQ is the vector space which has the paths of Q as its basis and multiplication given by the products of paths.

Elements of kQ are linear combinations of the paths, and are of the form $\lambda_1 x_1 + \lambda_2 x_2 + \ldots$, where $\lambda_1, \lambda_2, \ldots \in k$ and $x_1, x_2, \ldots$ are paths in Q. The elements can be added or multiplied together. For example, given two elements $\lambda_1 x + \lambda_2 y$ and $\lambda_3 x + \lambda_4 y$ of kQ, where x, y are paths, we have

$$(\lambda_1 x + \lambda_2 y) + (\lambda_3 x + \lambda_4 y) = (\lambda_1 + \lambda_3)x + (\lambda_2 + \lambda_4)y,$$

$$(\lambda_1 x + \lambda_2 y)(\lambda_3 x + \lambda_4 y) = (\lambda_1\lambda_3)xx + (\lambda_1\lambda_4)xy + (\lambda_2\lambda_3)yx + (\lambda_2\lambda_4)yy.$$

We study the dimensions of modules of the path algebra. First, we recall the standard definition of a left-module.

Definition 2.3. Given an algebra R, a *left-module* M over R is an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$.

A left-module M over R satisfies the following four equations for all $r, s \in R$ and $x, y \in M$:

1. $r \cdot (x + y) = r \cdot x + r \cdot y$,

2. $(r + s) \cdot x = r \cdot x + s \cdot x$,

3. $(rs) \cdot x = r \cdot (s \cdot x)$,

4. $1 \cdot x = x$.

A left-module M is *indecomposable* if $M \neq 0$ and $M = M_1 \bigoplus M_2$ implies $M_1 = 0$ or $M_2 = 0$.

Definition 2.4. The *Auslander-Reiten quiver* Γ of a quiver Q is the quiver with vertices corresponding to indecomposable left-modules of the path algebra kQ and arrows corresponding to irreducible morphisms between those modules.

We wish to bound the dimensions of the modules in the Auslander-Reiten quivers of $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ type quivers. Crawley-Boevey [7] stated that the category $Rep(Q)$ of finite representations of Q is equivalent to the category of left kQ-modules. Therefore, we may associate a unique indecomposable representation to each vertex of the Auslander-Reiten quiver of Q. Each indecomposable representation X has a unique dimension vector $\underline{\dim}\, X = \langle a_1, a_2, \ldots, a_n \rangle$, where each a_i is the dimension of the vector space assigned to vertex i of Q. The sum of the components of $\underline{\dim}\, X$ is equal to the dimension of the kQ-module that corresponds to X. Therefore, we may study the dimension vectors in the Auslander-Reiten quiver instead of the modules themselves.

2.1 The knitting algorithm

The knitting algorithm [8] allows us to recursively construct successive rows of the Auslander-Reiten quiver and calculate the dimension vector corresponding to a vertex from the dimension vectors corresponding to neighboring vertices. The algorithm is as follows:

Consider a vertex A with its corresponding dimension vector. Let $D = \{d_1, d_2, \ldots, d_k\}$ be the set of dimension vectors corresponding to vertices in the set $B = \{b_1, b_2, \ldots, b_k\}$ for which there exists an arrow in the Auslander-Reiten quiver pointing from A to b_i for each

$1 \leq i \leq k$. Then the dimension vector corresponding to the vertex A' two rows below A is

$$d_0' = \sum_{j=1}^{k} d_j - d_0, \tag{1}$$

and there exist arrows from the vertices B to the vertex A' (see Figure 4). Equation (1) is true for all vertices in an Auslander-Reiten quiver.

$$\begin{tikzcd}
 & & d_0 \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
d_1 \arrow[drr] & d_2 \arrow[dr] & \cdots & d_{k-1} \arrow[dl] & d_k \arrow[dll] \\
 & & d_0' & &
\end{tikzcd}$$

Figure 4: A sample application of the knitting algorithm

Crawley-Boevey [9] showed that the structure of the Auslander-Reiten quiver is independent of arrow orientation; if two quivers Q_1 and Q_2 have the same underlying graph, then there exists some finite k_1 and k_2 such that the Auslander-Reiten quiver of Q_1 after the k_1-row is identical in structure to the Auslander-Reiten quiver of Q_2 after the k_2^{th} row (although the dimension vectors at each vertex are not necessarily equal). We consider only this general graph structure of the Auslander-Reiten quiver. Then the recursive system of equations that the knitting algorithm yields is identical between quivers with the same underlying graph.

2.2 Notation of dimensions

We denote the total dimensions of each dimension vector in an Auslander-Reiten quiver as follows: the total dimension of the vector corresponding to the k^{th} from left vertex in the j^{th} row of the Auslander-Reiten quiver is $x_{i,k}$ if j is odd and is $y_{i,k}$ otherwise, where $i = \lceil \frac{j}{2} \rceil$ (see Figure 7). We define the first row to be the uppermost row such that the Auslander-Reiten quiver after and including that row has only the general structure described previously. In addition, we denote the sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ and $y_{1,k}, y_{2,k}, y_{3,k} \ldots$ for all k in an

Auslander-Reiten quiver to be the *dimension sequences* in that quiver. Our goal is to bound the dimension sequences in every Auslander-Reiten quiver of a $\tilde{D}_n$ or $\tilde{E}_{6,7,8}$ quiver.

3 Preliminaries

In this section, we describe the $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ quivers in greater detail and give examples of Auslander-Reiten quivers of each type. We also prove several lemmas that are used to prove linear bounds for dimension sequences in Auslander-Reiten quivers of $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ quiver.

We first show that all dimensions sequences in the Auslander-Reiten quiver of a $\tilde{D}_n$ or $\tilde{E}_{6,7,8}$ quiver are linearly bounded if the dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ are linearly bounded. Then it suffices to bound the only dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ in our later proofs.

Proposition 3.1. *If the dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ in the Auslander-Reiten quiver of a $\tilde{D}_n$ or $\tilde{E}_{6,7,8}$ quiver are linearly bounded, the dimension sequences $y_{1,k}, y_{2,k}, y_{3,k} \ldots$ are also linearly bounded.*

Proof. Given any integer k such that $y_{1,k}, y_{2,k}, y_{3,k} \ldots$ is a dimension sequence in the Auslander-Reiten quiver, we may use the knitting algorithm to find a set of positive integers S_k such that $y_{i,k} = \sum_{j \in S_k} x_{i,j} - y_{i-1,k}$ for all i. Then we have that $y_{i,k} < \sum_{j \in S} x_{i,j}$, so the dimension sequence $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ is linearly bounded. $\square$

3.1 $\tilde{D}_n$ quivers

A $\tilde{D}_n$ quiver exists only for $n \geq 4$ and has $n+1$ vertices. An example of a $\tilde{D}_6$ quiver can be seen in Figure 5.

The general graph structure of the Auslander-Reiten quiver of $\tilde{D}_n$ depends on its parity, as can be seen in Figures 8 and 9. We define a new variable $m = \lceil \frac{n}{2} \rceil - 2$ for convenience of notation.

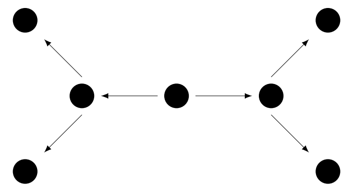

Figure 5: A $\tilde{D}_6$ quiver

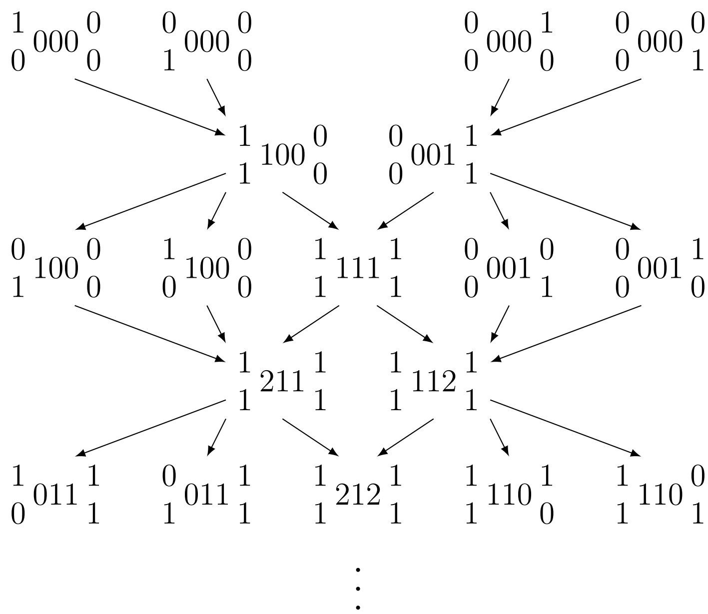

Figure 6: Auslander-Reiten quiver of the $\tilde{D}_6$ quiver in Figure 5

First, we show that every dimension in the Auslander-Reiten quiver can be expressed in terms of $x_{i,1}$.

Lemma 3.1. *Given the dimension sequences* $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ *and* $y_{1,k}, y_{2,k}, y_{3,k} \ldots$ *in the Auslander-Reiten quiver of a* $\tilde{D}_n$ *quiver, we have*

$$x_{i,k} = x_{i+k-1,1} - x_{i+k-2,1} + x_{i+k-3,1} - \ldots + x_{i-k+1,1} \tag{2}$$

and

$$y_{i,k+1} + y_{i-1,k+1} = x_{i+k-1,1} + x_{i-k+1,1} \tag{3}$$

for all $1 \leq i$ *and* $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

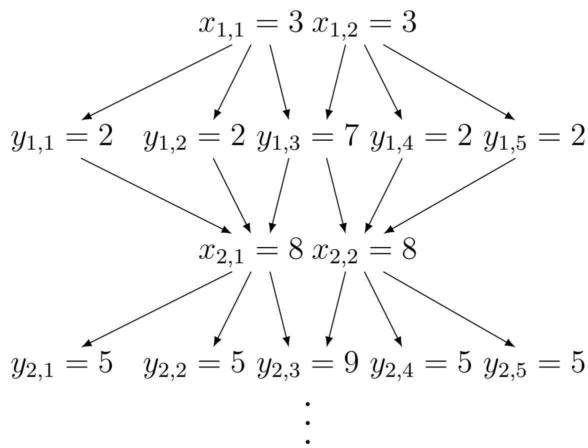

Figure 7: Total dimensions in the Auslander-Reiten quiver of the $\tilde{D}_6$ quiver in Figure 5

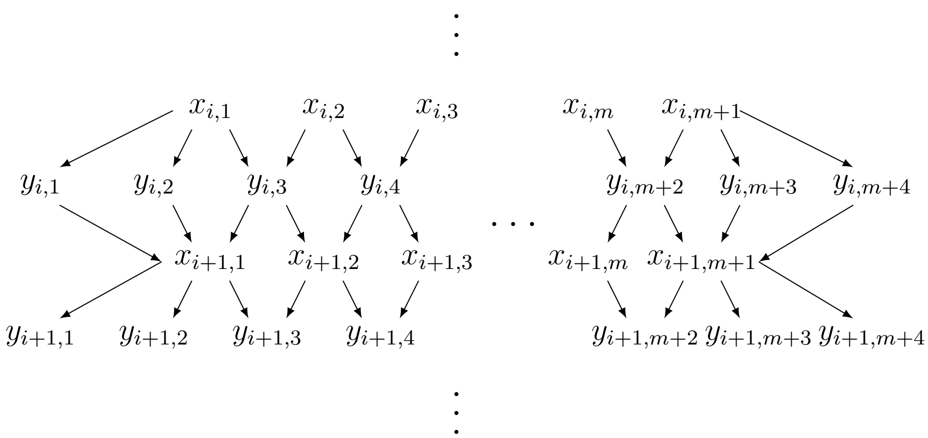

Figure 8: Auslander-Reiten quiver of $\tilde{D}_n$ for even n

Proof. We proceed with induction. We have $x_{i,1} = y_{i,1}+y_{i-1,1} = y_{i,2}+y_{i-1,2}$ from the knitting algorithm. Also,

$$x_{i+1,1} = y_{i,1} + y_{i,2} + y_{i,3} - x_{i,1},$$

$$x_{i,1} = y_{i-1,1} + y_{i-1,2} + y_{i-1,3} - x_{i-1,1},$$

so $y_{i,3} + y_{i-1,3} = x_{i+1,1} + x_{i-1,1}$.

Then since $y_{i,3} = x_{i,1} + x_{i,2} - y_{i-1,3}$, we have $x_{i,2} = x_{i+1,1} - x_{i,1} + x_{i-1,1}$.

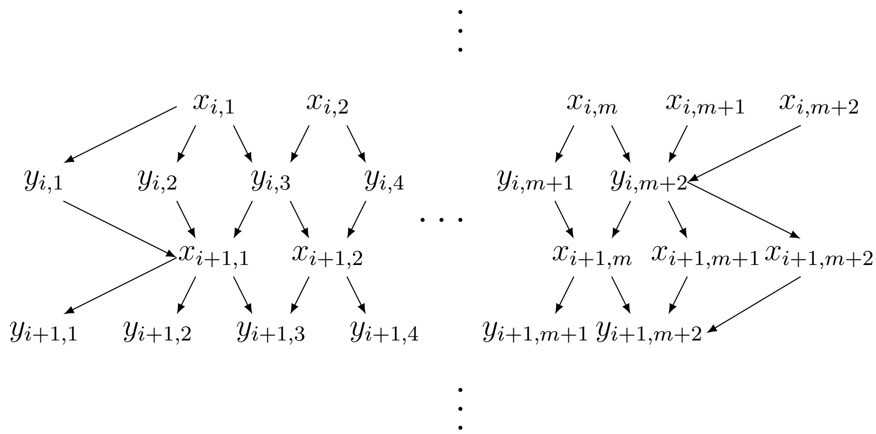

Figure 9: Auslander-Reiten quiver of $\tilde{D}_n$ for odd n

If Equations (2) and (3) are true for some $k \leq m$, then we have

$$y_{i,k+2} = x_{i,k} + x_{i+1,k} - y_{i,k+1} = x_{i+k,1} + x_{i-k,1},$$

$$x_{i,k+1} = y_{i,k+2} + y_{i-1,k+2} - x_{i,k} = x_{i+k,1} - x_{i+k-1,1} + x_{i+k-2,1} - ... + x_{i-k,1},$$

so by induction Equations (2) and (3) are true for all $k \leq m+1$.

□

We wish to show that all dimension sequences in the Auslander-Reiten quiver are linearly bounded. In fact, it suffices to bound only the dimension sequence $x_{1,1}, x_{2,1}, x_{3,1}, \dots$.

Lemma 3.2. *Consider the dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \dots$ and $y_{1,k}, y_{2,k}, y_{3,k} \dots$ in the Auslander-Reiten quiver of a $\tilde{D}_n$ quiver. If there exists constants A and B such that $Ai - B \leq x_{i,1} \leq Ai + B$ for all $i \geq 1$, then all dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \dots$ and all dimension sequences $y_{1,k}, y_{2,k}, y_{3,k} \dots$ are linearly bounded.*

Proof. Consider first when n is even. We have $-2B \leq x_{i,1} - x_{i-1,1}$, so from (2) we have

$$x_{i,k} = x_{i+k-1,1} - x_{i+k-2,1} + x_{i+k-3,1} - \dots + x_{i-k+1,1} \leq x_{i+k-1,1} + 2B(2k-3)$$

for all k. Therefore, all dimension sequences of the form $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ are linearly bounded.

Now let n be odd. From (2) we have

$$x_{i,k} \leq x_{i+k-1,1} + 2B(2k-3)$$

for all $k \leq m$. If $x_{i,1}$ is linearly bounded, then by symmetry $y_{i,m+2}$ is also linearly bounded. Then $x_{i,m+1} = y_{i-1,m+2} - x_{i-1,m+1} < y_{i-1,m+2}$ and $x_{i,m+2} = y_{i-1,m+2} - x_{i-1,m+2} < y_{i-1,m+2}$ are also linearly bounded.

Therefore, all dimension sequences of the form $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ are linearly bounded for both odd and even n. By Proposition 3.1, all dimension sequences of the form $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ are also linearly bounded. □

3.2 $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ quivers

The three types of $\tilde{E}$ graphs are $\tilde{E}_6$ (Figure 10), $\tilde{E}_7$ (Figure 12), and $\tilde{E}_8$ (Figure 14).

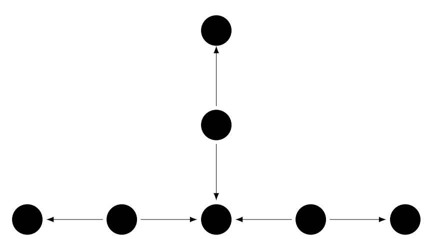

Figure 10: An $\tilde{E}_6$ quiver

Lemma 3.3. *If Q is an $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$ quiver, then The dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \ldots$ in the Auslander-Reiten quiver of Q is bounded linearly.*

Proof. For each $6 \leq k \leq 8$, applying the knitting algorithm to the Auslander-Reiten quiver of an $\tilde{E}_k$ quiver gives us $k+1$ recursive equations in terms of the labelled total dimensions. The complete list of these equations can be found in Appendix A. For each k, there exists a recursive equation in terms of only elements of the dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \ldots$

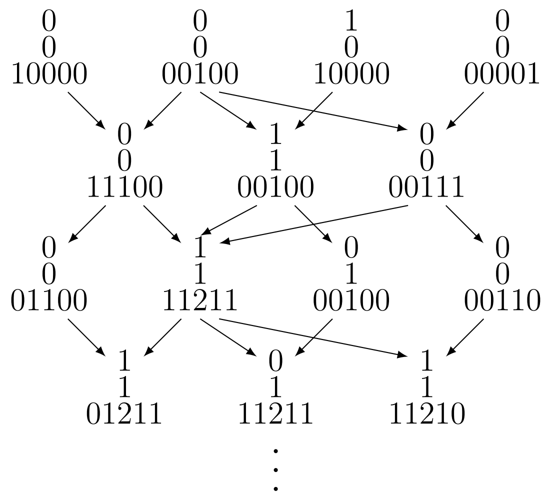

Figure 11: Auslander-Reiten quiver of the $\tilde{E}_6$ quiver in Figure 10

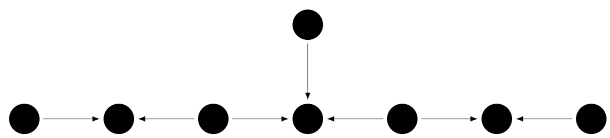

Figure 12: An $\tilde{E}_7$ quiver

that can be found through algebraic manipulation of the equations given by the knitting algorithm.

When $k = 6$, the recursive equation is

$$x_{i+1,2} + x_{i-3,2} = x_{i,2} + x_{i-2,2}, \tag{4}$$

so the characteristic polynomial of the dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \ldots$ in the Auslander-Reiten quiver of an $\tilde{E}_6$ quiver is

$$P(z) = (z + 1)(z - 1)^2.$$

Then there exists integer constants c_0, c_1, c_2, and $C = c_1 + |c_2|$ such that

$$x_{i,2} = c_0 i + c_1 + (-1)^i c_2 \leq c_0 i + C. \tag{5}$$

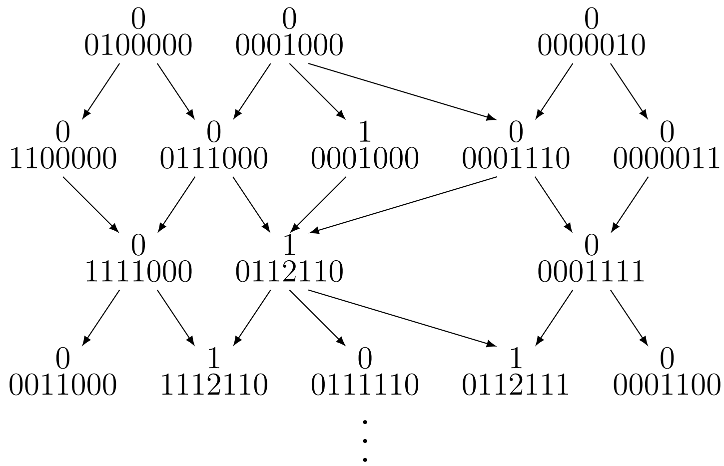

Figure 13: Auslander-Reiten quiver of the $\tilde{E}_7$ quiver in Figure 12

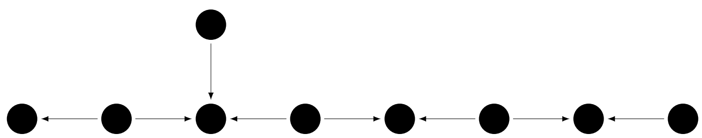

Figure 14: An $\tilde{E}_8$ quiver

When $k = 7$, the recursive equation is

$$x_{i+3,2} + x_{i-2,2} = x_{i+1,2} + x_{i,2}, \tag{6}$$

so the characteristic polynomial of the dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \ldots$ in the Auslander-Reiten quiver of an $\tilde{E}_7$ quiver is

$$P(z) = (z^3 - 1)(z^2 - 1).$$

The distinct roots $r_1, \ldots, r_4$ of $P(z)$ are roots of unity, with $r_1 = 1$ having multiplicity 2 and all other roots having multiplicity 1. Then we have

$$x_{i,2} = \sum_{j=1}^{4} c_j (r_j)^i + c_0 i$$

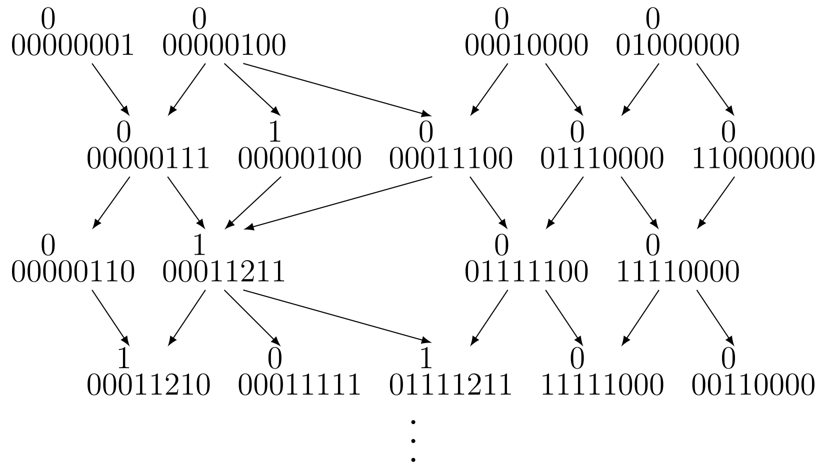

Figure 15: Auslander-Reiten quiver of the $\tilde{E}_8$ quiver in Figure 14

for some constants $c_0, \dots, c_4$. We may define a constant C such that $C = \sum_{j=1}^{4} |c_j|$, and so $x_{i,2} \leq c_0 i + C$.

When $k = 8$, the recursive equation is

$$x_{i+1,2} + x_{i-2,2} = x_{i+3,2} + x_{i-4,2}, \tag{7}$$

so the characteristic polynomial of the dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \dots$ in the Auslander-Reiten quiver of an $\tilde{E}_8$ quiver is

$$P(z) = (z^5 - 1)(z^2 - 1).$$

The distinct roots $r_1, \dots, r_6$ of $P(z)$ are roots of unity, with $r_1 = 1$ having multiplicity 2 and all other roots having multiplicity 1. Then we have

$$x_{i,2} = \sum_{j=1}^{6} c_j (r_j)^i + c_0 i$$

for some constants $c_0, \dots, c_6$. We may define a constant C such that $C = \sum_{j=1}^{6} |c_j|$, and so $x_{i,2} \leq c_0 i + C$. $\square$

4 Proof of linear bounds for $\tilde{D}_n$ quivers

We show our main result for dimension sequences in the Auslander-Reiten quivers of $\tilde{D}_n$-type quivers.

Theorem 4.1. *If Q is a $\tilde{D}_n$-type quiver, then all dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ and $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ in the Auslander-Reiten quiver of Q are linearly bounded.*

Proof. We consider the cases of n even and n odd separately. First, let n be even.

From Lemma 3.1, we have

$$x_{i,m+1} = x_{i+m,1} - x_{i+m-1,1} + x_{i+m-2,1} - \cdots + x_{i-m,1}$$

for all $i \geq 1$. Since the Auslander-Reiten quiver of $\tilde{D}_n$ is symmetric for even n, we may similarly state

$$x_{i,1} = x_{i+m,m+1} - x_{i+m-1,m+1} + x_{i+m-2,m+1} - \cdots + x_{i-m,m+1}.$$

Substitution yields the following:

$$x_{i,1} = x_{i+2m,1} - 2x_{i+2m-1,1} + 3x_{i+2m-2,1} - \cdots + (2m+1)x_{i,1} - \ldots + 3x_{i-2m+2,1} - 2x_{i-2m+1,1} + x_{i-2m,1}.$$

The characteristic polynomial of $x_{i,k}$ is therefore

$$P(z) = (z^{2m} - z^{2m-1} + z^{2m-2} - \cdots + 1)^2 - z^{2m}.$$

We can simplify $P(z)$ into

$$P(z) = \frac{(z^{m+1}+1)(z^{m+1}-1)(z^m+1)(z^m-1)}{(z+1)^2}. \tag{8}$$

Note that the roots of $z^{m+1}+1$, $z^{m+1}-1$, z^m+1, and z^m-1 are distinct, with the exception of $z=1$ and $z=-1$. Also, exactly two of those four polynomials have $z=-1$ as a root with multiplicity 1, so $z=-1$ is not a root of $P(z)$. Therefore, we have that $z=1$ is the only root of $P(z)$ with multiplicity greater than 1, and its multiplicity is 2.

Let $r_1, r_2, \ldots, r_{4m-1}$ be the distinct roots of $P(z)$, with $r_1 = 1$. Then we have

$$x_{i,1} = \sum_{j=1}^{4m-1} c_j (r_j)^i + c_0 i$$

for some constants $c_0, c_1, \ldots, c_{4m-1}$. The roots are complex numbers satisfying $|r_j| = 1$, so

$$c_0 i - C \leq x_{i,1} \leq c_0 i + C \tag{9}$$

for $C = \sum_{j=1}^{4m-1} |c_j|$.

Then by Lemma 3.2, the other dimensions in the Auslander-Reiten quiver are also linearly bounded.

Now let n be odd. From Lemma 3.1, we have $y_{i,m+2} + y_{i-1,m+2} = x_{i+m,1} + x_{i-m,1}$ and

$$x_{i+1,1} + x_{i-1,1} = y_{i,3} + y_{i-1,3} = y_{i+m-1,m+2} + y_{i-m,m+2}. \tag{10}$$

The right-hand side of Equation (10) can be expressed as

$$\sum_{j=1}^{2m-1} (-1)^{j+1} (y_{i+m-j,m+2} + y_{i+m-1-j,m+2}) = \sum_{j=1}^{2m-1} (-1)^{j+1} (x_{i+2m-j,1} + x_{i-j,1}).$$

Then we have

$$x_{i+1,1} + x_{i-1,1} = x_{i+2m-1,1} - x_{i+2m-2} + \cdots - x_{i+2,1} + x_{i+1,1} + x_{i-1,1} - \cdots + x_{i-2m+1,1},$$

and so the characteristic polynomial of $x_{i,1}$ is

$$P(z) = (z^{2m+1} - 1)(z - 1)(z^{2m-4} + z^{2m-6} + \cdots + 1) = \frac{(z^{2m+1} - 1)(z^{2m-2} - 1)}{z + 1}.$$

The $4m - 2$ roots of $P(z)$ are the $(2m + 1)^{th}$ and the $(2m - 2)^{th}$ roots of unity, not including $z = -1$.

If $3 \nmid 2m + 1$, then the only root with multiplicity is $z = 1$, which has multiplicity 2. Therefore, as in the n even case, there exists some constants c_0 and C such that $c_0 i - C \leq x_{i,1} \leq c_0 i + C$. If $3|2m + 1$, then the other two third roots of unity also have multiplicity 2. Let $r_1, \ldots, r_{4m-5}$ be the distinct roots of $P(z)$, with $r_1 = 1$ and r_2, r_3 the third roots of unity. Then we have

$$x_{i,1} = \sum_{j=1}^{4m-5} c_j (r_j)^i + (b_1 + b_2 r_2 + b_3 r_3) i$$

for some constants $c_1, \ldots, c_{4m-5}$ and b_1, b_1, b_2. Since $x_{i,1}$ is always real, the two values Im $(\sum_{j=1}^{4m-5} c_j (r_j)^i)$ and Im $(b_1 + b_2 r_2 + b_3 r_3)$ are always equal in magnitude. Then for all i, there exists real constants $c_0' =$ Re $(b_1 + b_2 r_2 + b_3 r_3)$ and $C_i =$ Re $(\sum_{j=1}^{4m-5} c_j (r_j)^i)$ such that

$$x_{i,1} = C_i + c_0' i. \tag{11}$$

We also have $|C_i| \leq C' = \sum_{j=1}^{4m-5} |c_j|$ for all i, so

$$c_0' i - C' \leq x_{i,1} \leq c_0' i + C'. \tag{12}$$

Then by Lemma 3.2, all dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ and all dimension sequences $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ in the Auslander-Reiten quiver are linearly bounded. $\square$

5 Proof of linear bounds for $\tilde{E}_{6,7,8}$ quivers

We now consider the dimension sequences in the Auslander-Reiten quivers of $\tilde{E}_{6,7,8}$-type quivers.

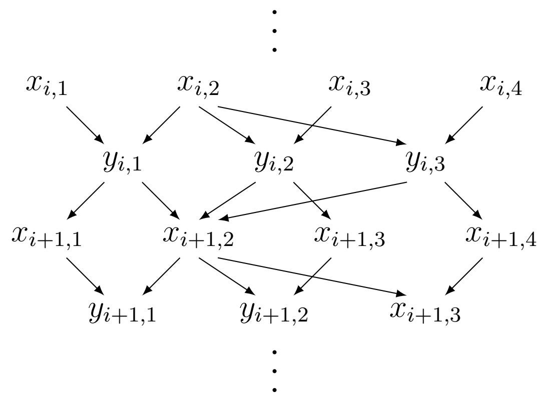

Figure 16: General structure of the Auslander-Reiten quiver of $\tilde{E}_6$

Theorem 5.1. *If Q is an $\tilde{E}_6$-type quiver, then the dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ and $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ in the Auslander-Reiten quiver of Q are all linearly bounded.*

Proof. We know that the dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \ldots$ is bounded linearly by Lemma 3.3.

From the knitting algorithm, we have $y_{i-1,1} = x_{i,1} + x_{i-1,1} > x_{i,1}$. Then $x_{i,2} = y_{i,1} + y_{i-1,1} - x_{i,1} > y_{i,1}$, so the dimension sequence $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ is linearly bounded for $k = 1$, and also for $k = 2$ and $k = 3$ by symmetry. Similarly, we have $y_{i,1} = x_{i+1,1} + x_{i,1} > x_{i,1}$, so the dimension sequence $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ is bounded linearly for $k = 1$, and also for $k = 3$ and $k = 4$ by symmetry. $\square$

Theorem 5.2. *If Q is an $\tilde{E}_7$-type quiver, then the dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ and $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ in the Auslander-Reiten quiver of Q are all linearly bounded.*

Proof. We know that the dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \ldots$ is bounded linearly by Lemma 3.3.

$$
\begin{tikzcd}[column sep=small]
 & & & & \vdots & & & & \\
 & x_{i,1} \arrow[dl] \arrow[dr] & & x_{i,2} \arrow[dl] \arrow[dr] \arrow[drrr] & & & & x_{i,3} \arrow[dl] \arrow[dr] & \\
y_{i,1} \arrow[dr] & & y_{i,2} \arrow[dl] \arrow[dr] & & y_{i,3} \arrow[dl] & & y_{i,4} \arrow[dlll] \arrow[dr] & & y_{i,5} \arrow[dl] \\
 & x_{i+1,1} \arrow[dl] \arrow[dr] & & x_{i+1,2} \arrow[dl] \arrow[dr] \arrow[drrr] & & & & x_{i+1,3} \arrow[dl] \arrow[dr] & \\
y_{i+1,1} & & y_{i+1,2} & & y_{i+1,3} & & y_{i+1,4} & & y_{i+1,5} \\
 & & & & \vdots & & & &
\end{tikzcd}
$$

Figure 17: General structure of the Auslander-Reiten quiver of $\tilde{E}_7$

Consider Equations (1), (2), and (5) in Appendix A.2. Substitution yields $x_{i,2} = x_{i+1,1} + x_{i-1,1} > x_{i-1,1}$, so the dimension sequence $x_{1,1}, x_{2,1}, x_{3,1}, \ldots$ is bounded linearly. Similarly, the dimension sequence $x_{1,3}, x_{2,3}, x_{3,3}, \ldots$ is bounded linearly. Then by Proposition 3.1, the dimension sequence $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ is linearly bounded for each integer $1 \leq k \leq 5$. $\square$

$$
\begin{tikzcd}[column sep=small]
 & & & & & \vdots & & & & \\
x_{i,1} \arrow[dr] & & x_{i,2} \arrow[dl] \arrow[dr] \arrow[drrr] & & & & x_{i,3} \arrow[dl] \arrow[dr] & & x_{i,4} \arrow[dl] \arrow[dr] & \\
 & y_{i,1} \arrow[dl] \arrow[dr] & & y_{i,2} \arrow[dl] & & y_{i,3} \arrow[dlll] \arrow[dr] & & y_{i,4} \arrow[dl] \arrow[dr] & & y_{i,5} \arrow[dl] \\
x_{i+1,1} \arrow[dr] & & x_{i+1,2} \arrow[dl] \arrow[dr] \arrow[drrr] & & & & x_{i+1,3} \arrow[dl] \arrow[dr] & & x_{i+1,4} \arrow[dl] \arrow[dr] & \\
 & y_{i+1,1} & & y_{i+1,2} & & y_{i+1,3} & & y_{i+1,4} & & y_{i+1,5} \\
 & & & & & \vdots & & & &
\end{tikzcd}
$$

Figure 18: General structure of the Auslander-Reiten quiver of $\tilde{E}_8$

Theorem 5.3. *If Q is an $\tilde{E}_8$-type quiver, then the dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ and $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ in the Auslander-Reiten quiver of Q are all linearly bounded.*

Proof. We know that the dimension sequence $x_{1,2}, x_{2,2}, x_{3,2}, \ldots$ is bounded linearly by Lemma 3.3. The dimension sequence $x_{1,1}, x_{2,1}, x_{3,1}, \ldots$ is bounded linearly by the argument used in the proof of Theorem 5.1. Substitutions with Equations (3), (4), (7), (8), (9) in Appendix A.3 yield

$$x_{i,2} = x_{i+2,4} + x_{i,4} + x_{i-2,4} > x_{i,4},$$

$$x_{i,2} = x_{i+2,4} + x_{i-1,3} > x_{i-1,3}.$$

Then the dimension sequences $x_{1,k}, x_{2,k}, x_{3,k}, \ldots$ are linearly bounded for all integers $1 \leq k \leq 4$. By Proposition 3.1, the dimension sequences $y_{1,k}, y_{2,k}, y_{3,k}, \ldots$ are also linearly bounded for integers $1 \leq k \leq 5$. $\square$

6 Conclusion

We studied the growth of module dimensions in the Auslander-Reiten quivers of $\tilde{D}_n$-type and $\tilde{E}_n$-type quivers. The goal was to determine if dimension sequences in the Auslander-Reiten quivers were linearly bounded. We constructed and solved systems of recursive equations using the knitting algorithm. We proved that all dimension sequences in Auslander-Reiten quivers of $\tilde{D}_n$-type and $\tilde{E}_n$-type quivers have linear bounds.

Euclidean quivers may be categorized into three disjoint types: $\tilde{A}_n$-type, $\tilde{D}_n$-type, and $\tilde{E}_{6,7,8}$-type quivers. A natural direction for future research is therefore to complete the investigation of Auslander-Reiten quivers of Euclidean quivers by studying the growth of module dimensions in the Auslander-Reiten quivers of $\tilde{A}_n$-type quivers. This problem is complicated by the existence of cycles, oriented or not, in $\tilde{A}_n$-type quivers. Another future step might include extending the results of this study to wild quivers with underlying graphs that lack cycles or multiple edges. It is likely that a linear bound may be similarly found for certain cases of wild quivers.

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Appendix A Recursive Equations for $\tilde{E}_n$

Appendix A.1 $\tilde{E}_6$ Equations

1. $$x_{i,1} + x_{i-1,1} = y_{i-1,1}$$
2. $$x_{i,2} + x_{i-1,2} = y_{i-1,1} + y_{i-1,2} + y_{i-1,3}$$
3. $$x_{i,3} + x_{i-1,3} = y_{i-1,2}$$
4. $$x_{i,4} + x_{i-1,4} = y_{i-1,3}$$
5. $$y_{i,1} + y_{i-1,1} = x_{i,1} + x_{i,2}$$
6. $$y_{i,2} + y_{i-1,2} = x_{i,3} + x_{i,2}$$
7. $$y_{i,3} + y_{i-1,3} = x_{i,4} + x_{i,2}$$

Appendix A.2 $\tilde{E}_7$ Equations

1. $$x_{i,1} + x_{i-1,1} = y_{i-1,1} + y_{i-1,2}$$
2. $$x_{i,2} + x_{i-1,2} = y_{i-1,2} + y_{i-1,3} + y_{i-1,4}$$
3. $$x_{i,3} + x_{i-1,3} = y_{i-1,4} + y_{i-1,5}$$
4. $$y_{i,1} + y_{i-1,1} = x_{i,1}$$
5. $$y_{i,2} + y_{i-1,2} = x_{i,1} + x_{i,2}$$
6. $$y_{i,3} + y_{i-1,3} = x_{i,2}$$
7. $$y_{i,4} + y_{i-1,4} = x_{i,3} + x_{i,2}$$
8. $$y_{i,5} + y_{i-1,5} = x_{i,3}$$

Appendix A.3 $\tilde{E}_8$ Equations

1. $$x_{i,1} + x_{i-1,1} = y_{i-1,1}$$

2. $$x_{i,2} + x_{i-1,2} = y_{i-1,1} + y_{i-1,2} + y_{i-1,3}$$

3. $$x_{i,3} + x_{i-1,3} = y_{i-1,3} + y_{i-1,4}$$

4. $$x_{i,4} + x_{i-1,4} = y_{i-1,4} + y_{i-1,5}$$

5. $$y_{i,1} + y_{i-1,1} = x_{i,1} + x_{i,2}$$

6. $$y_{i,2} + y_{i-1,2} = x_{i,2}$$

7. $$y_{i,3} + y_{i-1,3} = x_{i,3} + x_{i,2}$$

8. $$y_{i,4} + y_{i-1,4} = x_{i,3} + x_{i,4}$$

9. $$y_{i,5} + y_{i-1,5} = x_{i,4}$$