Distinct Distances Between Sets of Points on a Line and a Hyperplane in $\mathbb{R}^d$

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Abstract

A variant of Erdős's distinct distances problem considers two sets of points in Euclidean space $\mathcal{P}_1$ and $\mathcal{P}_2$, both of cardinality n, and asks whether we can find a superlinear bound on the number of distinct distances between all pairs of points with one in $\mathcal{P}_1$ and the other in $\mathcal{P}_2$. In 2013, Sharir, Sheffer, and Solymosi [8] showed a lower bound of $\Omega(n^{4/3})$ when $\mathcal{P}_1$ and $\mathcal{P}_2$ are both collinear point sets in $\mathbb{R}^2$, where the two lines defined by $\mathcal{P}_1$ and $\mathcal{P}_2$ are not orthogonal or parallel. Here, we contain $\mathcal{P}_1$ in a line l and $\mathcal{P}_2$ in a hyperplane in $\mathbb{R}^d$. We prove that the number of distinct distances in this case has a lower bound of $\Omega(n^{6/5})$ given some restrictions on l and $\mathcal{P}_2$.

Summary

In 1946, Erdős proposed his famous distinct distances problem in which he asked whether we can determine the minimum number of distinct distances between some points on a plane. Here we ask a slightly different question, where we consider some points on a line and some points on a hyperplane in n-dimensional Euclidean space. We find a lower bound for the number of distinct distances between all pairs of points with one on the line and one on the hyperplane.

1 Introduction

In a 1946 paper, Erdős [3] proposed the following problem: Given a set of n points in the plane, what is the minimum number of distinct distances between those points? In that same paper he derived a lower bound of $\sqrt{n-1/2}-3/4$ and used a $\sqrt{n}\times\sqrt{n}$ square lattice to derive an upper bound of $O(n/\sqrt{\log(n)})$. Since then, there has been a steady stream of papers achieving successively better lower bounds. The most recent improvement by Guth and Katz [7] gives a lower bound of $\Omega(n/\log(n))$, leaving a gap of $O(\sqrt{\log(n)})$ between the lower and upper bounds. Recently, people have begun to pose distinct distance problems with additional restrictions or in higher dimensions.

One version of the distinct distances problem asks what happens when we look for distinct distances between two sets of points $\mathcal{P}_1$, and $\mathcal{P}_2$ which both contain n points, such that all points in $\mathcal{P}_1$ lie on a line l_1 and all points in $\mathcal{P}_2$ lie on a line l_2. In 2013, Sharir, Sheffer, and Solymosi [8] showed that with this restriction, given that the lines are not orthogonal or parallel, the lower bound is $\Omega(n^{4/3})$. Shortly after, Charalambides [1] showed that m points on a real algebraic curve of degree d in $\mathbb{R}^n$ determine at least $c_{n,d}m^{5/4}$ distinct distances, given some restrictions on the shape of the curve, where $c_{d,n}$ is a constant that depends on the degrees of the curves. In 2015, Pach and de Zeeuw [5] examined the specific case where two sets of points $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{R}^2$ were each contained on an irreducible plane algebraic curve of degree at most d, and showed that the number of distinct distances is at least $c_d n^{4/3}$, again with some restrictions on the shapes of the curves.

The proofs of these results follow a general framework of converting the problem to one using incidence geometry by inversely relating the number of distinct distances to the number of incidences between a set of points and a set of curves, and using an upper bound on the number of such incidences to obtain a lower bound for the number of distinct distances. This method was first conceived by Elekes and Sharir [4] and was used by Guth and Katz [7] to prove the $\Omega(n/\log(n))$ lower bound on distinct distances in the plane. In both [8] and [5], the $n^{4/3}$ bound comes from the fact that the symmetry of the problem allows the set of curves $\mathcal{C}$ to have only two degrees of freedom, because the roles of $\mathcal{P}_1$ and $\mathcal{P}_2$ can be reversed in those situations. A full survey of distinct distance problem variants was compiled by Sheffer [9].

We consider the following extension of the distinct distances problem in $\mathbb{R}^d$. If there are two sets of points $\mathcal{P}_1$ and $\mathcal{P}_2$ such that the points of $\mathcal{P}_1$ are contained on a line and the points of $\mathcal{P}_2$ are contained on a hyperplane of dimension $d-1$ not containing that line, we consider the number of distinct distances

$$D(\mathcal{P}_1, \mathcal{P}_2) := \left|\{\|p-q\| \mid p \in \mathcal{P}_1,\ q \in \mathcal{P}_2\}\right|$$

where $\|p-q\|$ denotes the distance between p and q. In order to set up the framework for the proof of our main result and to gain some intuition for the special configurations of $\mathcal{P}_2$ in higher dimensions, we first prove the following superlinear lower bound on the number of distinct distances between points on a line and plane in $\mathbb{R}^3$.

Theorem 1.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two sets of points in $\mathbb{R}^3$ of cardinalities m and n, respectively, such that $\mathcal{P}_1$ is contained on a line l and $\mathcal{P}_2$ is contained on a plane Π, and l and Π are neither parallel or orthogonal. Also, let O be the intersection of l and Π. If no two points of $\mathcal{P}_2$ define a line that is perpendicular to the projection of l onto Π, and no two points of $\mathcal{P}_2$ can lie on the same ellipse centered at O with eccentricity $\csc(\alpha)$ whose semi-major axis lies along the projection of l onto Π, we have*

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(\min(m^{4/5}n^{2/5}, m^2, n^2)) .$$

We then prove the following extension on the number of distinct distances between points on a line and a hyperplane in $\mathbb{R}^d$ given some restrictions on the line and on $\mathcal{P}_2$.

Theorem 1.2 (Main Result)**.** *Let l be a line and Π be a hyperplane in $\mathbb{R}^d$, $d > 3$, such that l is not orthogonal to Π and l does not lie in any hyperplane with the same unit normal vector as Π. Also, let O be the intersection of l and Π. Then for any two sets of points $\mathcal{P}_1$ and $\mathcal{P}_2$ with cardinalities m and n, respectively, such that $\mathcal{P}_1$ is contained on l, $\mathcal{P}_2$ is contained on Π, and*

- *No two points in $\mathcal{P}_2$ can lie on the same flat of codimension 2 defined as the intersection of Π with a hyperplane orthogonal to l, and*
- *No two points in $\mathcal{P}_2$ can lie on the same manifold of degree 2 that is defined as the locus of points on Π that are some fixed distance R away from the line,*

we have

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(\min(m^{4/5}n^{2/5}, m^2, n^2)) \ .$$

The proofs of both Theorem 1.1 and Theorem 1.2 use a similar framework to those implemented in [8] and [5], examining quadruples of points and using Cauchy-Schwarz to transform the problem into one pertaining to incidences between curves and points. Note that our configuration is asymmetrical; that is, the objects that contain $\mathcal{P}_1$ and $\mathcal{P}_2$ are of different types. In that aspect our result differs strongly from past work done on distinct distances between points on two objects in Euclidean space, such as [8] and [5].

Theorems 1.1 and 1.2 immediately implies the following corollary.

Corollary 1.3. *If $\mathcal{P}_1$ lies on a line and $\mathcal{P}_2$ lies on a hyperplane in $\mathbb{R}^d$, and the cardinalities of $\mathcal{P}_1$ and $\mathcal{P}_2$ are both n, given the same conditions listed in Theorem 1.2, we have*

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(n^{\frac{6}{5}}) \ .$$

The above corollary is the first superlinear bound on the number of distinct distances between points on a line and hyperplane in $\mathbb{R}^d$.

In Section 2 we introduce the incidence bound that is crucial to the proof of our main result. In Section 3, we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2, our main result. In the Appendix, we will show work towards a superlinear bound on distinct distances between a parametric polynomial curve in $\mathbb{R}^3$ and a plane to help provide direction to future research on this type of distinct distance problem.

2 Incidence Bound

To prove Theorems 1.1 and 1.2, we use an incidence bound from Pach and Sharir [6]. Given a finite set of points $P \subset \mathbb{R}^2$ and a set of curves $\mathcal{C}$ in $\mathbb{R}^2$, we define

$$I(P, \mathcal{C}) = \{(p, \gamma) \in P \times \mathcal{C} \mid p \in \gamma\}$$

to be the set of *incidences* between P and $\mathcal{C}$. We say that P and $\mathcal{C}$ form a system with k degrees of freedom and multiplicity-type s if for any k points in P, there are at most s curves of $\mathcal{C}$ passing through them, and any pair of curves from $\mathcal{C}$ intersect at at most s points of P. In this paper, we cannot acheive two degrees of freedom in our system of points and curves because this problem lacks the symmetry found in the cases where $\mathcal{P}_1$ and $\mathcal{P}_2$ are contained on two lines or two plane algebraic curves, so we use the general theorem instead of the specific statement for two degrees of freedom.

Theorem 2.1 (Pach and Sharir [6]). *Let $\mathcal{P}$ be a finite set of points and $\mathcal{C}$ be a finite set of simple curves all lying in the plane. If $\mathcal{C}$ has k degrees of freedom and multiplicity-type s, then the number of incidences $|I(\mathcal{P}, \mathcal{C})|$ between the two has an upper bound of*

$$|I(\mathcal{P}, \mathcal{C})| \leq c(k, s) \left(|\mathcal{P}|^{k/(2k-1)} |\mathcal{C}|^{(2k-2)/(2k-1)} + |\mathcal{P}| + |\mathcal{C}|\right)$$

where c(k,s) is a constant that depends on k and s.

3 Proof of Theorem 1.1

Before proving Theorem 1.1, we consider the cases where the line is either orthogonal or parallel to the plane. In addition, we show why we restrict two specific configurations of the points in $\mathcal{P}_2$, as they allow for a relatively small amount of distinct distances. In both the orthogonal and parallel cases, we note that

$$D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m),$$

as any point in $\mathcal{P}_2$ gives at least $\lceil \frac{m}{2} \rceil$ distinct distances to the points in $\mathcal{P}_1$. Also, let $d(m, n)$ denote the minimum possible number of distinct distances between $|\mathcal{P}_1| = m$ points on l and $|\mathcal{P}_2| = n$ points on Π.

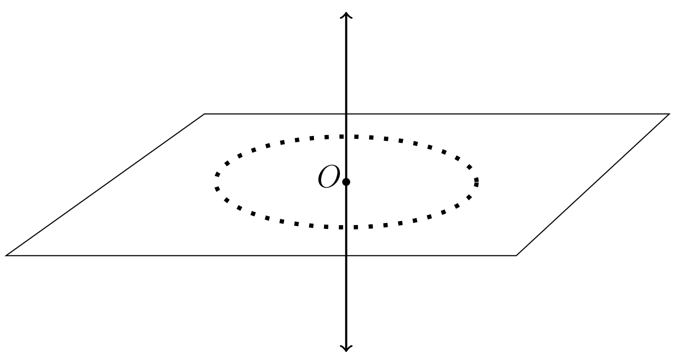

Figure 1: Orthogonal Case

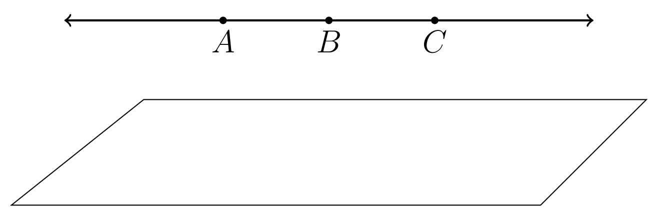

Figure 2: Parallel Case
(see Figure 3 for $\mathcal{P}_2$ configuration)

3.1 Bound on $d(m, n)$ for l orthogonal to Π

Let O be the point of intersection between l and Π. If l is normal to Π, then we can arrange $\mathcal{P}_2$ in a circle of arbitrary radius centered at O. From this we obtain $d(m, n) = \theta(m)$. If we add the restriction that any subset of $\mathcal{P}_2$ can have at most k points that are mutually equidistant from l, where k is some constant, then we have $d(m, n) = \theta(m + n)$ as we can put k points on each of $\lceil n/k \rceil$ circles of arbitrary distinct radii centered at O.

3.2 Bound on $d(m,n)$ for l parallel to Π

This case can be simplified to $\mathbb{R}^2$ as l can be placed or projected onto Π without changing the number of distinct distances. For $m = 1$, we have the following proposition.

Proposition 3.1. *If l is parallel to Π, we have $d(1,n) = 1$.*

Proof. $\mathcal{P}_2$ can be arranged in a circle of arbitrary radius whose center is the projection of the sole point in $\mathcal{P}_1$ onto Π. That gives us $d(1,n) = \theta(1)$. $\square$

For $m \geq 2$, the lower bound on $d(m,n)$ is $\sqrt{n}$, as each point in $\mathcal{P}_2$ is one of the two points determined by a pair of distances from two corresponding points.

Proposition 3.2. *If l is parallel to Π, we have $d(2,n) = d(3,n) = \theta(\sqrt{n})$.*

Proof. For $m = 2$, denote the two points in $\mathcal{P}_1$ as A, B and let the circles of radius r centered at A and B be a_r and b_r, respectively. Consider

$$\{a_x \cap b_y \mid x = d + p\epsilon,\ y = d + q\epsilon,\ 1 \leq p, q \leq \lceil \sqrt{n} \rceil\}\ ,$$

where

$$d > \frac{\mathrm{dist}(A,B)}{2}$$

and this gives us $d(2,n) = \theta(\sqrt{n})$.

For $m = 3$ we use an idea reminiscent of circle grids from [2]. If we let the three points in $\mathcal{P}_1$ be A, B, C and use the same convention for a_r, then if we place A, B, C in arithmetic progression in that order, we can consider

$$\{a_x \cap b_y \cap c_z \mid x = d\sqrt{p}, z = d\sqrt{q}, y = d\sqrt{\frac{p+q}{2} - 1}, 1 \leq p, q \leq \lceil \sqrt{n} \rceil\}\ ,$$

where $d = \mathrm{dist}(A,B)$ (Figure 3). This gives us $d(3,n) = \theta(\sqrt{n})$. $\square$

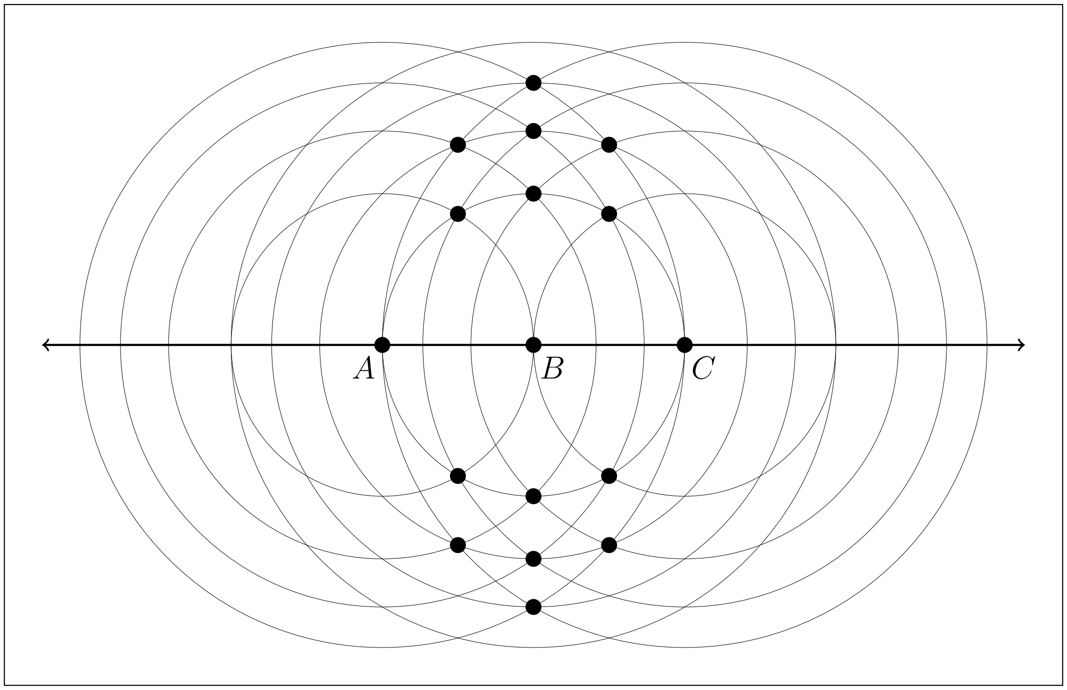

Figure 3: Example for $m = 3$

For $m > 3$, we have $d(m, n) = O(m + n)$, as we can always put the m points in an arithmetic progression and the n points in an arithmetic progression with the same difference in a line parallel to l.

3.3 Other Special Configurations

For the remainder of the proof of Theorem 1.1, we assume that l and Π are neither orthogonal nor parallel. We now discuss two specific constructions of $\mathcal{P}_2$ that achieve $D(\mathcal{P}_1, \mathcal{P}_2) = \theta(m + n)$ in order to show the necessity of the restrictions put forth in Theorem 1.1.

Proposition 3.3. *If $\mathcal{P}_2$ lies on a line perpendicular to the projection of l onto Π, we have $d(m, n) = \theta(m + n)$.*

Proof. Here we may ignore the distance between l and the line on which $\mathcal{P}_2$ lies, as it does not affect the number of distinct distances. We can then treat this problem as two perpendicular lines in $\mathbb{R}^2$ and set the lines as the axes in the Cartesian plane, and let

$\mathcal{P}_1 = \{(\sqrt{1}, 0), (\sqrt{2}, 0), ..., (\sqrt{m}, 0)\}$ and $\mathcal{P}_2 = \{(0, \sqrt{1}), (0, \sqrt{2}), ..., (0, \sqrt{n})$ and we obtain $d(m, n) = \theta(m + n)$.

□

Recall that O is defined as the intersection of l and Π. Throughout the rest of this paper we let $\|u\|$ denote the distance from O to u, and we let $\|u - v\|$ denote the distance between u and v, where u, v are any two points in $\mathbb{R}^d$.

Proposition 3.4. *If $\mathcal{P}_2$ lies on an ellipse of arbitrary semi-major axis centered at O and eccentricity $\csc(\alpha)$ whose semi-major axis lies along the projection of l onto Π, we have $d(m, n) = \theta(m + n)$.*

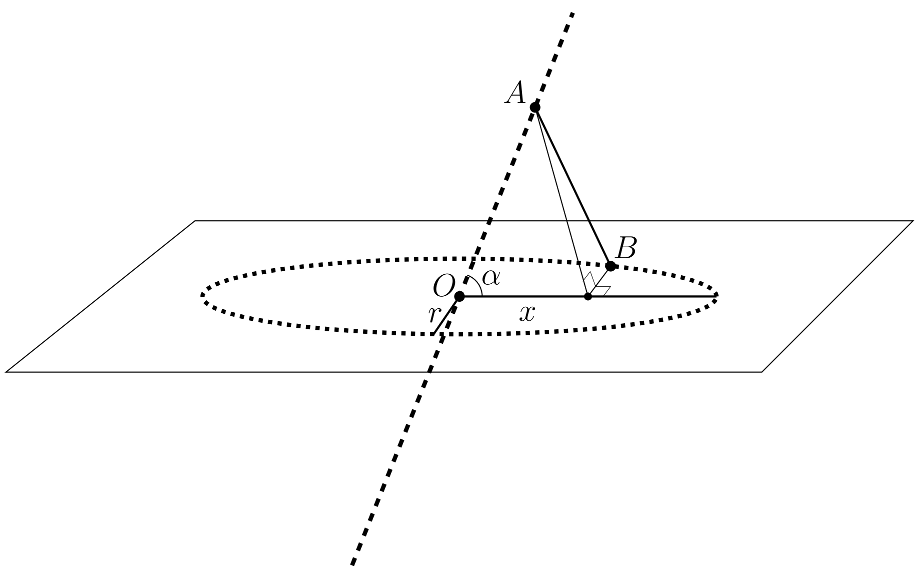

Figure 4: Ellipse case

Proof. Let r be the length of the semi-minor axis of our ellipse with eccentricity $\csc(\alpha)$ centered at O (Figure 4). Take two points $A \in \mathcal{P}_1$, $B \in \mathcal{P}_2$ and define (x, y) as the coordinates of B on the ellipse if $O = (0, 0)$ and the semi-major axis is oriented along the x-axis. By the

Law of Cosines,

$$\|A-B\|^2 = \|A-H\|^2 + \|H-B\|^2 = \|A\|^2 + x^2 - 2\|A\|x\cos(\alpha) + y^2 \ .$$

As the ellipse has eccentricity $\csc(\alpha)$, we see that $y^2 = r^2 - x^2\sin^2\alpha$, which gives us

$$\|A-B\|^2 = (\|A\| - x\cos(\alpha))^2 + r^2 \ .$$

So now if we place the points in $\mathcal{P}_1$ in arithmetic progression with common difference d and place the points in $\mathcal{P}_2$ along this type of ellipse with sufficiently large r such that the x-coordinates of those points form an arithmetic progression with common difference $\dfrac{d}{\cos\alpha}$, then it follows that $d(m,n) = \theta(m+n)$.

□

Remark 3.5. *This ellipse is the intersection of the plane* Π *and a cylinder of arbitrary radius* r *centered on* l*. This construction is therefore analogous to the parallel lines case in [8], as the cylinder is the locus of points with some constant distance to* l*. Note that as* α *tends to* 0 *we have exactly the two parallel lines case as* $\|A-B\|$ *will depend on* $\|A\| - x$*, and if* $\alpha = \frac{\pi}{2}$ *we arrive at the orthogonal case, where* $\|A-B\| = \sqrt{\|A\|^2 + r^2}$*, a constant.*

Now that we have shown the necessity of certain restrictions on l and $\mathcal{P}_2$, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let O be the point of intersection between l and Π. Without loss of generality, we may assume that all m points in $\mathcal{P}_1$ are on one side of O. Otherwise we can partition $\mathcal{P}_1$ into two subsets on either side of O and remove the subset that yields fewer distinct distances, reducing the total number by at most a constant factor of 2. Similarly, we may assume without loss of generality that all n points in $\mathcal{P}_2$ lie in one of the four quadrants determined by l', the projection of l onto Π, and the line on Π going through O that is perpendicular to l', because otherwise we can remove the three that yield fewer distinct distances to reduce by a constant factor of at most 4.

Consider quadruples of points (a, b, p, q) such that $a, b \in \mathcal{P}_1,\ p, q \in \mathcal{P}_2$. We would like to relate the minimum number of distinct distances to occurrences of quadruples of points (a, b, p, q) such that $\|a - p\| = \|b - q\|$. Now, let

$$S \coloneqq \{d_i \in \mathbb{R} \mid p \in \mathcal{P}_1, q \in \mathcal{P}_2, d_i = \|p - q\|\} \ .$$

Let $N = |S|$, and let the sets E_i, where i ranges from 1 to n, contain all pairs of points (u, v) with $u \in \mathcal{P}_1, v \in \mathcal{P}_2$ such that $\|u - v\| = d_i$. Let Q be the set of all quadruples (a, b, p, q) with $a, b \in \mathcal{P}_1,\ p, q \in \mathcal{P}_2$ such that $\|a - p\| = \|b - q\|$, and $(a, p) \neq (b, q)$. Now we use Cauchy-Schwarz in order to relate the number of these quadruples to m, n and the number of distinct distances. We have

$$|Q| = 2\sum \binom{|E_i|}{2} \geq \sum (|E_i| - 1)^2 \geq \frac{1}{N}\left(\sum (|E_i| - 1)\right)^2 = \frac{(mn - N)^2}{N} \ . \tag{1}$$

Let $\angle aOp = \beta$ and $\angle bOq = \gamma$ for some element of Q (Figure 5). We have by Law of Cosines

$$\begin{aligned}\|a - p\|^2 &= \|a\|^2 + \|p\|^2 - 2\|a\|\|p\|r_p \\ \|b - q\|^2 &= \|b\|^2 + \|q\|^2 - 2\|b\|\|q\|r_q \ ,\end{aligned}$$

where $r_p = \cos(\beta), r_q = \cos(\gamma)$. Since $(a, b, p, q) \in Q$, we must have

$$\|a\|^2 + \|p\|^2 - 2\|a\|\|p\|r_p = \|b\|^2 + \|q\|^2 - 2\|b\|\|q\|r_q \ .$$

The above equation can be rewritten to give

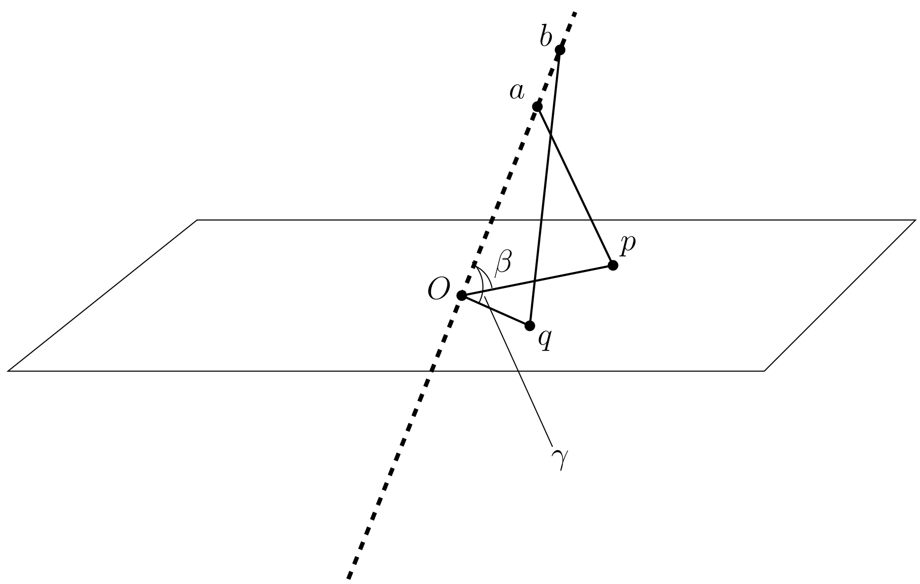

Figure 5: General case

$$(\|a\| - \|p\|r_p)^2 - (\|b\| - \|q\|r_q)^2 = \|q\|^2(1 - r_q^2) - \|p\|^2(1 - r_p^2) \ . \tag{2}$$

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the sets of all ordered pairs of distinct points in $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. For every $(p, q) \in \mathcal{V}_2$ such that $p \neq q$, there is a curve $\gamma_{p,q}$ corresponding to that pair of the form

$$(x - \|p\|r_p)^2 - (y - \|q\|r_q)^2 = \|q\|^2(1 - r_q^2) - \|p\|^2(1 - r_p^2) \ ,$$

where the values of $\|p\|, \|q\|, r_p, r_q$ are fixed. Note that $\|p\|r_p$ is equivalent to the distance between O and the projection of p onto l. Because no two points define a line on Π perpendicular to the projection of l onto Π, we have that all values of $\|p\|r_p$ are distinct. From here we see that all curves $\gamma_{p,q}$ are distinct, and in fact are hyperbolas unless

$$\|q\|^2(1 - r_q^2) - \|p\|^2(1 - r_p^2) = 0 \ ,$$

which gives us a degenerate hyperbola. However, because $\|p\|^2(1 - r_p^2)$ is exactly the square of the distance from p to the line l we see that we can only arrive at a degenerate hyperbola if both p, q lie on the same ellipse of eccentricity $\csc(\alpha)$ centered at O with semi-major axis on the projection of l onto Π, which is forbidden. Now, let $\mathcal{C}$ be the set of non-degenerate hyperbola $\mathcal{C} = \{\gamma_{p,q} | (p,q) \in \mathcal{V}_2, p \neq q\}$. Any curve $\gamma_{p,q}$ in $\mathcal{C}$ is incident to a point $(a,b) \in \mathcal{V}_1$ if and only if (a,b,p,q) satisfies the condition in (2). First, let us examine the number of elements of Q not counted by incidences between $\mathcal{C}$ and $\mathcal{V}_1$, which happens when $p = q$. If we consider that there are at most $2\lfloor \frac{m}{2} \rfloor$ ordered pairs of points equidistant to some point in $\mathcal{P}_2$, we obtain an upper bound of $2mn$ on the missing elements.

Because the curves in $\mathcal{C}$ are hyperbolas, and the coefficients of x^2, y^2, xy are fixed, $\mathcal{C}$ has 3 degrees of freedom and multiplicity-type 2. Now, applying Theorem 2.1 to $\mathcal{V}_1$ and $\mathcal{C}$ with $k = 3, s = 2$ and adding in the missing elements gives us

$$\|q\| = O(m^{\frac{6}{5}} n^{\frac{8}{5}} + m^2 + n^2 + mn) \ . \tag{3}$$

Combining this with (1) gives us

$$\frac{(mn - N)^2}{N} = O(m^{\frac{6}{5}} n^{\frac{8}{5}} + m^2)$$

if $m \geq n$, and a similar expression for $n \geq m$.

Since $N \leq mn$ this gives us

$$N = \Omega(\min(m^{\frac{4}{5}} n^{\frac{2}{5}}, n^2))$$

and combining this with the version for $n \geq m$ gives us Theorem 1.1.

$\square$

4 Proof of Theorem 1.2

Set the origin of the vector space $\mathbb{R}^d$ to be the intersection of l and Π, and let Π be the hyperplane $x_d = 0$. Let the unit direction vector of l be $\hat{l} = (a_1, a_2, ..., a_d)$. We consider separately the cases where l orthogonal to Π, which happens iff $a_d = 1$, or where l is contained in a hyperplane with the same unit normal vector as Π, which happens iff $a_d = 0$. Similarly to the $\mathbb{R}^3$ case, we must also consider special configurations of the points in $\mathcal{P}_2$ that give relatively few distinct distances.

4.1 Special Configurations

Recall that $d(m, n)$ denotes the minimum possible number of distinct distances between $|\mathcal{P}_1| = m$ points on l and $|\mathcal{P}_2| = n$ points on Π. Note that for any m, n, we must have $d(m, n) = \Omega(m)$ as any point in $\mathcal{P}_2$ gives at least $\lceil \frac{m}{2} \rceil$ distinct distances to the points in $\mathcal{P}_1$, since there can be at most two points on l that are some fixed distance R away from a point on Π.

Proposition 4.1. *If l is orthogonal to Π, we have $d(m, n) = \theta(m)$.*

Proof. If $a_d = 1$, we can place all n points of $\mathcal{P}_2$ in a $(d-2)$-sphere of arbitrary radius centered at O that lies in Π. From this we obtain $d(m, n) = \theta(m)$. □

Proposition 4.2. *If l is contained in a hyperplane with the same unit normal vector as Π, we have $d(m, n) = \theta(m)$.*

Proof. If $a_d = 0$, we can simplify this case as we did in Subsection 3.2 by placing or projecting l onto Π, which does not change the number of distinct distances. Label the m points of $\mathcal{P}_1$ $v_1, v_2, ..., v_m$. If we treat l as the real number line, and place v_i at $(-1)^i \sqrt{\lfloor \frac{i}{2} \rfloor}$, we can contain $\mathcal{P}_2$ in the $(d-3)$-sphere defined as the intersection of the $(d-2)$-spheres with one centered at each v_i with radius $\sqrt{\lfloor \frac{i}{2} \rfloor + 1}$ to obtain $d(m, n) = \theta(m)$. □

For the remainder of this paper, we assume l and Π are neither orthogonal nor parallel. Now, we examine special constructions of $\mathcal{P}_2$ that give few distinct distances.

Proposition 4.3. *If $\mathcal{P}_2$ lies on a flat of codimension 2 that is defined as the intersection of Π and a hyperplane ϱ orthogonal to l, then we have $d(m,n) = O(m+n)$.*

Proof. Let p be the intersection of l and ϱ, and let l_0 be the image of l under the projection $\psi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ given by $(x_1, x_2, ..., x_d) \mapsto (x_1, x_2, ..., x_{d-1})$. Define $p' = l_0 \cap \Pi \cap \varrho$, and let l' be the image of l under a translation that sends p to p'. We can then use the same construction as in the proof of Proposition 3.4, using l' and some line perpendicular to l' in $\Pi \cap \varrho$ going through p' to obtain $d(m,n) = O(m+n)$, as our translation does not affect the number of distinct distances. □

Proposition 4.4. *If $\mathcal{P}_2$ lies on the locus of points on Π that are some fixed distance R away from the line, and we forbid more than two points of $\mathcal{P}_2$ from lying in the flat of codimension 2 that is defined as the intersection of Π and a hyperplane orthogonal to l, then we have $d(m,n) = O(m+n)$.*

Proof. The distance R from l to a point p on Π defined as $(x_1, x_2, ..., x_{d-1}, 0)$ is given by

$$R^2 = \sum_{1}^{d-1}(1-a_i^2)x_i^2 - 2\sum_{1\le i<j\le d-1} a_i a_j x_i x_j$$

If we fix R, we see that this is a surface of degree 2 and codimension 2 contained in Π.

Note that this surface, defined as the locus of points on Π that are the same distance R from l, is equivalent to the intersection of Π and the Cartesian product of l with a $(d-2)$ sphere of radius R centered at O in the hyperplane orthogonal to l containing O. This is true because a $(d-2)$ sphere of radius R centered at some point s on l in the hyperplane orthogonal to l containing s is also the locus of points at a distance R from l such that the foot of the perpendiculars from those points to l is s. Define S_p as the $(d-2)$ sphere of radius R centered at p in the hyperplane orthogonal to l containing p. Now, to achieve few distinct distances, let us define a sequence of points $p_1, p_2, ..., p_{\max(m,n)}$ that lie in arithmetic progression on l. Let $\mathcal{P}_1 = \{p_1, p_2, ...p_m\}$ and $\mathcal{P}_2 = \{q_1, q_2, ..., q_n\}$ such that $q_i \in S_{p_i} \cap \Pi$. This construction gives us $d(m,n) = O(m+n)$. $\square$

Proof of Theorem 1.2. Without loss of generality, we may assume that $\mathcal{P}_1$ lies on one side of O for the same reasoning given in the proof of Theorem 1.1. We again consider quadruples (a, b, p, q) with $a, b \in \mathcal{P}_1$, $p, q \in \mathcal{P}_2$, such that $\|a - p\| = \|b - q\|$, and $(a, p) \neq (b, q)$. If Q is the set of all such quadruples and N is the number of distinct distances, we have already seen that

$$|Q| \geq \frac{(mn-N)^2}{N} .$$

Let the vector $\hat{p} = (x_1, x_2, ..., x_{d-1}, 0)$ be the vector associated with a point p on Π. Now, recall that $\|p\|$ denotes the distance from O to the point p, and let $r_p = \cos(\angle aOp) = \dfrac{\hat{l} \cdot \hat{p}}{\|\hat{l}\| \|p\|}$ where a is some point on l that is not O. Since $\|a - p\| = \|b - q\|$, we have

$$\|a\|^2 + \|p\|^2 - 2\|a\|\|p\|r_p = \|b\|^2 + \|q\|^2 - 2\|b\|\|q\|r_q$$

by Law of Cosines, and this can be rewritten to give

$$(\|a\| - \|p\|r_p)^2 - (\|b\| - \|q\|r_q)^2 = \|q\|^2(1 - r_q^2) - \|p\|^2(1 - r_p^2) . \tag{4}$$

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the sets of all ordered distinct pairs of points in $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. For every $(p, q) \in \mathcal{V}_2$ such that $p \neq q$, there is a curve $\gamma_{p,q}$ corresponding to that pair of the form

$$(x - \|p\|r_p)^2 - (y - \|q\|r_q)^2 = \|q\|^2(1 - r_q^2) - \|p\|^2(1 - r_p^2) ,$$

where the values of $\|p\|, \|q\|, r_p, r_q$ are fixed. Now, since

$$\|p\|r_p = \frac{\hat{l} \cdot \hat{p}}{\|\hat{l}\|} = \sum_{i=1}^{d-1} a_i x_i \ ,$$

if we have two points p and p' such that $\|p\|r_p = \|p'\|r_{p'}$, then by the above equation we see that they must lie in the same flat of codimension 2 defined as the intersection of Π and a hyperplane ϱ of the form $\sum_{i=1}^{d-1} a_i x_i = C$ for some real constant C. Because no two points of $\mathcal{P}_2$ can lie on this type of flat, all curves $\gamma_{p,q}$ are distinct and are hyperbolas unless $\|q\|^2(1-r_q^2) - \|p\|^2(1-r_p^2) = 0$, giving us a degenerate hyperbola. We see that

$$\|p\|^2(1-r_p^2) = \|p\|^2 \sin^2(\angle aOp)$$

is equivalent to the square of the distance from the point p to the line l, so we can only arrive at a degenerate hyperbola if p and q both lie on the locus of points on Π that are some fixed distance R from l, which is forbidden. Now, let $\mathcal{C}$ be the set of curves $\mathcal{C} = \{\gamma_{p,q} | (p,q) \in \mathcal{V}_2, p \neq q\}$. Any curve $\gamma_{p,q}$ in $\mathcal{C}$ is incident to a point $(a,b) \in \mathcal{V}_1$ if and only if (a,b,p,q) satisfies the condition in (2). First, let us examine the number of elements of Q not counted by incidences between $\mathcal{C}$ and $\mathcal{V}_1$, which happens when $p = q$. If we consider that there are at most $2\lfloor \frac{m}{2} \rfloor$ ordered pairs of points equidistant to some point in $\mathcal{P}_2$, we obtain an upper bound of $2mn$ on the missing elements.

We now finish the proof of Theorem 1.2 in exactly the same fashion as the proof of Theorem 1.1.

$\square$

5 Conclusion and Direction for Future Research

We considered the number of distinct distances between elements of two sets of points, $\mathcal{P}_1$ and $\mathcal{P}_2$. We proved a superlinear lower bound of $\Omega(n^{6/5})$ for the number of distinct distances between points of $\mathcal{P}_1$ and $\mathcal{P}_2$ when $\mathcal{P}_1$ is contained in a line and $\mathcal{P}_2$ is contained in a plane in $\mathbb{R}^3$, given some restrictions on the line and $\mathcal{P}_2$. This required showing that the configurations we restricted led to relatively few distinct distances. Furthermore, we generalized this result to obtain the same lower bound when $\mathcal{P}_1$ is still contained in a line but $\mathcal{P}_2$ is instead contained in a hyperplane in $\mathbb{R}^d$. Our results can most likely be improved upon, as we believe that this bound can be improved to $\Omega(n^{2-\epsilon})$ for any $\epsilon > 0$.

Future work on asymmetrical bipartite distinct distance problems may include showing superlinear bounds for the generalizations where $\mathcal{P}_1$ is contained on an algebraic curve of degree d and $\mathcal{P}_2$ is contained on a hyperplane in $\mathbb{R}^n$, or where $\mathcal{P}_1$ is contained on a line and $\mathcal{P}_2$ is contained on a hypersurface of degree d in $\mathbb{R}^n$. To effectively use the Elekes-Sharir framework to solve those problems, it is necessary to show that the resulting curves $\gamma_{p,q}$ cannot have infinite intersection very often. This could be done by considering under what conditions our curves can have infinite intersection and partitioning the set of curves to achieve finite degrees of freedom, or by examining the conditions that many curves having infinite intersection impose on the curve or hypersurface containing $\mathcal{P}_1$ or $\mathcal{P}_2$ and using those to reach a contradiction.

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Appendix A Parametric Polynomial Curves

If we have two sets of points $\mathcal{P}_1$ and $\mathcal{P}_2$ such that the points of $\mathcal{P}_1$ lie on a parametric polynomial curve C in $\mathbb{R}^3$ and the points of $\mathcal{P}_2$ lie on a plane Π, we look at the number of distinct distances from a point in $\mathcal{P}_1$ to a point in $\mathcal{P}_2$. We aim to use the same general framework for this situation.

Let $C(t) = (f_x(t), f_y(t), f_z(t))$, where each of f_x, f_y, f_z are polynomials, and let the plane Π be the one defined by the equation $ax+by+cz = D$. Also, let $p = (p_x, p_y, p_z), q = (q_x, q_y, q_z)$. We examine quadruples of points (t_1, t_2, p, q) such that $\|C(t_1) - p\| = \|C(t_2) - q\|$. Any such quadruple must satisfy

$$(f_x(t_1) - p_x)^2 + (f_y(t_1) - p_y)^2 + (f_z(t_1) - p_z)^2 = (f_x(t_2) - q_x)^2 + (f_y(t_2) - q_y)^2 + (f_z(t_2) - q_z)^2 \ .$$

If we let

$$f_x(t) = a_d t^d + a_{d-1} t^{d-1} + ... + a_0$$
$$f_y(t) = b_d t^d + ... + b_0$$
$$f_z(t) = c_d t^d + ... + c_0$$
$$f(t) = f_x(t)^2 + f_y(t)^2 + f_z(t)^2$$

then we can construct curves $\gamma_{p,q}$ in the xy-plane of the form

$$f(x) - f(y) + 2\sum_{i=0}^{d} \big((a_i q_x + b_i q_y + c_i q_z)t_2^i - (a_i p_x + b_i p_y + c_i p_z)t_1^i\big) + p_x^2 + p_y^2 + p_z^2 - q_x^2 - q_y^2 - q_z^2 = 0$$

If these curves are not distinct, then there must be points p, q, p', q' such that $\gamma_{p,q} = \gamma_{p',q'}$, with $(p, q) \neq (p', q')$. We then must have

$$a_i p_x + b_i p_y + c_i p_z = a_i p'_x + b_i p'_y + c_i p'_z$$

$$a_i q_x + b_i q_y + c_i q_z = a_i q'_x + b_i q'_y + c_i q'_z$$

for all i such that $1 \leq i \leq d$. If $p \neq p'$ and $q \neq q'$, this implies that for two vectors $\hat{u} = \langle p'_x - p_x, p'_y - p_y, p'_z - p_z \rangle$ and $\hat{v} = \langle q'_x - q_x, q'_y - q_y, q'_z - q_z \rangle$, we have $\hat{u} \cdot \langle a_i, b_i, c_i \rangle = \hat{v} \cdot \langle a_i, b_i, c_i \rangle = 0$ for $1 \leq i \leq d$. It follows that $C(t)$ is actually a line, unless $\hat{u} = k\hat{v}$ for some non-zero scalar.

If we can figure out under what conditions the curves $\gamma_{p,q}$ have infinite intersection, we conjecture that we will be able to make the instances of infinite intersection bounded to a small number so that we can again apply the incidence bound result to obtain

$$N = \Omega(\min(m^{4d/4d+1} n^{2/4d+1}, m^2, n^2))$$

or

$$N = \Omega(n^{4d+2/4d+1})$$

when $m = n$.

Remark Appendix A.1. *If the parametric polynomial curve (which could also be generalized to an irreducible plane algebraic curve) lies in a plane, then we can use the same methods as in [5] to acheive some superlinear bound.*