Homomesy of Alignments in Perfect Matchings

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Abstract

We investigate the existence of a group action τ that is homomesic with respect to alignments, a type of statistic in perfect matchings. Homomesy is defined as the consistency of an average, and perfect matchings are defined as the set of all partitions of 1 to $2n$ into pairs. We take advantage of the bijection between labeled Dyck paths and perfect matchings to investigate to investigate the possibility of defining τ inductively. We also relate perfect matchings to the oscillating tableaux and discover relationships that support the possibility of the existence of τ but do not prove it. We find some surprisingly clean formulae for statistical averages of the analogue to the number of alignments for oscillating tableaux of arbitrary shape, which suggests that τ may exist for more general contexts than just perfect matchings.

Summary

We investigate the existence of a function that preserves a certain characteristic of a set of combinatorial objects. After failing at proving its existence inductively, we relate the function to a different set of objects that have similar properties. We discover new connections that support the possibility of the function's existence. The existence of such a function is applicable in algebra and group theory, and may potentially reveal a hidden symmetry in perfect matchings, which investigate the relationships of groups of pairings in certain sets of objects.

1 Introduction

Combinatorics, the study of discrete mathematical structures, is useful in every branch of mathematics. One particularly interesting subfield of combinatorics, homomesy, deals with symmetries in statistics of combinatorial arguments and specifically. Specifically, in this paper, we find homomesy of alignments in perfect matchings. As their names suggest, homomesy is the consistency of the average and perfect matchings are partitions into pairs. James Propp and Tom Roby coined the term homomesy in 2013 [1]. Since then, research on homomesy has been discovered in a variety of combinatorial structures. For example, rowmotion in order ideals of posets [1] and promotion of Young tableaux have a beautiful homomesic structure. Perfect matchings are used in Kekulé structures of aromatic compounds in order to show the carbon skeleton.

Our main goal is to find a group action that exhibits homomesy with respect to the alignment statistic of perfect matchings. In Section 2, we introduce some preliminary definitions and give a more rigorous formulation of the problem. In Section 3, we outline a different representation of perfect matchings. In Section 4, we describe our inductive approach to finding the solution and prove some lemmata relating Dyck paths to statistics on perfect matchings. However, we show that there are some serious obstructions to obtaining any recursive construction, and switch to a more algebraic approach. In Section 5, we describe the limitations of the approach in Section 4 towards finding homomesic maps. In Section 6, we derive numerous results on the relationship between oscillating tableaux and perfect matchings.

2 Preliminaries

Definition 1. *A statistic $f : S \to \mathbb{N}$ on a set of combinatorial objects S is* homomesic *with respect to the action of a group G on S if there is some constant $c \in \mathbb{R}$ such that the average* *of f along each G-orbit is c. The G-orbit of an element $s \in S$ is the set of all elements that s can map to after the action of a group element $g \in G$ is applied on S.*

Definition 2. *A* perfect matching *of order n is the set of all partitions of $\{1, 2, \ldots, 2n\}$ into pairs. We denote this by $\mathcal{PM}(n)$.*

Past research on homomesy has been restricted to the case where the group G is cyclic, i.e., $G = \langle \varphi \rangle$ for an invertible map $\varphi : S \to S$ where S is an arbitrary set. Generally, the φ investigated have been well-studied procedures such as rowmotion in order ideals of posets [1, 2] or promotion of Young tableaux [3]. The goal in such cases is to understand which natural statistics on S are homomesic with respect to φ.

Homomesy can be very useful in understanding the behavior of a statistic f on a set S. If the triple (S, f, G) exhibits homomesy, where G is some group, then the denominator in the average of f across a G-orbit can be at most be the size of the orbit. By the Orbit Stabilizer Theorem, the size of the orbit must divide the size of G. If the average values of f across all G-orbits are equal, then they are equal to the average of f across all elements of S. As a result, the denominator in this average is bounded by the size of G, and in particular, G can bound the denominator in the average of the statistic f over all elements of S.

The set of combinatorial objects we focus on here is the set of perfect matchings.

A matching can be graphically represented by arranging the numbers $1, 2, \ldots, 2n$ in order on a line and connecting the pairs in the matching by arcs. An example of a matching on six elements can be seen in Figure 1 [4].

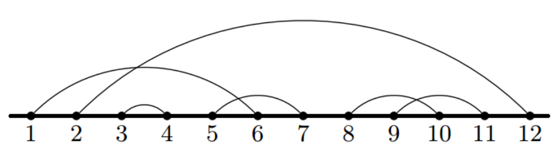
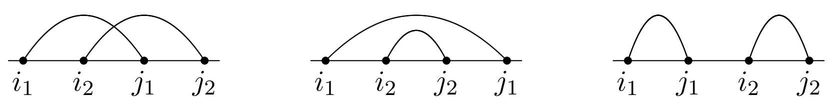


Figure 1: Example of a matching of order 6

Between each pair of arcs in a matching, there are three possible relationships: crossing, nesting, or alignment, as shown in Figure 2 [5]. The numbers of crossings, nestings, and alignments of a matching M are denoted by $\mathrm{cr}(M)$, $\mathrm{ne}(M)$, and $\mathrm{al}(M)$ respectively. Many authors in the past have studied the statistics cr, ne, al on perfect matchings [6, 7, 8, 5]. We can make a couple of simple observations about these functions. For any matching $M \in \mathcal{PM}(n)$ we have $\mathrm{cr}(M) + \mathrm{ne}(M) + \mathrm{al}(M) = \binom{n}{2}$, because each pair of arcs must either cross, nest, or align, and there are $\binom{n}{2}$ such pairs. Two randomly chosen arcs in a randomly chosen $M \in \mathcal{PM}(n)$ are equally likely to cross, nest, or align. Thus, the average number of crossings, 0, and alignments across all matchings in $\mathcal{PM}(n)$ is $\binom{n}{2}/3$. It is this small denominator 3 that we aim to explain via homomesy.



(i) *(ii)* *(iii)*

Figure 2: Crossing, nesting and alignments of two edges

Definition 3. *Given a finite set S of combinatorial objects and statistics $f_1, \ldots, f_k \colon S \to \mathbb{N}$, we say that $f_1, \ldots, f_k$ are* symmetrically distributed *if*

$$\sum_{s \in S} x_1^{f_1(s)} x_2^{f_2(s)} \cdots x_k^{f_k(s)} = \sum_{s \in S} x_{\pi(1)}^{f_1(s)} x_{\pi(2)}^{f_2(s)} \cdots x_{\pi(k)}^{f_k(s)}$$

for all permutations $\pi \in S_k$, where S_k is the group of all permutations of $\{1, 2, \ldots, k\}$.

The statistics cr and ne are symmetrically distributed over $\mathcal{PM}(n)$. This follows from the existence of a certain invertible map $\sigma \colon \mathcal{PM}(n) \to \mathcal{PM}(n)$ defined by de Médicis-Viennot [6] (see [5] for a description of σ in English). The map σ is an involution $\sigma^2 = \mathrm{id}$, and it interchanges crossings and nestings in the sense that $\mathrm{cr}(\mathcal{M}) = \mathrm{ne}(\sigma(\mathcal{M}))$ and $\mathrm{ne}(\mathcal{M}) = \mathrm{cr}(\sigma(\mathcal{M}))$ for all $M \in \mathcal{PM}(n)$.

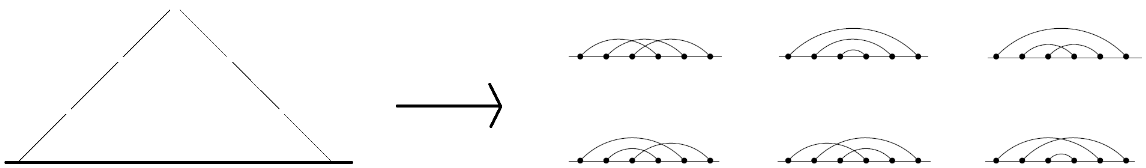


Figure 3: Dyck path shape and corresponding arc diagrams

Note that the three statistics cr, ne, and al are not symmetrically distributed. Nevertheless, there is the possibility that there exists some small group G acting on $\mathcal{PM}(n)$ such that all three of these statistics are homomesic with respect to the action. In fact, it is highly likely that there exists such a G with $G \simeq S_3$. If so, homomesy would reveal a S_3-symmetry relating cr, ne, and al that is more nuanced than symmetric distribution. This would also be the first example of homomesy involving a nonabelian group that we are aware of.

Our goal is to establish an S_3 homomesy on matchings. This reduces to finding some invertible map $\tau\colon \mathcal{PM}(n) \to \mathcal{PM}(n)$ with $\tau^3 = \mathrm{id}$ such that al is homomesic with respect to τ. If $(\sigma\tau)^2 = \mathrm{id}$, then $G = \langle \tau, \sigma \rangle$ is isomorphic to S_3 as desired.

3 Dyck path Representation

Definition 4. *A* Dyck path *is a walk from* $(0,0)$ *to* (n,n) *that lies strictly below, but may touch the diagonal* $y = x$*.*

Dyck paths are one of the many representations of Catalan numbers. Catalan numbers will play an important role in the proof of Lemma 1.

Definition 5. *Catalan numbers are given by the sequence* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *for* $n \geq 1$*.*

A matching can also be represented by a *labeled Dyck path*. The beginning of an arc in an arc diagram is represented by an up-step in the Dyck path, and the end of an arc is represented by a down-step. Some Dyck path shapes correspond to multiple arc diagrams, depending on which down-step corresponds to which up-step. For example, the Dyck path in Figure 3 can correspond to any of the six arc diagrams. Thus, a dot of a certain height is placed at each down-step, indicating which up-step the down-step is connected to. If the down-step is connected to the first available up-step, then the dot is placed on the axis. If the down-step is connected to the second available up-step, the dot is placed one unit above the axis. In general, if the down-step is connected to the n^{th} available up-step, the dot is placed $n-1$ units above the axis. A clearer explanation can be given by using Figure 4 as an example.

Figure 4: labeled Dyck path and the corresponding arc diagram

We see in Figure 4 that the first node to contain a down-step is node 4, so the first dot on the Dyck path has x-coordinate 4. The corresponding up-step begins at node 2 and so the first dot on the Dyck path is at $(4,1)$. The next down-step ends at node 5 and corresponds to the up-step at node 1, so the second dot on the Dyck path is placed at $(5,0)$. The final down-step ends at node 6 and begins at node 3, the first node available. Because 3 is the first node available, the y-coordinate of the dot is $1-1=0$. Thus the third dot on the Dyck path is placed at $(6,0)$.

Recall that the map σ is an involution, meaning $\sigma^2 = \mathrm{id}$. The map σ can also be graphically represented; it is equivalent to taking the height of the down-step, subtracting the height of the label, and placing the new dot at that height. This action successfully interchanges crossings and nestings and does not affect the number of alignments.

However, because we are only concerned about alignments, and as such the shape of the Dyck path, we disregard the labels. To keep track of the number of labeled Dyck paths that correspond to a given Dyck path, we can simply multiply the heights of the down-steps.

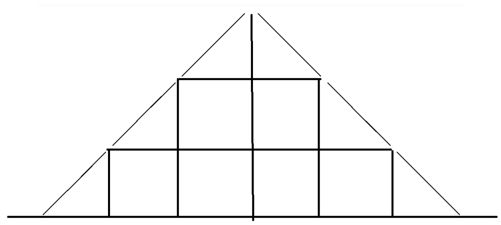


Figure 5: Area under Dyck path

We define the area of a Dyck path to be the number of whole squares that can be drawn under the path. An example of a Dyck path with area 6 is shown Figure 5. The number of alignments in a matching is proportional to the area of the Dyck path. We denote the area under a Dyck path of a matching with $A_{\mathcal{M}}$.

Lemma 1. *For all matchings $\mathcal{M}$, we have* $\mathrm{al}(\mathcal{M}) = \binom{n}{2} - \frac{1}{2}A_{\mathcal{M}}$.

Proof. In Dyck paths, every down-step has an up-step of corresponding height. This comes from the representation of Catalan numbers, because no initial segment of the sequence of up-steps and down-steps has more up-steps than down-steps. Thus, the heights of the down-steps equal half of the area. We also know that

$$\begin{aligned} \frac{1}{2}A_{\mathcal{M}} &= \sum \text{heights of down steps} \\ &= \mathrm{cr}(\mathcal{M}) + \mathrm{ne}(\mathcal{M}) \end{aligned}$$

from [6]. Recall that $\mathrm{cr}(\mathcal{M}) + \mathrm{ne}(\mathcal{M}) + \mathrm{al}(\mathcal{M}) = \binom{n}{2}$, where $\mathcal{M} \in \mathcal{PM}(n)$, as shown in [5]. Thus, $\frac{1}{2}A_{\mathcal{M}} = \binom{n}{2} - \mathrm{al}(\mathcal{M})$. Rearranging, we obtain $\mathrm{al}(\mathcal{M}) = \binom{n}{2} - \frac{1}{2}A_{\mathcal{M}}$.

$\square$

4 Inductive Approach to Finding τ

We begin by enumerating $\mathcal{PM}(n)$. Note that $|\mathcal{PM}(n)| = (2n-1)\cdot(2n-3)\cdots 3\cdot 1$. This product formula for the number of perfect matchings follows easily by observing how matchings can be built up inductively by adding in arcs; see [8] for the resulting tree of perfect matchings. We attempt to define the partition of $\mathcal{PM}(n)$ into triples that we desire by inductively adding in arcs in this manner one at a time. We do this by adding an arc that starts at the beginning of a matching and ends anywhere within the matching, thus turning a matching of order n to one of order $n+1$. If we do this to all matchings of order n, we are able to create to all matchings of order $n+1$.

Definition 6. *Given a matching $\mathcal{M} \in \mathcal{PM}(n)$, we associate to $\mathcal{M}$ the string of alignment numbers $(a_1, a_2, a_3, ..., a_{2n+1})$, where a_i is the number of additional alignments that we obtain if we end the arc between the $i-1^{th}$ point and the i^{th} point.*

An example of alignment numbers for a Dyck path of order 3 is shown in Figure 6.

Lemma 2. *The series of alignment numbers for any matching is always decreasing.*

Proof. Every time the end of the arc crosses the beginning of a new arc, or an up-step on a Dyck path, we lose one alignment. If we represent the Dyck path as a series of -1's and 0's, with an up-step being a -1 and a down-step being a 0, we can quickly calculate the alignment numbers for a specific Dyck path. We have $a_1 = n$ and $a_{i+1} = a_i + \delta_i$, where $\delta_i = -1$ if the i^{th} step is up, and $\delta_i = 0$ if the i^{th} step is down for all sequences of alignment numbers. □

There is also a relationship between the sum of alignment numbers, which we denote by $S_{\mathcal{M}}$, and $\mathrm{al}(\mathcal{M})$.

Lemma 3. *For all matchings of order n, $S_{\mathcal{M}} = \mathrm{al}(\mathcal{M}) + \binom{n}{2}$.*

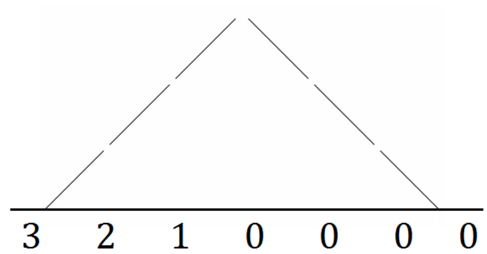


Figure 6: Alignment Numbers for a Dyck path of Order 3

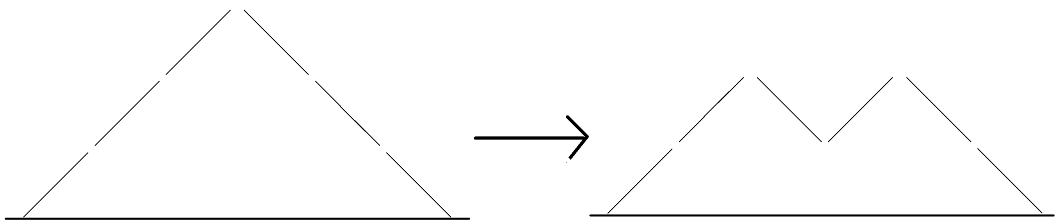


Figure 7: Depressing a Peak

Proof. If we take any sequence of up-down from a matching's Dyck path and switch it to a sequence of down-up, we subsequently decrease the area of the Dyck path by 2, increase the number of alignments by 1, and increase the sum of the alignment numbers by 1. Looking at this switch graphically, the switch inverts a peak of the Dyck path. An inversion is illustrated in Figure 7.

For a Dyck path of order n that resembles the one in Figure 6, the sum of the alignment numbers is equal to $n + (n-1) + (n-2) + \cdots + 1 + 0 + 0 + \cdots + 0 = \binom{n+1}{2}$, where there are $n+1$ 0's. Note that there are 0 alignments in the matching that corresponds to a Dyck path with only one peak. Since the number alignments and the sum of the alignment numbers increase at the same rate when we do the switch, $S_{\mathcal{M}} = \binom{n+1}{2} + \text{al}(\mathcal{M})$. □

We want a partition of $\mathcal{PM}(n)$ into orbits $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_k$, such that $|\mathcal{O}_i = 3|$ and the average number of alignments in perfect matchings in each $\mathcal{O}_i$ is the same. As mentioned in

Section 2, the average number of alignments over every element in $\mathcal{PM}(n)$ is $\frac{1}{3}\binom{n}{2}$. This is equivalent to saying

$$\frac{1}{|\mathcal{O}_i|}\sum_{\mathcal{M}\in\mathcal{O}_i}\mathrm{al}(\mathcal{M}) = \frac{1}{|\mathcal{PM}(n)|}\sum_{\mathcal{M}\in\mathcal{PM}(n)}\mathrm{al}(\mathcal{M})$$
$$= \frac{1}{3}\binom{n}{2}.$$

Therefore, the total number of alignments in each orbit for matchings of order n is $\binom{n}{2}$. The existence of a homomesic triple $(\mathrm{al}, \mathcal{PM}(n), \tau)$ such that $\tau^3 = \mathrm{id}$ is equivalent to the existence of a partition of $\mathcal{PM}(n)$ into triples such that for each triple $\mathcal{O}_i$, we have

$$\sum_{M\in\mathcal{O}}\mathrm{al}(M) = \binom{n}{2}.$$

The difference between the total number of alignments in an orbit of matchings of order $n+1$ and order n is then $\binom{n+1}{2} - \binom{n}{2} = n$. This shows us how to choose where to end the arcs in the matchings of an orbit. We group triples of Dyck numbers together, one from each matching, such that they sum to n. This guarantees that the orbits of the perfect matchings of order $n+1$ are also homomesic.

Lemma 4. *In the partition of $\mathcal{PM}(n)$ into orbits of size 3 such that it is homomesic, the sum of all alignment numbers in an orbit must equal $n(2n+1)$.*

Proof. Since there are 3 sets of alignments numbers in each orbit–one per matching–and $2n+1$ alignments numbers per element, there are $2n+1$ triples of numbers. Since each triple sums to n, the total sum of all alignment numbers is $n(2n+1)$. □

When we inductively add arcs, the sum of the alignment numbers is equal to

$$3\binom{n+1}{2} + \sum \mathrm{al}(\mathcal{M}) = n(2n+1).$$

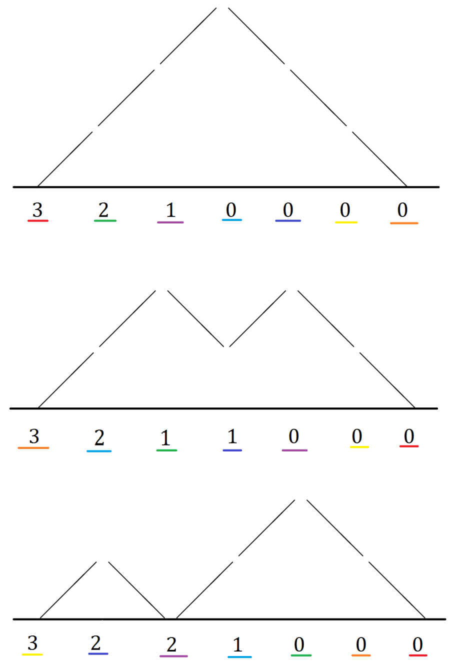


Figure 8: Tripling in an orbit of Dyck path of order 3

Thus, we have the necessary conditions for the tripling to exist. We can perform a greedy algorithm to find these triples, but it may not succeed, as Section 5 explains. An example of the tripling can be seen in Figure 8.

5 Counterexample to Inductive Approach

Computations carried out with Sage mathematical software using integer programming verify the existence of such a partition for all $2 \leq n \leq 14$.

There are cases where the tripling approach does not work. In one case with an orbit of the perfect matchings of order 4, the Dyck numbers can never be fully tripled. The orbit is shown in Figure 9. The method of inductively showing that τ exists has failed, although we must keep in mind that this is a naïve inductive approach. Despite the existence of a counterexample, the success of the Sage program in finding partitions suggests a more refined

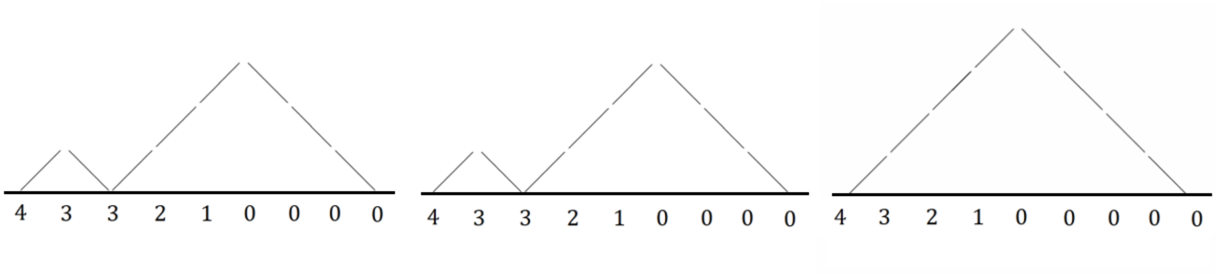


Figure 9: Counterexample to Greedy Algorithm

argument network.

6 Oscillating Tableaux

Another set of combinatorial objects closely related to perfect matchings of order n is the oscillating tableaux of length $2n$. The oscillating tableaux of length $2n$ are the walks in the Hasse diagram of Young's lattice from the empty partition $\emptyset$ to itself taking $2n$ steps. Young's lattice is the partially ordered set, or poset, of partitions by inclusion, as shown in Figure 10.

We denote the set of oscillating tableaux of length $2n$ by $\mathcal{OT}(n)$. More generally, the set of all walks of Young's lattice of length l that start with $\emptyset$ and end with some tableau λ are denoted by $OT(\lambda, l)$. Perfect matchings of order n are in bijection with oscillating tableaux of length $2n$. An explicit bijection is given in [9], and this relationship is also illustrated in Figure 11.

A matching's corresponding oscillating tableaux is created in the following manner. Essentially, we add a square to the tableau at the beginning of an arc, and we take away a square at the end of an arc. We label the ends of the arcs from left to right with $\bar{1}$, $\bar{2}$, $\bar{3}$,...,$\bar{n}$ and label the beginnings of the arcs with the corresponding number that they are connected

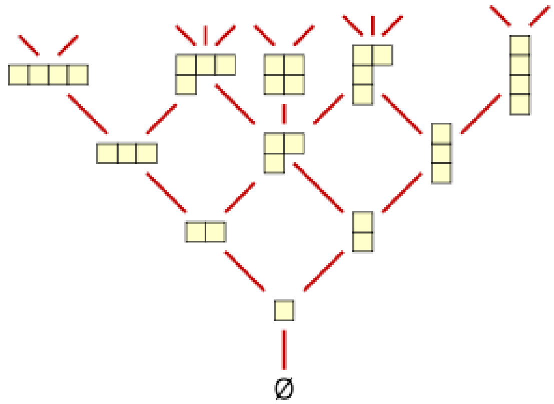


Figure 10: Young's Lattice

to, with $1, 2, 3, ..., n$. When we add a square at the beginning of an arc, we can only place numbers in order, from least to greatest from top to bottom and left to right. When we take away a square at the end of an arc, we shuffle the rest of the squares such that they remain in order from least to greatest from top to bottom and left to right. For example, an arc diagram and its corresponding tableau is shown in Figure 12.

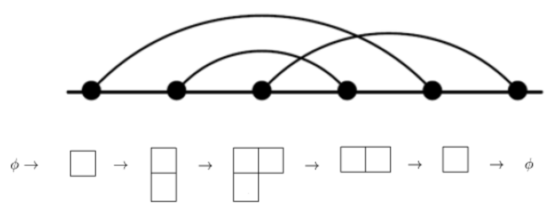


Figure 11: Matching and Corresponding Tableau

Definition 7. *The* weight *of a partition is the number of squares it contains. The weight of a tableau $T \in OT(n)$ is the sum of the weights of all its partitions. We denote it by $wt(T)$.*

For example, the size of the partitions in Figure 11 are, from left to right, $0, 1, 2, 3, 2, 1, 0$, and $wt(T) = 9$. We can then relate weights of oscillating tableaux to the Dyck path area of their corresponding matching.

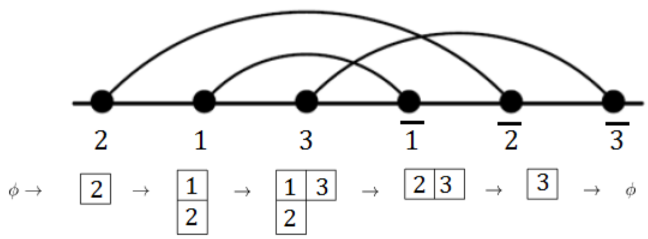


Figure 12: Positioning of squares in tableaux

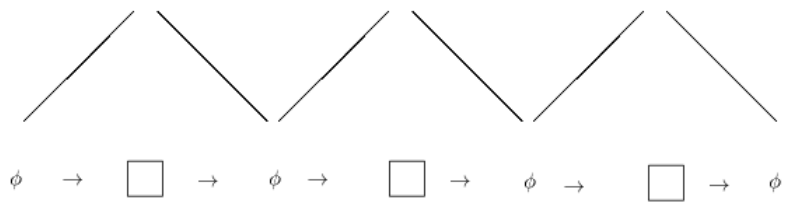


Figure 13: Weights of a Dyck path

Theorem 1. *For all $T \in OT(n)$, $wt(T) = A_{\mathcal{M}} + n$.*

Proof. We approach this proof in a manner similar to that used in the proof of Lemma 2. For Dyck paths of order n, similar to the one in Figure 13, the weights of the oscillating tableaux are always alternating 0's and 1's, starting with 0, ending with 0, and summing to n. Every time we take the opposite of an inversion or take a down-up sequence and change it into an up-down sequence, we increase $A_{\mathcal{M}}$ by 2. The total weight of the oscillating tableau also increases by 2, because a down-step is replaced by an up-step. Since we are adding a square instead of taking one away, $wt(T)$ increases by 2. The total weight is the least when it resembles Figure 13 and slowly increases as we increase the peak. Because $A_{\mathcal{M}}$ and $wt(T)$ increase at the same rate, $wt(T) = n + A_{\mathcal{M}}$

$\square$

Corollary 2. *For all $T \in OT(n)$ and matchings that correspond to the tableau, we have $\mathcal{M}_T$, $wt(T) = n^2 - 2\mathrm{al}(\mathcal{M}_T)$.*

Proof. This corollary follows directly from Theorems 1 and 2. $\square$

Thus, the average weight of all $T \in OT(n)$ is $n^2 - 2\mathrm{avgal}(\mathcal{PM}(n)) = \frac{n(2n+1)}{3}$, where $\mathrm{avgal}(\mathcal{PM}(n))$ refers to the average number of alignments across perfect matchings. We can also find the average weight of all $T \in OT(\lambda, l)$. Table 1, which was computed using Sage, suggests that a clean formula for this average may exist. We use the theory of differential posets, developed by Richard P. Stanley. Accordingly, we use the linear operators, U and D, defined in Stanley's book [10].

Definition 8. *We write $\mu \lessdot \nu$ to mean that ν covers μ in Young's lattice; in other words, we write $\mu \lessdot \nu$ when ν is obtained from μ by adding a box. Let V be the vector space of linear combinations of partitions. Define two linear operators $U, D \colon V \to V$ by*

$$U(\mu) := \sum_{\mu \lessdot \nu} \nu$$

and

$$D(\mu) := \sum_{\nu \lessdot \mu} \nu$$

.

Let λ be a partition of size k, and let $\theta := OT(\lambda, k + 2n)$. Additionally, let $[\lambda]$ denote the coefficient of λ of some expression. We find the number of ways to reach λ from $\emptyset$ by using a method similar to generating functions by looking at $[\lambda](U + D)^l \cdot \emptyset$.

We define integers $b_{ij}(l)$ by

$$(U + D)^l = \sum_{ij} b_{ij} U^i D^j$$

and

$$c_{ij}(l) := \frac{d}{dy}(x^{i-j}q_{ij}(l)),$$

where polynomials $q_{ij}(l)$ are defined by

$$(y^{xd/dx}xU + y^{xd/dx}x^{-1}D)^l = \sum_{ij} x^{i-j}q_{ij}(l)U^iD^j.$$

k \ n	0	1	2	3	4	5	n
0	0	1	10/3	7	12	55/3	$\frac{n(2n+1)}{3}$
1	1	10/3	7	12	55/3		$\frac{(n+1)(2n+3)}{3}$
2	3	20/3	35/3	18	77/3		$\frac{(n+3)(2n+3)}{3}$
3	6	11	52/3	25			$\frac{(n+2)(2n+9)}{3}$
4	10	49/3	24	33			$\frac{(n+6)(2n+5)}{3}$
5	15	68/3	95/3				$\frac{(n+3)(2n+15)}{3}$
6	21	30	121/3				$\frac{(n+9)(2n+7)}{3}$
k even							$\frac{(n+\frac{3}{2}k)(2n+k+1)}{3}$
k odd							$\frac{(n+\frac{k+1}{2})(2n+3k)}{3}$

Table 1: Average weight of $T \in OT(\lambda, 2n+k)$

Ultimately, we find a closed-form formula for the average weight of all tableaux. Before we can prove the formula, we must first prove the following three lemmas.

Lemma 5. *For all $T \in OT(n)$,* $\text{avgwt}(T) = \frac{c_{k0}(2n+k)}{b_{k0}(2n+k)}$, *where* $\text{avgwt}(T)$ *is the average weight of all $T \in OT(n)$.*

Proof. We note that

$$\text{avgwt}(T) = \frac{\sum_{T\in\theta} wt(T)}{|\theta|}.$$

We already have $|\theta| = [\lambda](U + D)^{2n+k} \cdot \emptyset$ from [10]. From $(U + D)^l = \sum b_ij(l)U^iD^j$, we can simplify that expression to obtain

$$
\begin{aligned}
[\lambda](U+D)^{2n+k} \cdot \emptyset &= \sum_{i,j} b_{ij}(2n+k)[\lambda]U^i D^j \cdot \emptyset \\
&= b_{k0}(2n+k)[\lambda]U^k \cdot \emptyset.
\end{aligned}
$$

The last line follows because any positive power of D applied to $\emptyset$ returns 0.

We claim that

$$\sum_{T \in \emptyset} wt(T) = [\lambda]\frac{d}{dy}(y^{xd/dx}xU + y^{xd/dx}x^{-1}D)^{k+2n} \mid_{x=1,y=1} \cdot \emptyset,$$

where x keeps track of the size of the partition, and y keeps track of the weight of the path. This expression can be simplified using $(y^{xd/dx}xU + y^{xd/dx}x^{-1}D)^l = \sum_{i,j} x^{i-j}q_{ij}(l)U^i D^j$ and $c_{ij} = \frac{d}{dy}(x^{i-j}q_{ij}(l)) \mid_{x=1,y=1}$:

$$
\begin{aligned}
\sum_{T \in \emptyset} wt(T) &= [\lambda]\frac{d}{dy}(y^{xd/dx}xU + y^{xd/dx}x^{-1}D)^{k+2n} \mid_{x=1,y=1} \cdot \emptyset \\
&= [\lambda]\frac{d}{dy}(\sum_{i,j} x^{i-j}q_{ij}(l)U^i D^j) \mid_{x=1,y=1} \cdot \emptyset \\
&= \sum_{i,j} \frac{d}{dy}(x^{i-j}q_{ij}(l)) \mid_{x=1,y=1} U^i D^j[\lambda] \cdot \emptyset \\
&= \sum_{i,j} c_{ij}(2n+k)[\lambda]U^i D^j \cdot \emptyset \\
&= c_{k0}(2n+k)[\lambda]U^k \cdot \emptyset.
\end{aligned}
$$

Again, the last line follows from the fact that any positive power of D applied to $\emptyset$ returns 0. Thus, $\text{avgwt}(T) = \frac{\sum_{T \in \theta} wt(T)}{\theta} = \frac{c_{k0}(2n+k)}{b_{k0}(2n+k)}$. $\square$

Lemma 6. *For all polynomials $q_{ij}(l)$,*

$$q_{ij}(l+1) = y^{i-j}(q_{i-1j}(l) + q_{ij-1}(l) + q_{i+1j}(l)(i+1)).$$

Proof.

$$\begin{aligned}\sum_{i,j} x^{i-j} q_{ij}(l+1) U^i D^j &= (y^{xd/dx} xU + y^{xd/dx} x^{-1} D)^{l+1} \\ &= (y^{xd/dx} xU + y^{xd/dx} x^{-1} D)(y^{xd/dx} xU + y^{xd/dx} x^{-1} D)^{l} \\ &= (y^{xd/dx} xU + y^{xd/dx} x^{-1} D) \sum_{i,j} x^{i-j} q_{ij}(l) U^i D^j \\ &= \sum_{i,j} y^{xd/dx} x^{i-j+1} q_{ij}(l) U^{i+1} D^j + y^{xd/dx} x^{i-j-1} q_{ij}(l) D U^i D^j.\end{aligned}$$

Using $y^{xd/dx} x^n = x^n y^n$ and the fact from [10] that $DU^i = U^i D + iU^{i-1}$, we obtain

$$\begin{aligned}&\sum_{i,j} y^{xd/dx} x^{i-j+1} q_{ij}(l) U^{i+1} D^j + y^{xd/dx} x^{i-j-1} q_{ij}(l) D U^i D^j \\ =&\sum_{i,j} x^{i-j+1} y^{i-j+1} q_{ij}(l) U^{i+1} D^j + \sum_{i,j} x^{i-j-1} y^{i-j-1} q_{ij}(l) U^i D^{j+1} + \sum_{i,j} x^{i-j-1} y^{i-j-1} q_{ij}(l) i U^{i-1} D^j \\ =&\sum_{i'-1,j} x^{i'-j} y^{i'-j} q_{i-1j}(l) U^{i'} D^j + \sum_{i,j'-1} x^{i-j'} y^{i-j'} q_{ij'-1}(l) U^i D^{j'} + \sum_{i'+1,j} x^{i'-j} y^{i'-j} q_{i'+1j}(l)(i'+1) U^{i'} D^j.\end{aligned}$$

Since the coefficients of $U^i D^j$ must be equal on both sides, we have

$$q_{ij}(l+1) = y^{i-j}(q_{i-1j}(l) + q_{ij-1}(l) + q_{i+1j}(l)(i+1)).$$

□

Lemma 7. *For all constants $c_{i0}(l)$ and $b_{i0}(l)$,*

$$c_{i0}(l+1) = ib_{i-10}(l) + ib_{i+10}(l)(i+1) + c_{i-10}(l) + c_{i+10}(l)(i+1).$$

Proof. From

$$q_{ij}(l+1) = y^{i-j}(q_{i-1j}(l) + q_{ij-1}(l) + q_{i+1j}(l)(i+1)),$$

we can set $j = 0$. The middle term cancels out because $q_{i0} = 0$ and we obtain

$$q_{i0}(l+1) = y^{i}(q_{i-10}(l) + q_{ij-1}(l) + q_{i+10}(l)(i+1)).$$

Taking the derivative of both sides, the equation becomes

$$\begin{aligned} c_{i0}(l+1) &= iy^{i-1}q_{i-10}(l) + iy^{i-1}q_{i+10}(l)(i+1) + y^{i}c_{i-10}(l) + y^{i}c_{i+10}(l)(i+1) \mid_{y=1} \\ &= ib_{i-10}(l) + ib_{i+10}(l)(i+1) + c_{i-10}(l) + c_{i+10}(l)(i+1) \end{aligned}$$

as desired. □

Now that we have the necessary equations, we can proceed to prove the closed-form formula for the average weight of all tableaux.

Theorem 3. *For all* $T \in OT(n)$*,* $\text{avgwt}(T) = \frac{1}{6}(3k^2 + 8kn + 3k + 4n^2 + 2n)$ *over all* $\mathcal{OT}(\lambda, k+2n)$.

Proof. First, we substitute in $i = k$ and $l = 2n + k$ into Lemma 6, where $|\lambda| = k$ and n is some positive integer used to keep the parity of i and l the same. We rewrite this expression in terms of $b_{ij}(l)$ using $\text{avgwt}(T) = \frac{c_{k0}(2n+k)}{b_{k0}(2n+k)}$ from Lemma 4. Note that $\frac{1}{6}(3k^2+8kn+4n^2+2n)$ is also equal to $(n + \frac{3k}{2})(2n+k+1)3$ and to $(n + \frac{k+1}{2})(2n+3k)3$.

On the right side (RHS), we obtain

$$\begin{aligned} c_{k0}(2n+k) &= b_{k0}(2n+k)\frac{(n+\frac{3}{2}k)(2n+k+1)}{3} \\ &= \frac{1}{6}(3k^2 + 8kn + 3k + 4n^2 + 2n)b_{k0}(2n+k). \end{aligned}$$

On the left side (LHS) obtain

$$
\begin{aligned}
&kb_{k-10}(2n+k-1)+kb_{k+10}(2n+k-1)(k+1)+b_{k-10}(2n+k-1)\frac{(n+\frac{k}{2})(2n+3k-3)}{3}\\
&+b_{k+10}(2n+k-1)(k+1)\frac{(n-1+\frac{k+2}{2})(2n+3k+1)}{3}.
\end{aligned}
$$

We group like terms and obtain

$$
\begin{aligned}
&b_{k-10}(2n-k-1)(k+\frac{(n+\frac{k}{2})(2n+3k-3)}{3}\\
&\quad+b_{k+10}(2n+k-1)(k+1)(k+\frac{(n-1+\frac{k+2}{2})(2n+3k+1)}{3}\\
&=\frac{1}{6}(3k^2+8kn+3k+4n^2-6n)b_{k-10}(2n+k-1)\\
&\quad+\frac{1}{6}(3k^2+8kn+7k+4n^2+2n)(k+1)b_{k+10}(2n+k-1)\\
&=(\frac{1}{6}(3k^2+8kn+3k+4n^2+2n)-\frac{8}{6}n)b_{k-10}(2n+k-1)\\
&\quad+(\frac{1}{6}(3k^2+8kn+3k+4n^2+2n)+\frac{2}{3}k)(k+1)b_{k+10}(2n+k-1)\\
&=\frac{1}{6}(3k^2+8kn+3k+4n^2+2n)(b_{k-10}(2n-k-1)\\
&\quad+(k+1)b_{k+10}(2n+k-1))-(\frac{4}{3}nb_{k-10}(2n-k-1)\\
&\quad-\frac{2}{3}k(k+1)b_{k+10}(2n+k-1)).
\end{aligned}
$$

We also know that $b_{k-10}(2n-k-1)+(k+1)b_{k+10}(2n+k-1)=b_{k0}(2n+k)$ from [10], so the equation simplifies even further to

$$
\begin{aligned}
&\frac{1}{6}(3k^2+8kn+3k+4n^2+2n)(b_{k0}(2n+k))\\
&\quad-(\frac{4}{3}nb_{k-10}(2n-k-1)-\frac{2}{3}k(k+1)b_{k+10}(2n+k-1))\\
&=c_{k0}(2n+k)-(\frac{4}{3}nb_{k-10}(2n-k-1)\\
&\quad-\frac{2}{3}k(k+1)b_{k+10}(2n+k-1)).
\end{aligned}
$$

Moreover, from [10], $b_{i0}(l) = \binom{l}{i}(l-i-1)!!$. Applying that to the last term of our expression, we get

$$\begin{aligned}
&\frac{4}{3}nb_{k-10}(2n-k-1) - \frac{2}{3}k(k+1)b_{k+10}(2n+k-1) \\
&= \frac{2}{3}(k(k+1)\binom{2n+k-1}{k+1}(2n-3)!! - 2n\binom{2n+k-1}{k-1}(2n-1)!!) \\
&= \frac{2}{3}(2n-3)!!(k(k+1)\binom{2n+k-1}{k+1} - 2n\binom{2n+k-1}{k-1}(2n-1)).
\end{aligned}$$

After some computer-assisted calculation, we find that it indeed equals 0. Since $c_{00}(0) = 0$, which agrees with our formula for the average, we are done by induction $\square$

The existence of this theorem suggests that homomesy may exist for oscillating tableaux of arbitrary shape, because of the small denominator of 3.

7 Conclusion and Open Problem

We have shown the possibility of the existence of a function that exhibits homomesy in perfect matchings and oscillating tableaux. In addition, we introduced an inductive approach to constructing a homomesic group action. Despite the existence of counterexamples, this approach shows potential for further refinement. For oscillating tableaux, we obtained the average that will be conserved if the function exists.

In the future, we hope to find a more detailed algorithm to find triples of alignment numbers. We conjecture that the homomesic group action can be found for oscillating tableaux.

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