Arrangements and Amounts of Equal Minors in Totally Positive Matrices

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Abstract

Understanding the positive Grassmannian has been a highly significant question in mathematics for several decades. We discuss arrangements of equal minors in the totally positive Grassmannian. It was previously shown that arrangements of equals minors of largest value correspond to the simplices in Sturmfels triangulation. Here we discuss arrangements of equals minors of second largest value, and show that they correspond to the facets of Sturmfels triangulation. We then define the notion of cubical distance and obtain a general conjecture that provides a correspondence between minors of m-largest value and maximal simplices of certain cubical distance in Sturmfels triangulation. We prove this conjecture for the case $m = 3$, and also for certain cases in $Gr^{+}(2, n)$.

Summary

We study relationships between minors, or determinants of square submatrices, of totally positive matrices. Recent studies in this topic have found interesting properties and bounds on maximal and minimal minors in matrices. In this paper, we expand these results to other ranks of minors, such as second largest, third largest, etc. We find some interesting properties and surprising relationships between minors in totally positive matrices, which could prove to be highly applicable in future advances in computer science, mathematics, and physics. In particular, understanding these relationships helps explain underlying processes in many physical processes, such as scattering of light.

1 Introduction

We study the relations between minors of totally positive matrices and triangulations of the hypersimplex. This study is strongly tied to various combinatorial objects such as the *positive Grassmannian* [1], alcoved polytopes and Sturmfels' triangulation [2].

Totally positive matrices have been extensively studied ever since the notion of total positivity was first introduced by Schoenberg [3] and Gantmacher and Krein [4] in the 1930s, and they have a variety of applications in computer science and mathematics [5, 6]. An m-by-n matrix is called totally positive (TP), if every minor of it is positive. Similarly, a matrix is called totally non-negative (TN) if every minor of it is non-negative. Therefore, studying the structure of the minors is key for understanding the properties of TP matrices. For example, it is well known that minors of TP matrices possess an intriguing combinatorial structure that corresponds to boundary measurements in planar acyclic networks [1]. Recently, the number and positioning of equal minors in TP matrices was studied. In [7], it was shown that the number of equal entries in a TP $n \times n$ matrix is $O(n^{4/3})$. The authors also discussed positioning of equal entries in such a matrix and obtained relations to the Bruhat order of permutations. In [8] it was shown, using incidences between points and hyperplanes, that the maximal number of equal $k \times k$ minors in a $k \times n$ TP matrix is $O(n^{k-\frac{k}{k+1}})$.

Inequalities between products of two minors in TP matrices have been widely studied as well [9, 10], and have close ties with Temperley-Lieb Immanants. Recently there has been also a study of products of three minors in such matrices [11], that related such products with dimers. Despite all of the above, not much is known about the inequalities between the minors themselves. What is the full structure of all the possible inequalities between minors in TP matrices? The only part of this problem that has been solved discusses the structure of the minors with largest value and smallest value [12], while the rest of the problem remains open. It was shown in [12] that arrangements of minors of largest value are in bijection with *sorted sets,* which appeared in the context of *alcoved polytopes* and Gröbner bases. Maximal arrangements of such minors correspond to maximal simplices of the Sturmfels triangulation of the hypersimplex, and their number equals the *Eulerian number*. It was also conjectured, and proved in various cases, that arrangements of equal minors of smallest value are exactly the *weakly separated sets*, which were originally introduced by Leclerc and Zelevinsky. They are intimately related to the *positive Grassmannian* and the associated *cluster algebra.*

In this paper we delve into the structure of r-th largest minors (for some positive integer r), and present a surprising relation between this structure and the Sturmfels triangulation of the hypersimplex. As we mentioned, it was shown in [12] that the case $r = 1$ corresponds to the maximal simplices in such a triangulation. In section 3 we show that the case $r = 2$ corresponds the common facets of such simplices, and in section 4 we show that the case $r = 3$ corresponds to a certain distance between maximal simplices. Finally,in section 5 we also form a conjecture for a general r, and prove some additional cases of this conjecture.

2 Definitions and known results

2.1 Preliminary definitions

For $n \geq k \geq 0$, let the *Grassmannian* $Gr(k, n)$ (over $\mathbb{R}$) be the manifold of k-dimensional subspaces $V \subset \Re^n$. It can be identified with the space of real $k \times n$ matrices of rank k modulo row operations. (The rows of a matrix span a k-dimensional subspace in $\mathbb{R}^n$.). Here we assume that the subspace V associated with a $k \times n$-matrix A is spanned by the row vectors of A. For such a matrix A and a k-element subset $I \subset [n] := \{1, 2, 3 \ldots, n\}$, we denote by A_I the $k \times k$-submatrix of A in the column set I, and let $\Delta_I(A) := \det(A_I)$. The coordinates Δ_I form projective coordinates on the Grassmannian, called the *Plücker coordinates.*

In [1], the *totally positive (totally nonnegative) Grassmannian* $Gr^+(k, n)$ ($Gr^{\geq}(k, n)$) was defined to be the subset of $Gr(k,n)$ whose elements can be represented by $k \times n$ matrices A with strictly positive (nonnegative) Plücker coordinates: $\Delta_I > 0$ for all I.

For a $k \times m$ matrix B, and the subsets $I = \{i_1, \ldots, i_r\} \subset [k]$ and $J = \{j_1, \ldots, j_r\} \subset [m]$, we denote by $\Delta_{I,J}(B)$ the minor in the row set I and column set J. The space of totally positive $k \times m$ matrices $A = (a_{ij})$ can be embedded into the totally positive Grassmannian $Gr^+(k,n)$ with $n = m + k$ via an embedding *phi* described in [1]. This construction provides a bijection between totally positive matrices and the totally positive positive Grassmannian. Moreover, under this map, all minors (of all sizes) of the $k \times m$-matrix A are equal to the *maximal* $k \times k$-minors of the $k \times n$ matrix $\phi(A)$. In particular, we have

$$\Delta_{I,J}(A) = \Delta_{([k]\setminus\{k+1-i_r,\ldots,k+1-i_1\})\cup\{j_1+k,\ldots,j_r+k\}}(\phi(A)).$$

Thus, from now on, instead of discussing inequalities between minors, we will discuss inequalities between maximal minors (which are just the plücker coordinates) of the totally positive Grassmannian. We see that this point of view reveals symmetries which are hidden on the level of matrices. One such symmetry that is useful for us is the cyclic symmetry. Let $[v_1, \ldots, v_n]$ denote a point in $Gr(k,n)$ given by n column vectors $v_1, \ldots, v_n \in \mathbb{R}^k$. Then the map

$$[v_1, \ldots, v_n] \mapsto [(-1)^{k-1}\, v_n, v_1, v_2, \ldots, v_{n-1}]$$

preserves the totally positive Grassmannian $Gr^+(k,n)$. This defines the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on the totally positive Grassmannian $Gr^+(k,n)$.

One of the main purposes of the paper is to present a novel connection between inequalities on minors of the totally positive Grassmannian and triangulations of the hypersimplex. For fixed integers $0 < k < n$, let us denote by $\binom{[n]}{k}$ the collection of k-element subsets of $[n]$. With each $I \in \binom{[n]}{k}$ we associate the $0,1$-vector $\epsilon_I = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ such that $\epsilon_i = 1$ iff $i \in I$, and otherwise $\epsilon_i = 0$. The *hypersimplex* $\Delta_{k,n} \in \Re^n$ is the convex polytope defined as the convex hull of the points $\{\epsilon_I\}_{I\in\binom{[n]}{k}}$ (all those points are actually vertices of the hypersimplex). $\Delta_{k,n}$ is an $(n-1)$-dimensional polytope, which is clear from the following alternative definition: $\Delta_{k,n} = \{(x_1,\ldots,x_n)|0 \leq x_1,\ldots,x_n \leq 1; x_1+x_2+\ldots+x_n = k\}$. It was shown in [13, 14] that the normalized volume of $\Delta_{k,n}$ equals the Eulerian number $A(n-1,k-1)$, that is, the number of permutations w of size $n-1$ with exactly $k-1$ descents. In [2] four triangulations of the hypersimplex into $A(n-1,k-1)$ unit simplices are presented: Stanleys triangulation [13], Sturmfels triangulation [14], Alcove triangulation and Circuit triangulation. It was shown in [2] that these four triangulations coincide. We describe here the Sturmfels triangulation, which will be used in the next sections.

Sturmfels' triangulation. This triangulation naturally appears in the context of Gröbner bases. For a multiset S of elements from $[n]$, we define $Sort(S)$ the non-decreasing sequence obtained by ordering the elements of S. Let $I,J \subset \binom{[n]}{k}$ and let $Sort(I\cup J) = (a_1,a_2,\ldots,a_{2k})$. Then we denote by $sort_1(I,J), sort_2(I,J)$ the following subsets in $\binom{[n]}{k}$: $sort_1(I,J) := \{a_1,a_3,\ldots,a_{2k-1}\}$ and $sort_2(I,J) := \{a_2,a_4,\ldots,a_{2k}\}$. A pair $\{I,J\}$ is called *sorted* if $sort_1(I,J) = I$ and $sort_2(I,J) = J$, or $sort_1(I,J) = J$ and $sort_2(I,J) = I$. A collection $W = \{I_1,I_2,\ldots,I_r\}$ of elements in $\binom{[n]}{k}$ is called sorted if I_i, I_j are sorted, for any pair $1 \leq i < j \leq n$. For such a collection W, we denote by $\bigtriangledown_W$ the $(r-1)$-dimensional simplex with the vertices $\epsilon_{I_1},\ldots,\epsilon_{I_r}$.

Theorem 2.1. *[14] The collection of simplices $\bigtriangledown_W$ where W varies over all sorted collections of k-element subsets in $[n]$, is a simplicial complex that forms a triangulation of the hypersimplex $\Delta_{k,n}$.*

From Theorem 2.1, it follows that the maximal by inclusion sorted collections correspond to the maximal simplices in the triangulation are all of size n. We define the dual graph $\Gamma_{(k,n)}$ of the Sturmfels triangulation to be the graph whose vertices are the maximal simplices, and two maximal simplices are adjacent by an edge if they share a common facet. The maximal degree of a vertex in $\Gamma_{(k,n)}$ is n ([2]). An example of $\Gamma_{(k,n)}$ is shown in Figure 9 in the Appendix.

Another triangulation mentioned above is the Circuit triangulation. We describe a correspondence between maximal sorted collections and circuits of certain kind of graphs, that appears as part of the triangulation. For a full description of the triangulation, see [2]. We define $G_{k,n}$ to be the directed graph whose vertices are $\{\epsilon_I\}_{I\in\binom{[n]}{k}}$, and two vertices $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ and ϵ' are connected by an edge oriented from ϵ to ϵ' if there exists some $i \in [n]$ such that $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$ and the vector ϵ' is obtained from ϵ by switching $\epsilon_i, \epsilon_{i+1}$ (and leaving all the other coordinates unchanged, so the 1 is "shifted" one place to the right). When considering $i \in [n]$ we refer to i as $i \mod n$, and thus if $i = n$, we have $i + 1 = 1$. A circuit in $G_{k,n}$ of minimal possible length must be of length n, and it is given by a sequence of shifts of 1's so that the first 1 in ϵ moves to the position of the second 1, the second 1 moves to the position of the third 1, and so on. Finally, the last 1 cyclically moves to the position of the first 1. Figure 1 presents an example of a minimal circuit in $G_{3,6}$. Let $\{\epsilon_{J_i}\}_{i=1}^{n}$ be the set of vertices in a minimal circuit (that is, circuit of length n). Then by [2], the collection $W = \{J_1, \ldots, J_n\}$ is a maximal sorted collection. The other direction also holds - every maximal sorted collection can be realized via a minimal circuit in the graph $G_{k,n}$.

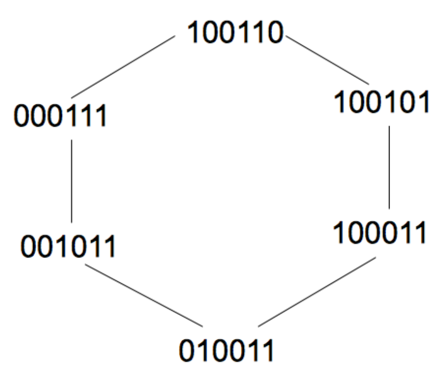

Figure 1: Example of a minimal circuit in $G_{3,6}$.

The following definition and problem were presented in [12].

Definition 2.2. Let $U = (U_0, U_1, \ldots, U_l)$ be an ordered set-partition of the set $\binom{[n]}{k}$ of all k-element subsets in $[n]$. Let us subdivide the nonnegative Grassmannian $Gr^{\geq}(k,n)$ into the strata S_U labelled by such ordered set partitions U and given by the conditions:

1. $\Delta_I = 0$ for $I \in U_0$,

2. $\Delta_I = \Delta_J$ if $I, J \in U_i$,

3. $\Delta_I < \Delta_J$ if $I \in U_i$ and $J \in U_j$ with $i < j$.

An *arrangement of minors* is an ordered set-partition U such that the stratum S_U is not empty.

Problem 2.3. *Describe combinatorially all possible arrangements of minors in $Gr^{\geq}(k,n)$. Investigate the geometric and combinatorial structure of the above stratification.*

We discuss the case of the totally positive Grassmannian $Gr^{+}(k,n)$, that is, we assume that $U_0 = \emptyset$. The combinatorial description of the sets U_1 and U_l (assuming $U_0 = \emptyset$) was given in [12], and is summarized in the following theorem and conjecture:

Theorem 2.4. *[12] A subset of $\binom{[n]}{k}$ is an arrangement of largest minors in $Gr^{+}(k,n)$ if and only if it is a sorted subset. Maximal arrangements of largest minors contain exactly n minors. The number of maximal arrangements of largest minors in $Gr^{+}(k,n)$ equals the Eulerian number $A(n-1,k-1)$.*

Conjecture 2.5. *[12] A subset of $\binom{[n]}{k}$ is an arrangement of smallest minors in $Gr^{+}(k,n)$ if and only if it is a weakly separated subset.*

The forward direction of Conjecture 2.5 was proven, and the other direction was proven for the cases $k = 1, 2, 3, n-1, n-2, n-3$ ([12]).

Our purpose is to give a partial combinatorial description for U_j for $j \notin \{1, l\}$. We focus on the case in which U_l is of maximal size (that is, n) and present a description of the sets

U_{l-1}, U_{l-2}. For particular values of k we obtain a description for additional values of j. We then present a conjecture that generalizes our claims for any j. Finally, we discuss the case in which U_l is not maximal.

3 The second largest minors

In this section, we concentrate on the structure of U_{l-1}, and show that it corresponds to the edges of the graph $\Gamma_{(k,n)}$.

Claim 3.1. *Let $\mathcal{I}$ and J be adjacent in $\Gamma_{(k,n)}$. Then the circuits $C_\mathcal{I}$ and C_J are different in exactly one vertex. Furthermore, C_J can be obtained from $C_\mathcal{I}$ by removing a vertex and the two edges next to it and adding a new vertex and a pair of edges, as shown by the example in Figure 2.*

Definition 3.2. Let $C_\mathcal{I}$ be a circuit. We say that C_J is obtained from $C_\mathcal{I}$ by a *detour* if $\mathcal{I}$ and J are adjacent in $\Gamma_{(k,n)}$. The detour P is the path formed by the two new edges together with the new vertex. The vertex that appears in C_J but not in $C_\mathcal{I}$ is called v_p. An example of a detour is provided in Figure 2.

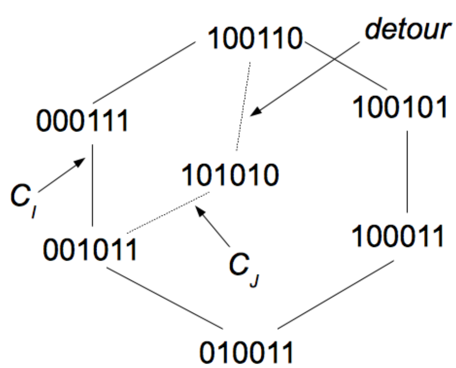

Figure 2: An example of a new path formed by a detour from the original.

We use the following theorem due to Skandera [9] extensively to prove our result:

Theorem 3.3. *Let $I, J \in \binom{[n]}{k}$ be a pair which is not sorted, and let $sort_1(I,J), sort_2(I,J)$ be the sorting of the pair I, J. Then we have the strict inequality $\Delta_{sort_1(I,J)}\ \Delta_{sort_2(I,J)} > \Delta_I \Delta_J$ for points of the positive Grassmannian $Gr^+(k,n)$.*

Lemma 3.4. *If $a \in \binom{[n]}{k}$ is sorted with $c \in \binom{[n]}{k}$, while $b \in \binom{[n]}{k}$ is not sorted with a, and b is not sorted with c, then the set $\{a, c, sort_1(b,c), sort_2(b,c), sort_1(a,b), sort_2(a,b)\}$ can not be sorted.*

Lemma 3.5. *If $a \in \binom{[n]}{k}$ is sorted with $c \in \binom{[n]}{k}$, while $b \in \binom{[n]}{k}$ is not sorted with a, and b is not sorted with c, then the set $\{a, c, sort_1(b,c), sort_2(b,c), sort_1(a,b), sort_2(a,b)\}$ can not be sorted.*

Proof. We consider the vectors ϵ_a, ϵ_b, ϵ_c. Let $a_{ij} = \sum_{t=i}^{j} (\epsilon_a)_t$, let $b_{ij} = \sum_{t=i}^{j} (\epsilon_b)_t$, and let $c_{ij} = \sum_{t=i}^{j} (\epsilon_c)_t$. For example, if $a = \{1,3,5,7,8\}$, then $\epsilon_a = 1010101100$, and $a_{37} = 3$. Define $\alpha_{ij} = a_{ij} - b_{ij}$, $\beta_{ij} = c_{ij} - b_{ij}$ for all $1 \le i \le j \le n$. Therefore without loss of generality,

$sort_1(b,c)_{i,j} = \lceil b_{i,j} + \frac{\beta_{i,j}}{2} \rceil$, $sort_2(b,c)_{i,j} = \lfloor b_{i,j} + \frac{\beta_{i,j}}{2} \rfloor$, $sort_1(b,a)_{i,j} = \lceil b_{i,j} + \frac{\alpha_{i,j}}{2} \rceil$, and $sort_2(b,a)_{i,j} = \lfloor b_{i,j} + \frac{\alpha_{i,j}}{2} \rfloor$ (there are four cases in total, all of which are handled in the same way. Therefore, we indeed can assume that this is without loss of generality). Note that $sort_1(b,c)_{i,j}$ and $sort_2(b,c)_{i,j}$, and $sort_1(b,a)_{i,j}$ and $sort_2(b,a)_{i,j}$, can not differ by more than 1. In addition, since a and c are sorted, a_{ij} and c_{ij} can not differ by more than 1 ([2]). Thus the following properties hold:

1. $\alpha_{i,j} \le 2$.
2. $\beta_{i,j} \le 2$.
3. If $|\alpha_{i,j}| = 2$, then $\alpha_{i,j} = \beta_{i,j}$.
4. If $|\beta_{i,j}| = 2$, then $\alpha_{i,j} = \beta_{i,j}$.
5. If $|\alpha_{i,j}| = 1$, then $\alpha_{i,j} = \beta_{i,j}$ or $\beta_{i,j} = 0$.

6. If $|\beta_{i,j}| = 1$, then $\alpha_{i,j} = \beta_{i,j}$ or $\alpha_{i,j} = 0$.

We consider some pair $\{i, j\}$ for which $a_{i,j} = b_{i,j} + 2$. In addition, we impose two extra rules. Namely, that $i = 1$, which can be achieved by the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on the totally positive Grassmannian, and that j is maximal, that is, there is no $k > j$, in which $a_{i,k} = b_{i,k} + 2$.

We consider three cases:

1. $k = m$: Consider the next element u. It must be in at least one of a, b, or c, but not all 3 (if it is in all 3, we do not consider it because it does not in any way affect the sortedness of the sets). If u is in b, we contradict rule 2 for the pair $[j+1, u]$. If u is in a and not in c, or vice-versa, we contradict rules 3 and 4 in the interval $[j+1, u]$. Therefore, u must be in both a and c.

2. $k = m - 1$: Consider the next element u. If u is in b, then we violate rule 2 in the interval $[j+1, u]$. If u is in a and not in c, then we violate rules 5 and 6 for the interval $[j+4, m]$. Thus, u must be in both a and c.

3. $k = m + 1$: If the next element u is in b, then we violate rule 2. If it is in a but not in c, we violate rules 5 and 6 for the interval $[j+4, m]$. Thus, u must be in both a and c.

We can do an analogous argument considering the sequence between i and j, from which it follows that the set $\{a, c, sort_1(b, c), sort_2(b, c), sort_1(a, b), sort_2(a, b)\}$ is sorted only if $a = c$, and we reach a contradiction. $\square$

Thus we obtain the following theorem:

Theorem 3.6. *Let $A \in Gr^+(k, n)$ for which U_L is of maximal size, that is, $|U_L| = n$. Let $I \in \binom{[n]}{k}$ such that, for any maximal sorted collection $\mathcal{I}$ that contains I, $\mathcal{I}$ is not adjacent to U_L in $\Gamma_{(k,n)}$. Then $I \notin U_{L-1}$ (that is, Δ_I cannot be the second largest minor).*

Since the maximal degree of a vertex in $\Gamma_{(k,n)}$ is at most n ([2]), we get that if $|U_L| = n$, then $|U_{L-1}| \leq n$, therefore the number of second largest minors can not exceed n. For the case in which U_L is not maximal, we obtain the following:

Theorem 3.7. *Let $A \in Gr^+(k,n)$, and let $I \in \binom{[n]}{k}$. If there exist J_1, $J_2 \in U_L$ such that I is not sorted with J_1 and I is not sorted with J_2, then $I \notin U_{L-1}$.*

Claim 3.8. *Let $W = \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4\}$ be four distinct maximal sorted collections. We say that W forms a square in $\Gamma_{(k,n)}$ if both $\mathcal{I}_2$, $\mathcal{I}_3$ are adjacent to $\mathcal{I}_1$ and to $\mathcal{I}_4$. In general, a d-dimensional cube in $\Gamma_{(k,n)}$ is formed by starting from a certain circuit $C_{\mathcal{I}_1}$, identifying a collection of d detours $\{P_i\}$ such that for any pair $\{v_{p_i}, v_{p_j}\}$, the distance between v_{p_i} and v_{p_j} is at least two, and then constructing the additional 2^{d-1} circuits $\{\mathcal{I}_j\}_{j=2}^{2^d}$ formed by all possible combinations of the detours. Then the set $\{\mathcal{I}_j\}_{j=1}^{2^d}$ forms a d-dimensional cube in $\Gamma_{(k,n)}$.*

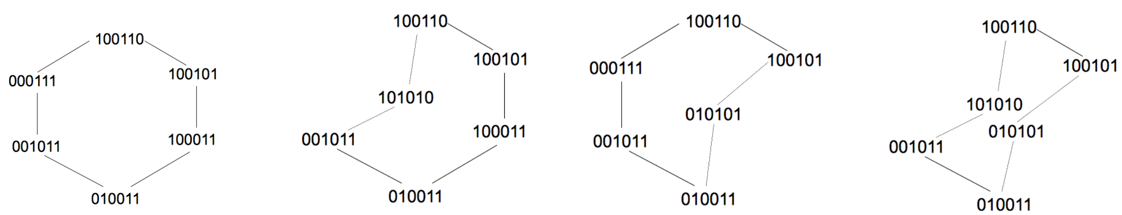

Figure 3: An example of a square in $\Gamma_{(k,n)}$: From left to right, $C_{\mathcal{I}_1}$, $C_{\mathcal{I}_2}$, $C_{\mathcal{I}_3}$, and $C_{\mathcal{I}_4}$. We have: $\mathcal{I}_1 = \{145, 146, 156, 256, 356, 456\}$, $\mathcal{I}_2 = \{145, 146, 156, 256, 356, 135\}$, $\mathcal{I}_3 = \{145, 146, 246, 256, 356, 456\}$, and $\mathcal{I}_4 = \{145, 146, 246, 256, 356, 135\}$.

Definition 3.9. Two vertices u, $v \in \Gamma_{(k,n)}$ are of *cubical distance*, denoted *cube_d*, a if one can arrive from u to v by moving along a cubes in $\Gamma_{(k,n)}$, and a is minimal with respect to this property.

An example of cubical distance is given in Figure 11.

In the context of this new definition and claim, we can rephrase the theorem that follows from Lemma 3.5 as follows:

Theorem 3.10. *Let $\mathcal{I}$ and J be of cubical distance one, and let $I \in J$ such that $I \notin \mathcal{I}$. Then there exists $\mathcal{I}_1$ that is adjacent to $\mathcal{I}$ in $\Gamma_{(k,n)}$ such that $I \in \mathcal{I}_1$.*

4 The third largest minors

In this section, we concentrate on the structure of U_{l-2}, and show that it corresponds to distances along the graph $\Gamma_{(k,n)}$.

Theorem 4.1. *Let $A \in Gr^{+}(k,n)$ for which U_L is of maximal size, that is, $|U_L| = n$. Let $I \in \binom{[n]}{k}$ such that, for any maximal sorted collection $\mathcal{I}$ that contains I, $\mathcal{I}$ is of cubical distance at least 3. Then $I \notin U_{L-1}, U_{L-2}$ (that is, Δ_I cannot be the second or third largest minor).*

Proof. There are six cases for us to consider, all shown in Figure 4. We consider all of the following six cases:

1. From the Skandera inequality we have $\Delta_{a_6} \times \Delta_{a_4} < \Delta_{a_5} \times \Delta_{a_7}$, from which it follows that $\Delta_{a_6} < \Delta_{a_7}$. Similarly, we get that $\Delta_{a_7} < \Delta_{a_8}$ and $\Delta_{a_8} < \Delta_{a_1}$. Therefore, Δ_{a_6} can not be the third largest minor.

2. The same way as in case 1, we get that $\Delta_{a_9} < \Delta_{a_8}$ and $\Delta_{a_8} < \Delta_{a_7}$. From the Skandera inequality we obtain $\Delta_{a_{10}} \times \Delta_{a_4} < \Delta_{a_9} \times \Delta_{a_{11}}$ since we know that $\Delta_{a_4} > \Delta_{a_{11}}$, we get $\Delta_{a_{10}} < \Delta_{a_9} < \Delta_{a_8} < \Delta_{a_7}$, and thus $\Delta_{a_{10}}$ can not be the third largest minor.

3. Similarly to case 2, we have $\Delta_{a_8} < \Delta_{a_9} < \Delta_{a_{10}} < \Delta_{a_1}$ and thus Δ_{a_8} can not be third largest.

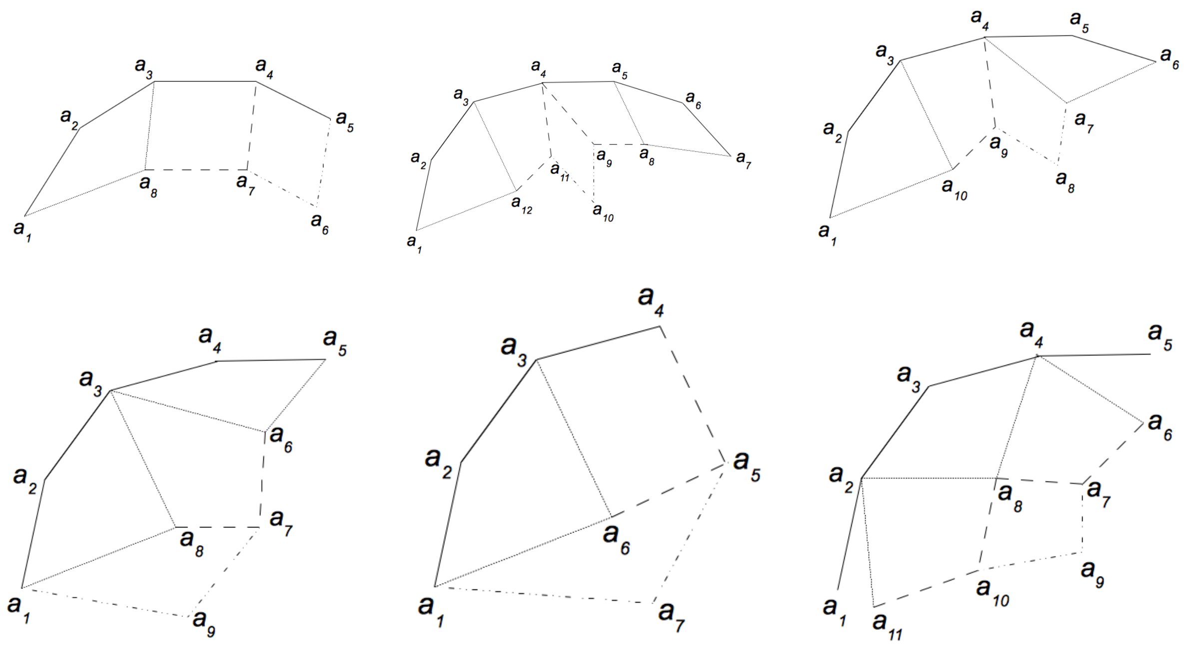

Figure 4: From left to right, and then top to bottom, case 1 to case 6. Fine dotted line = 1^{st} detour, dashed line = 2^{nd} detour, two dots two dashes = 3^{rd} detour.

4. Here we must consider sorting of sets. In fact, there are six cases to consider but here we only show one of them as the others are very similar. As shown in Figure 5 we have assigned the minors indices. We have chosen to examine the case in which $a < b < c$. By sorting, we obtain that $\Delta_{a,b,c+1} \times \Delta_{a+1,b,c} < \Delta_{a,b,c} \times \Delta_{a+1,b,c+1}$. Since $\Delta_{a+1,b,c} = \Delta_{a,b,c}$, we have $\Delta_{a,b,c+1} < \Delta_{a+1,b,c+1}$. By sorting, we also obtain $\Delta_{a+1,b+1,c} \times \Delta_{a+1,b,c+1} < \Delta_{a+1,b,c} \times \Delta_{a+1,b+1,c+1}$, and therefore $\Delta_{a,b,c+1} < \Delta_{a+1,b,c+1} < \Delta_{a+1,b+1,c+1} < 1$.

5. It is not possible using Skandera inequalities to show that Δ_{a_7} can not be the third largest minor. Instead we show that there is a shorter path to get to a_7, such that it is only two detours away from the original path. As shown in Figure 6, we have assigned indices to the vertices. This is without loss of generality, as we do not indicate the order in which a, b, and c appear. We first assign a, b, c as the indices for one of the vertices and then move up to the right. Without loss of generality, we can assign it to

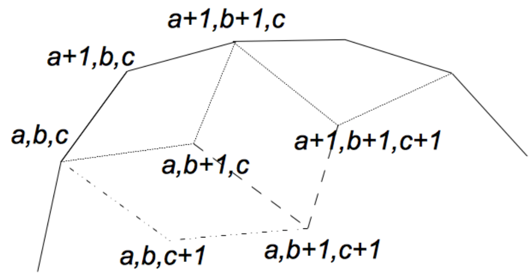

Figure 5: Case 4 with indices assigned.

be $a+1, b, c$. Moving to the next vertex, we assign it to be $a+1, b+1, c$ since an index other than a must be moved. If a is moved, then no detour can exist. Similarly, all of the other vertices can be assigned indices uniquely and without loss of generality. In Figure 7, however, we have shown a shorter path to get to the minor $\delta_{a,b,c+1}$, and thus it is not actually three, but only two detours away and we no longer must consider it.

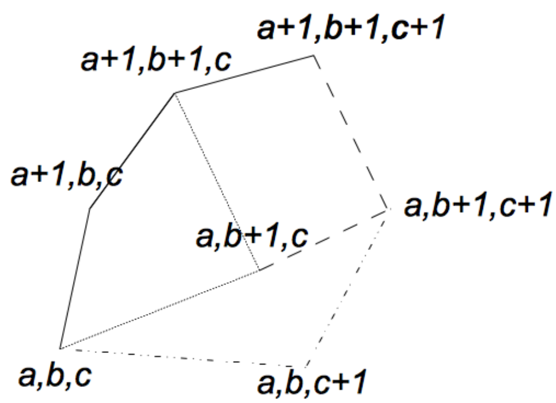

Figure 6: Case 5 with indices assigned.

6. We have $\Delta_{a_9} \times \Delta_{a_8} < \Delta_{a_7} \times \Delta_{a_{10}}$, giving us that $\Delta_{a_9} < \Delta_{a_7}$. Similarly, $\Delta_{a_7} < \Delta_{a_6} < \Delta_{a_4}$ and thus Δ_{a_9} can not be third largest.

Therefore, we have examined all cases in which a certain minor is three detours away from the original maximal sorted collection, and in each of those cases we have proved that

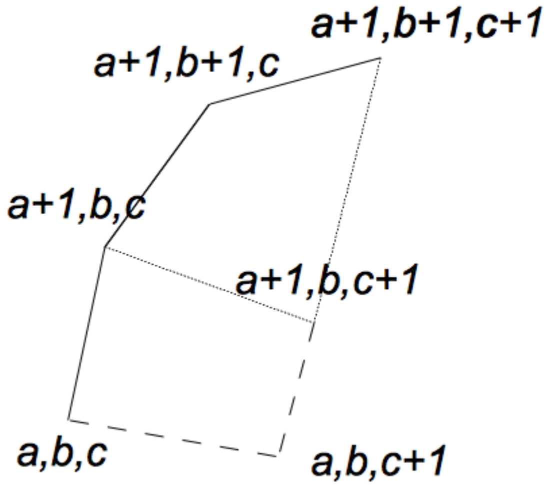

Figure 7: Case 5: shorter path to minor.

such a minor can not possibly be a third largest minor.

□

5 The general case

Here we present a conjecture for the general r.

Conjecture 5.1. *Let $A \in Gr^+(k,n)$ for which U_L is of maximal size. Let $I \in \binom{[n]}{k}$ such that for any $\mathcal{I} \in V(\Gamma_{(k,n)})$, $cube\_d(\mathcal{I}, U_L) \geq b$ and there exists $J \in V(\Gamma_{(k,n)})$, $I \in J$ such that $cube\_d(J, U_L) = b$. Then $I \notin U_i$ for all $i \in \{L-1, L-2, \ldots, L-b+1\}$ (that is, I is at most the $b+1^{th}$ largest).*

We prove one case of the conjecture: the one in which $k = 2$ and no cubes are present.

Theorem 5.2. *Let $A \in Gr^+(2,n)$ for which U_L is of maximal size. Let $\mathcal{I} \in \binom{[n]}{2}$ such that for any $\mathcal{I} \in v(\Gamma_{(2,n)}) cube\_d(\mathcal{I}, U_L) \geq b$ and there exists $J \in v(\Gamma_{(2,n)})$ such that $cube\_d(J, U_L) = b$. If the graph theoretical distance of J and U_L equals b as well, then $I \notin U_i$ for all $i \in \{L-1, L-2, \ldots, L-b+1\}$ (that is, I is at most the $b+1^{th}$ largest).*

Proof. In the $2 \times n$ case, we only have two indices a and b which we can move. We can assign one of the vertices without loss of generality, the index a, b. Then, moving to the right, we can call the vertex next to it $a+1, b$. The next vertex to the right can either be $a+1, b+1$ or $a+2, b$. However, since we have elected for there to be another path coming from a, b, the option $a+2, b$ is impossible. Thus we have now established four of the indices shown in Figure 8: a, b, $a+1, b$, $a+1, b+1$, and $a, b+1$. Now, we add a second path on to the first. This one must be either directly to the left or to the right of the first minor. Without loss of generality we choose to go to the right. The new minor formed which must connect to $a, b+1$ must either shift a forward or $b+1$. Since the vertex, $a+1, b+1$ has already been assigned, though, we know that only $b+1$ can be shifted, thus obtaining two new vertices: $a, b+2$ and $a+1, b+2$. To add on a third detour on top of the first two, we must either go to the right or to the left. We first consider going to the left. We must form a new path from $a, b+2$ to a, b. But such a path, other than the one involving $a, b+1$, does not exist. Thus we must go to the right again. From $a, b+2$ we must move to $a, b+3$ as $a+1, b+2$ has already been assigned.

In general, we can see that a new path formed from the old ones can only move to the right, and new vertices are of the form $a, b+k$, where a, b is the first vertex from which the path started. The entire path consists of squares which lead up from a, b to $a, b+k$, as shown in Figure 8. From this construction, we can use the inequalities of the form: $\Delta_{a,b+k} \times \Delta_{a+1,b+k-1} < \Delta_{a+1,b+k} \times \Delta_{a,b+k-1}$ to obtain $\Delta_{a,b+k} < \Delta_{a,b+k-1} < \Delta_{a,b+k-2} < \ldots < \Delta_{a,b+1} < \Delta_{a,b}$.

$\square$

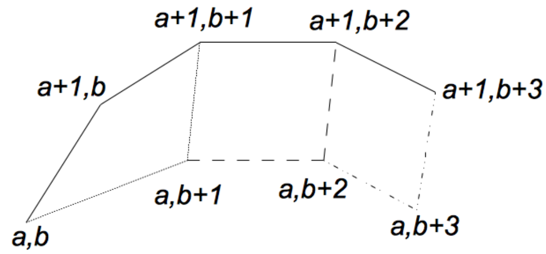

Figure 8: Without Loss of generality, the assignment of indices to the vertices in a $2 \times n$ matrix.

6 Concluding remarks

In this paper, we have found and proved important properties and relationships between arrangements of minors in TP matrices and other important combinatorial objects. We formed a conjecture for the general case, and showed that the conjecture is true in several subcases. It seems that the rest of the conjecture may be within reach using methods similar to those used in this paper. The surprising relationships we found between totally positive matrices and Sturmfels' triangulation could pave the way for future advances in physics, computer science, and mathematics. The results found in this paper have undoubtedly advanced us much closer to gaining a complete understanding of the positive Grassmannian.

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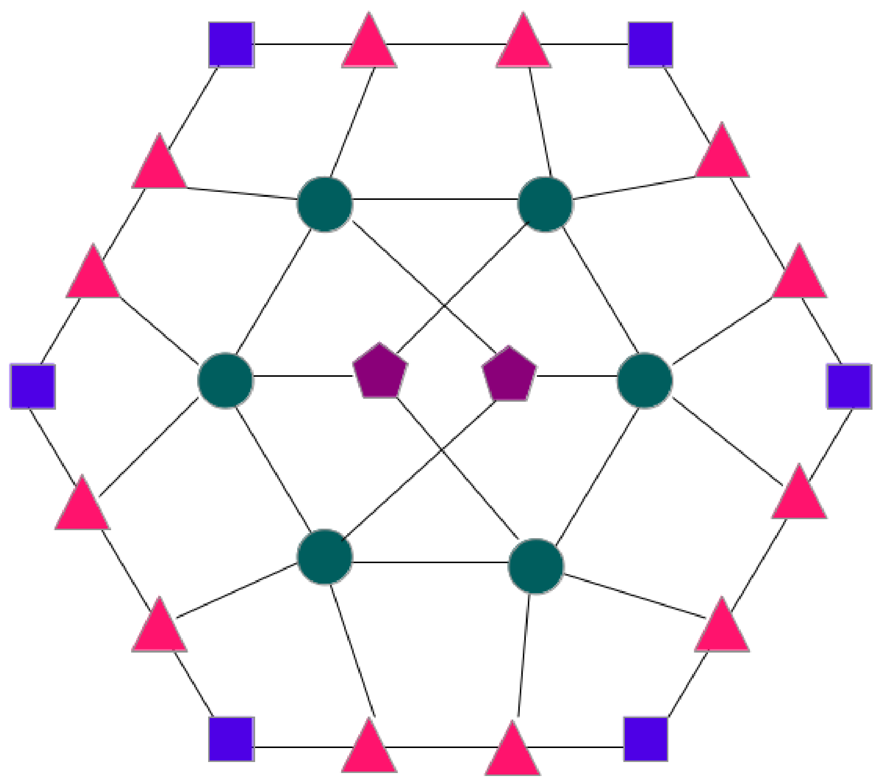
A Sample Diagrams

Figures 9, 10, and 11 are all sample diagrams intended to clarify concepts discussed in this paper which are difficult to visualize.

Figure 9: The graph $\Gamma_{(k,n)}$ of the Sturmfels' Triangulation



In Figure 11, we can see that $cube\_d(1,4) = 3$, going by the path from 1 to 2 to 3 to 4.

Figure 10: key for Figure 9

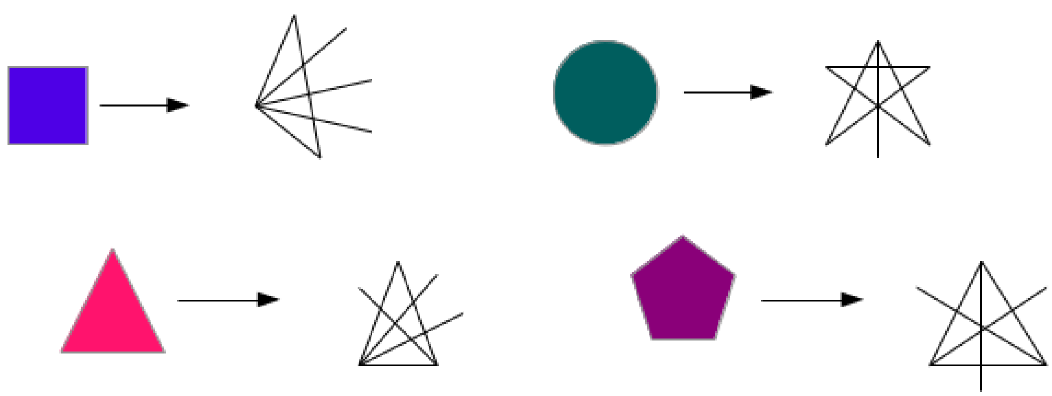


Figure 11: The graph $\Gamma_{(k,n)}$ of the Sturmfels' Triangulation with diagonals representing cubical distance.