Acyclic Colorings and Subgraphs of Directed Graphs

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Abstract

The *acyclic chromatic number* of a directed graph D, denoted $\chi_A(D)$, is the minimum positive integer k such that there exists a decomposition of the vertices of D into k disjoint sets, each of which induces an acyclic subgraph. We show that for all digraphs D without directed 2-cycles, we have $\chi_A(D) \leq \frac{4}{5} \cdot \bar{\Delta}(D) + o(\bar{\Delta}(D))$, where $\bar{\Delta}(D)$ denotes the maximum arithmetic mean of the out-degree and the in-degree of a vertex in D. This result significantly improves a bound of Mohar and Harutyunyan. A related question to finding $\chi_A(D)$ is to find the maximum size of an acyclic induced subgraph of D. We partially resolve a conjecture of Harutyunyan that all planar digraphs on n vertices have an acyclic induced subgraph of size $3n/5$. We also improve several existing lower bounds on the size of an acyclic induced subgraph for general digraphs.

Summary

We investigate *directed graphs* which are collections of nodes and arrow pointing from one node to another. Specifically, we study the question of removing *directed cycles*, or collections of nodes and arrows that eventually loop back around to themselves. This question has applications in scheduling and deadlock resolution. In a computer, actions can be represented by nodes in a directed graph, and a directed cycle in the directed graph corresponds to a situation where each action is preventing the next from finishing. The result is *deadlock*, a situation when no actions can finish. Therefore, studying how to remove cycles from directed graphs can lead to more efficient deadlock resolution. We improve upon several bounds relating to this question and partially answer several conjectures in this field.

1 Introduction

A proper *vertex coloring* of an undirected graph G partitions the vertices into independent sets. It is natural to try to extend this notion to directed graphs (digraphs). An *acyclic set* in a digraph is a set of vertices whose induced subgraph contains no directed cycle. The *acyclic chromatic number* of a digraph D, denoted $\chi_A(D)$, is the minimal number of acyclic sets into which the vertices of D can be partitioned. In this paper, we consider *oriented graphs*, which are digraphs such that at most one edge connects any pair of vertices.

Studying the acyclic subgraphs of digraphs has applications in scheduling and deadlock resolution. For instance, Jain et al. [1] studied acyclic subgraphs to develop efficient approximation algorithms for deadlock resolution. Actions in a computer may be represented as vertices in a digraph. A directed edge is drawn from action A to action B if A is waiting for B to finish. This situation can happen for example if B is holding up a processor that A needs to terminate. Therefore, a directed cycle in such a digraph represents a situation where none of the events in the cycle can finish since each is waiting for the following event to be complete. This situation is called deadlock, and one way to resolve the deadlock is to study acyclic subgraphs of a digraph. Each acyclic subgraph contains no cycles, and therefore no deadlock, so obtaining bounds on the size of acyclic subgraphs can lead to improvements in efficient deadlock resolution.

1.1 Preliminary definitions

Given a directed graph D, we let $V(D)$ denote the set of vertices of D, and $E(D)$ denote the set of edges of D. For any vertex v in a directed graph, the *out-neighborhood* $N^+(v)$ of v is the set of all vertices u for which vu is an edge, and the *in-neighborhood* $N^-(v)$ of v is the set of all vertices u for which uv is an edge. The *out-degree* of v is $d^+(v) = |N^+(v)|$, and the *in-degree* of v is $d^-(v) = |N^-(v)|$. The *digirth* of a directed graph is the length of its shortest directed cycle. All logarithms in this paper are taken base 2.

In this paper we use several notions of maximal degree of a vertex in a digraph. Given a digraph D, $\Delta^+(D)$ denotes the maximum out-degree of D, and $\Delta^-(D)$ denotes the maximum in-degree of D. $\tilde{\Delta}(D)$ is the maximum geometric mean of the in-degree and the out-degree of a vertex in D, and $\bar{\Delta}(D)$ is the maximum arithmetic mean of the in-degree and the out-degree of a vertex in D. Notice that if the out-degrees and in-degrees of all vertices in D are equal, then $\Delta^+(D) = \Delta^-(D) = \tilde{\Delta}(D) = \bar{\Delta}(D)$.

1.2 Chromatic number from maximal degree

Although recent results [2, 3, 4, 5] suggest that the acyclic chromatic number in digraphs behaves similarly to the chromatic number in undirected graphs, much still remains to be learned. For instance, it is well that $\chi_A(D) \leq \Delta^+(D) + 1$; this is easily proved using the greedy algorithm. However, this bound is not tight for most digraphs D; in fact, Harutyunyan and Mohar [5] credit McDiarmid and Mohar with the following conjecture

Conjecture 1.1 ([5])**.** *Every oriented graph D and with maximum total degree $\Delta(D)$ has $\chi_A(D) = O\left(\frac{\Delta(D)}{\log \Delta(D)}\right)$.*

Conjecture 1.1 seems to be relatively difficult, so Harutyunyan and Mohar [6] conjectured a relaxation of Conjecture 1.1. The following conjecture is paraphrased from [6].

Conjecture 1.2 ([6])**.** *Let D be an oriented graph. Then*

$$\chi_A(D) \leq \left\lfloor \frac{\bar{\Delta}(D)}{2} \right\rfloor.$$

Harutyunyan and Mohar proved that $\chi_A(D) \leq (1 - e^{-13})\tilde{\Delta}(D)$. In Theorem 2.1, we prove an upper bound on a generalization of the acyclic chromatic number. It follows from

Figure 1: On the left is shown a directed graph which is not acyclic. As shown on the right, by removing a vertex and all edges incident to it (dashed), we can delete all directed cycles. The remaining vertices form an acyclic set of maximum size.

Theorem 2.1 that we can significantly improve Harutyunyan and Mohar's bound, as shown in Corollary 2.9, which we restate below:

Corollary. *For an oriented graph D with $\bar{\Delta}(D) \geq 2$,*

$$\chi_A(D) \leq \left\lfloor \frac{4}{5} \cdot \bar{\Delta}(D) + \frac{2}{5} \right\rfloor + 1.$$

1.3 Large acyclic subgraphs

A quantity related to the acyclic chromatic number of a digraph D is the size of a largest induced acyclic subgraph of D, which is denoted by $\alpha_A(D)$. As shown in Figure 1, finding the maximum value of $\alpha_A(D)$ is equivalent to finding the minimum number of vertices such that when these vertices are deleted, the subgraph induced by the remaining vertices is acyclic. Given a digraph D with n vertices, the values $\chi_A(D)$ and $\alpha_A(D)$ are related by the inequality $\alpha_A(D) \geq n/\chi_A(D)$, so finding an upper bound on $\chi_A(D)$ in turn gives a lower bound on $\alpha_A(D)$.

The values $\chi_A(D)$ and $\alpha_A(D)$ have been studied much for *planar digraphs*, digraphs which can be embedded, or drawn, in the plane so that no 2 edges cross each other. As shown in Figure 2, the directed 4-cycle is planar because there exists an embedding in the plane with no crossing edges. Notice that not all embeddings in the plane have no crossing edges.

Bokal et al. [3] credit Škrekovski with the conjecture that all oriented planar graphs have

Figure 2: The directed 4-cycle is planar because there exists an embedding in the plane with no crossing edges, as shown on the left. However, as shown on the right, not all embeddings in the plane have no crossing edges.

acyclic chromatic number at most 2.

Conjecture 1.3 ([3]). *If D is an oriented planar graph, $\chi_A(D) \leq 2$.*

Bokal et al. [3] showed that all oriented planar graphs are acyclically 3-colorable. One approach to Conjecture 1.3 has been to look for lower bounds on $\alpha_A(D)$ for planar digraphs D. Borodin [7] showed that in any planar oriented graph there exists an acyclic subset containing at least 2/5 of the vertices. Harutyunyan and Mohar [8] ask whether, in any planar oriented graph, there exists an acyclic subset of at least 1/2 of the vertices. Note that this fact would follow immediately from Conjecture 1.3. Harutyunyan [9] recently conjectured an even stronger bound on the maximum size of an acyclic set:

Conjecture 1.4 ([5]). *If D is an oriented planar graph on n vertices, then $\alpha_A(D) \geq \frac{3n}{5}$.*

Recently, Harutyunyan and Mohar [8] proved that every oriented planar graph of digirth 5 is acyclically 2-colorable, using a complex vertex-discharging method. They posed the problem of finding a simpler proof of the acyclic 2-colorability of planar digraphs of large digirth, and we partially answer this question in Theorem 3.5 by giving a short proof that in a planar digraph of digirth g, there exists an acyclic set of vertices of size at least $n - 3n/g$. We restate Theorem 3.5 below:

Theorem. *If D is a planar digraph with digirth g on n vertices, then $\alpha_A(D) \geq n - 3n/g$. Moreover, if $g = 4$, then $\alpha_A(D) \geq 5n/12$, and if $g = 5$, then $\alpha_A(D) \geq 7n/15$.*

It follows immediately from Theorem 3.5 that Conjecture 1.4 is true for digraphs of digirth at least 8.

Lower bounds on $\alpha_A(D)$ have also been studied for general digraphs D. Ben-Eliezer et al. [10] showed that if D has n vertices, then $\alpha_A(D) \geq \log n$. Moreover, Aharoni et al. [2] showed that for any digraph D with n vertices and m edges,

$$\alpha_A(D) \geq \frac{n^2}{m+n}. \tag{1}$$

Notice that when $m \leq n^2/\log n$, the bound in (1) is tighter than $\alpha_A(D) \geq \log n$. In fact, Aharoni et al. [2] conjectured that $\alpha_A(D)$ is nearly bounded below by a product of these two bounds.

Conjecture 1.5 ([2])**.** *If D is a digraph with n vertices and m edges, then*

$$\alpha_A(D) \geq (1 + o(1))\frac{n^2}{m} \log \frac{m}{n}.$$

This conjecture remains open, even up to a constant factor, but Ben-Eliezer et al. [10] recently proved that there is an absolute constant c such that

$$\alpha_A(D) \geq c \cdot \frac{n^2}{m} \cdot \frac{\log n}{\log (n^2/m)}. \tag{2}$$

Ben-Eliezer et al. [10] then used the bound in (2) to find a lower bound on the size Ramsey number of a directed path. A proof of Conjecture 1.5 would lead to improved lower bounds on this size Ramsey number. In Theorem 4.1, we give an improvement on the bound for $\alpha_A(D)$ in inequality (2) for a certain class of digraphs which we call *weakly regular digraphs*. We define a digraph to be *weakly regular* if the sums of the out-degree and in-degree of each vertex are all equal. In Theorem 4.1, restated below, we improve the bound in (2) of Ben-Eliezer et al. by a factor of nearly 10 for weakly regular graphs.

Theorem. *Suppose D is a weakly regular digraph with n vertices and at most ϵn^2 edges, and $\epsilon \geq 1/n$. Then D contains an acyclic set of size $\frac{A \log n}{\epsilon \log 1/\epsilon}$, where A is any constant strictly less than 1.*

In Theorem 4.2, restated below, we also improve the bound in (1) by a constant factor of 2 for all digraphs D:

Theorem. *If D is a digraph on n vertices and with average out-degree d, then $\alpha_A(D) \geq \frac{2n}{d+4}$.*

The organization of this paper is as follows. In Section 2, we prove Theorem 2.1, giving an upper bound on $\chi_A(D)$ for any digraph D in terms of $\bar{\Delta}(D)$. In Section 3, we prove Theorem 3.5, showing that in planar oriented graphs of large digirth, there exists a large induced acyclic subgraph. In Section 4, we improve the lower bounds for $\alpha_A(D)$ in inequalities (1) and (2) by constant factors in Theorems 4.1 and 4.2, respectively.

2 Acyclic colorings

Recall that $\tilde{\Delta}(D)$ is the maximum geometric mean of the in-degree and the out-degree of a vertex in D. Harutyunyan and Mohar [6] proved that given a digraph D, if $\tilde{\Delta}(D)$ is large enough, then $\chi_A(D) \leq (1 - e^{-13})\tilde{\Delta}(D)$. They used a non-constructive method to do so, and posed the problem of improving this bound, remarking that a different technique may be necessary. We constructively find a significantly stronger upper bound on $\chi_A(D)$ in Theorem 2.1, thus making progress towards Conjecture 1.2. The outline of our proof is somewhat similar to that of an undirected analogue proved by Borodin [11].

We begin by introducing some notation and definitions that will be useful in this section. Given a digraph D and $u \in V(D)$, we denote the subgraph induced on $V(D)\backslash\{u\}$ by $D - u$. A *strongly connected component* of a digraph D is an induced subgraph H such that for any $u, v \in H$, there are directed paths from u to v and from v to u.

A digraph D is said to be *weakly m-degenerate* if for every induced subgraph of D, there is a vertex of out-degree or in-degree strictly less than m. Therefore, a digraph is weakly 1-degenerate if and only if it is acyclic. A (k, m)*-degenerate coloring* of D is a partition of $V(D)$ into k sets, each of which is weakly m-degenerate. We denote the smallest k such that D has a (k, m)-degenerate coloring by $\psi_m(D)$. Given D, a *critical vertex* is a vertex $v \in V(D)$ such that $\psi_m(D - v) < \psi_m(D)$. If every vertex of D is critical and $\psi_m(D) = k$, then we define D to be a (k, m)*-critical digraph.* We can now state Theorem 2.1:

Theorem 2.1. *Let m be a positive integer and $t = 2m$. Suppose we are given a digraph D with $\bar{\Delta}(D) \geq t$. Then*

$$\psi_m(D) \leq \left\lfloor \frac{\bar{\Delta}(D) - \left(\frac{1}{2}\right) \left\lfloor \frac{\bar{\Delta}(D)+1/2}{t+1/2} \right\rfloor}{m} \right\rfloor + 1.$$

The following Lemmas 2.2 through 2.5, generalize a directed graph analogue of Brook's theorem [12] due to Mohar [13]. Our proofs follow similar outlines to those of Mohar.

Lemma 2.2 shows that critical vertices in a digraph must have large in-degree and out-degree.

Lemma 2.2. *Suppose v is a critical vertex in a digraph D, and $\psi_m(D) = k$. Then $d^+(v) \geq (k-1)m$ and $d^-(v) \geq (k-1)m$.*

Proof. Suppose for the purpose of contradiction that $d^+(v) < (k-1)m$. We will show that we can find a $(k-1, m)$-degenerate coloring of D, a contradiction to the fact that $\psi_m(D) = k$. Since v is (k, m)-critical, we can find a $(k-1, m)$-degenerate coloring of $D - v$. At least one color class c must be represented in less than m out-neighbors of v because otherwise v would have at least $(k-1)m$ out-neighbors. Now we color v color c, and claim that the subgraph H induced by all vertices of color c is m-degenerate. To see this, let H' be an induced subgraph of H. If $v \in V(H')$, then notice that v has at most $m-1$ out-neighbors in H'. Otherwise, note that H' is a subset of a color class in a $(k-1,m)$-degenerate coloring of $D-v$, meaning that there is some vertex in H' of in-degree or out-degree less than m.

A similar proof shows that $d^-(v) \geq (k-1)m$. □

Proposition 2.3 is the bulk of the proof of Lemma 2.5. It states that the in-degree and out-degree of every vertex cannot be too small in a (k,m)-critical oriented graph. Moreover, it generalizes a theorem of Mohar [13], who proved the case $m=1$.

Proposition 2.3. *Suppose that D is a (k,m)-critical oriented graph in which each vertex v satisfies $d^+(v) = d^-(v) = (k-1)m$. Then $k \leq 2$.*

To prove Proposition 2.3, we assume that $k \geq 3$ for the purpose of contradiction and create a linear ordering of the vertices of D, as follows. Pick any vertex u_n, and let two of its out-neighbors or two of its in-neighbors be u_1, u_2. Let $D' = D - u_n$. Now, since u_n has at least 2 in-neighbors, there is some $u_{n-1} \in D'$ apart from u_1, u_2 such that u_n is an out-neighbor of u_{n-1}. Thus, u_{n-1} has at most $(k-1)m-1$ out-neighbors in D'. Now let $D'' = D' - u_{n-1}$ and continue in a similar manner, counting down from u_n. In the ith step $(3 \leq i \leq n-1)$, find a vertex $u_i \neq u_1, u_2$ such that u_i has some in-neighbor or out-neighbor among $\{u_{i+1}, \dots, u_n\}$. Thus, u_i must have in-degree or out-degree less than $m(k-1)$ in the digraph $D - \{u_{i+1}, \dots, u_n\}$. Finding u_i, however, is possible if and only if there is some edge between $\{u_{i+1}, \dots, u_n\}$ and $\{u_3, \dots, u_i\}$. We therefore establish that this fact is true for some appropriate choice of u_1, u_2 and for all i $(3 \leq i \leq n-1)$ in the following Lemma 2.4 before we prove Proposition 2.3.

Given a digraph D, we call two vertex-disjoint subgraphs D_1 and D_2 *non-adjacent* if there is no edge of D with one endpoint in D_1 and the other endpoint in D_2. A *weakly connected* component of a digraph is an induced subgraph H such that for any $u, v \in V(H)$, there is a path (not necessarily directed) from u to v.

Lemma 2.4. *Suppose D is a (k,m)-critical digraph. Assume that for any choice of vertices $u_1, u_2 \in D$ such that u_1 and u_2 are both out-neighbors or both in-neighbors of another vertex of D, there is some i, $3 \le i \le n$, such that the subgraphs induced by the sets $\{u_{i+1}, \dots, u_n\}$ and $\{u_3, \dots, u_i\}$, as formed in the process above, are non-adjacent. Then $k \le 2$.*

Proof. Assume for the purpose of contradiction that $k \ge 3$. Pick any two vertices u_1, u_2 which are both in-neighbors or out-neighbors of another vertex. Let D_1 denote the digraph induced by $\{u_1, u_2, \dots, u_i\}$ and D_2 denote the digraph induced by $\{u_1, u_2, u_{i+1}, u_{i+2}, \dots, u_n\}$. We claim that both D_1 and D_2 are weakly connected. Suppose first that D_1 is not weakly connected for the purpose of contradiction. There are 2 cases to consider:

Case 1. u_1 and u_2 are in the same weakly connected component of D_1.

Let w be a vertex in D_1 which is not in the same weakly connected component as u_1 and u_2. Any path (not necessarily directed) from w to u_1 or u_2 must contain some vertex in D_2 before going through u_1 or u_2 because otherwise there would be a path from w to u_1 or u_2. But this contradicts the fact that D_1 and D_2 are non-adjacent.

Case 2. u_1 and u_2 are in different weakly connected components of D_1.

Notice that Case 2 cannot hold for D_2 also, or else D would not be connected. Let F_1 be the weakly connected component of D_1 that contains u_1 and F_2 be the weakly connected component of D_1 that contains u_2. By the (k,m)-criticality of D, we can find a $(k-1,m)$-degenerate coloring of each of F_1 and F_2. Moreover, we can rearrange the colors on F_1 so that u_1 and u_2 are the same color and the coloring remains $(k-1,m)$-degenerate.

We now extend this coloring to $\{u_{i+1}, \dots, u_n\}$ so that it remains $(k-1,m)$-degenerate. Notice that for $i+1 \le j \le n-1$, u_j has fewer than $(k-1)m$ in-neighbors or out-neighbors among $\{u_1, \dots, u_{j-1}\}$ because it has at least one out-neighbor or in-neighbor among $\{u_{j+1}, \dots, u_n\}$. Hence, for any $(k-1,m)$-degenerate coloring of $\{u_1, \dots, u_{j-1}\}$, we can color u_j so that it has fewer than m in-neighbors or out-neighbors of the same color among $\{u_1, \dots, u_{j-1}\}$. Thus, we can color the vertices of D_2 in the order $\{u_{i+1}, u_{i+2}, \dots, u_{n-1}\}$ so that the coloring remains $(k-1,m)$-degenerate with each additional vertex we color. When we reach u_n, since two of its out-neighbors or in-neighbors are of the same color, we can also color u_n so that it has fewer than m in-neighbors or out-neighbors of the same color. This completes a $(k-1,m)$-degenerate coloring of D, contradicting the fact that $\psi_m(D)=k$. Therefore, D_1 is weakly connected.

To show that D_2 is weakly connected, notice that by construction, for all j where $i+1 \leq j \leq n$, there is some $k > j$ such that there is an edge connecting u_j and u_k. Therefore, there is a path (not necessarily directed) connecting u_j to u_n.

We have that for any initial choice of $\{u_1, u_2\}$ and for any D_1, D_2 chosen according to the process above, D_1 and D_2 are both weakly connected. In the rest of the proof below, we will only use the weakly connectedness of D_1 and D_2 and will therefore treat them identically.

Suppose without loss of generality that $|V(D_1)| \leq |V(D_2)|$. If there exists a vertex $u \in V(D_1)$ with two out-neighbors or two in-neighbors u_1' and u_2' which are in $D_1 - u_1 - u_2$, then we consider the set $\{u_1', u_2'\}$, and repeat the process of ordering the vertices. By assumption, there is some i' so that the subgraph D_1' induced by $\{u_1', u_2', \ldots, u_{i'}'\}$ and the subgraph D_2' induced by $\{u_1', u_2', u_{i'+1}', u_{i'+2}' \ldots, u_n'\}$ are non-adjacent. Notice that either $|V(D_1')|$ or $|V(D_2')|$ is larger than $|V(D_2)|$ since D_1 is weakly connected. We now repeat the above process with D_1' and D_2'; at the jth step of the iteration, either $D_1^{(j)}$ or $D_2^{(j)}$ is not weakly connected, contradicting the (k,m)-criticality of D, or the positive difference between the $|V(D_1^{(j+1)})|$ and $|V(D_2^{(j+1))})|$ increases from the positive difference between $|V(D_1^{(j)})|$ and $|V(D_2^{(j)})|$, so the process must eventually end at the tth iteration, for some positive integer t. That is, there must be some $u_1^{(t)}$, $u_2^{(t)}$ and a corresponding $D_1^{(t)}$ and $D_2^{(t)}$ with, without loss of generality, $|V(D_1^{(t)})| \leq |V(D_2^{(t)})|$, such that there is no vertex $u \in V(D_1^{(t)})$ such that u has 2 out-neighbors or 2 in-neighbors in $D_1^{(t)}$. However, this implies that $k=3, m=1$, and each vertex in $D_1^{(t)}$ has u_1 as an in-neighbor and u_2 as an out-neighbor, or vise versa. Thus, $D_1^{(t)}$ has at most 2 vertices. However, we can now clearly find a 2-coloring of $D_1^{(t)} + u_1 + u_2$ to ensure that there are no monochromatic cycles in D. $\square$

We now prove Proposition 2.3, using the fact established in Lemma 2.4 that there is an ordering of the vertices of D, $\{u_1, u_2, \ldots, u_n\}$, such that for $1 \leq i \leq n-1$, u_i has an out-neighbor or in-neighbor among $u_{i+1}, \ldots, u_n$.

Proof. Pick some ordering $\{u_1, u_2, \ldots, u_n\}$, chosen according to the process described above. We color the vertices of D as follows: we give u_1, u_2 the same color. Then for $3 \leq i \leq n-1$, we note that in the subgraph of D induced by $\{u_1, \ldots, u_i\}$, u_i has in-degree or out-degree less than $(k-1)m$. Therefore, in the $(k-1, m)$-degenerate coloring of $\{u_1, \ldots, u_{i-1}\}$, one of the color classes contains no in-neighbors or out-neighbors of u_i. We now color u_i this color. Finally, since u_n has two out-neighbors or two in-neighbors of the same color, we can find a color represented among less than m out-neighbors or in-neighbors of u_n, and color u_n this color.

At the end of the process, we claim that each color class c is m-degenerate. To show this, for any color class c and subset S of the vertices colored c, pick $u_i \in S$ so that i is as large as possible. Then by the construction of the coloring, u_i has at most $m-1$ in-neighbors or out-neighbors in S, completing the proof. $\square$

Lemma 2.5 uses Lemma 2.2 to extend Proposition 2.3 to digraphs that are not k-critical. This is possible because intuitively, k-critical digraphs are the worst case for finding an acyclic coloring with few colors.

Lemma 2.5. *Suppose that $\psi_m(D) = k+1 \geq 3$, for some integer k, and that D is an oriented graph. Then $\bar{\Delta}(D) > km$.*

Proof. Suppose for the purpose of contradiction that for some $k \geq 2$, there is an oriented graph D with as few vertices as possible, such that $\bar{\Delta}(D) \leq km$. Notice that if D' is any induced subgraph of D, then $\bar{\Delta}(D) \geq \bar{\Delta}(D')$. Notice that if D were not $(k+1, m)$-critical, we could remove some vertex v to form $D' = D - v$, and we would have $\psi_m(D') = k+1$ and $\bar{\Delta}(D') \leq \bar{\Delta}(D) \leq km$. This contradicts the fact that D has as few vertices as possible such that $\bar{\Delta}(D) \leq km$ holds. Hence D is $(k+1, m)$-critical.

By Lemma 2.2, for each $v \in V(D)$, we have that $d^+(v) \geq km$ and $d^-(v) \geq km$. In order to have $\bar{\Delta}(D) \leq km$, we must have $d^+(v) = d^-(v) = km$ for all $v \in V(D)$. But, then by Proposition 2.3, we have that $k+1 \leq 2$, contradicting the fact that $k+1 \geq 3$. $\square$

The following corollary follows immediately from Proposition 2.5.

Corollary 2.6. *If D is an oriented graph such that $\bar{\Delta}(D) > km$, and if $\bar{\Delta}(D) \geq 2m$, then $\psi_m(D) \leq \left\lceil \frac{\bar{\Delta}(D)}{m} \right\rceil$.*

To prove Theorem 2.1, we also use a theorem of Lovász [14], which states that the vertices of a graph can be decomposed into sets so that the sum of the maximal degrees of all the sets is less than the maximal degree of the graph.

Theorem 2.7 (Lovász [14])**.** *For an undirected graph G, suppose that $\Delta(G)+1 = \sum_{i=1}^{s}(\Delta_i + 1)$, with Δ_i being nonnegative integers, and $s \geq 1$. Then there is a covering of $V(G)$ with s subgraphs G_i $(1 \leq i \leq s)$, so that $\Delta(G_i) \leq \Delta_i$ for $1 \leq i \leq s$.*

We now deduce as a corollary of Theorem 2.7 a version for directed graphs. To do so, we need an analogue of maximal degree in undirected graphs. Recall that $\bar{\Delta}(D)$ is the maximum, over all vertices v of D, of the average of the out-degree and the in-degree of v.

Corollary 2.8. *For a digraph D and positive integer s, suppose $\bar{\Delta}(D) = \sum_{i=1}^{s} \bar{\Delta}_i + \frac{s-1}{2}$. Then there is a covering of $V(D)$ with s subgraphs D_i $(1 \leq i \leq s)$ such that $\bar{\Delta}(D_i) \leq \bar{\Delta}_i$.*

Proof. Given a digraph D, consider the underlying undirected graph G. Notice that $\Delta(G) = 2\bar{\Delta}(D)$. By Theorem 2.7, we can partition G into sets of vertices $G_1, \ldots, G_s$, so that the maximum total degree in each set is at most $2\bar{\Delta}_i$. For $1 \leq i \leq s$, we let D_i be the directed subgraph of D induced by $V(G_i)$. Thus, for $1 \leq i \leq s$, $\bar{\Delta}(D_i) \leq \bar{\Delta}_i$. $\square$

We now prove Theorem 2.1, which uses Lemma 2.5 and Corollary 2.8 to find an upper bound on $\psi_m(D)$.

Proof of Theorem 2.1. Set

$$s = \left\lfloor \frac{\bar{\Delta} + 1/2}{t + 1/2} \right\rfloor,$$

and $r = \bar{\Delta}(D) + 1/2 - s(t + 1/2)$. Then $\bar{\Delta}(D) = \sum_{i=1}^{s} t + (r - \frac{1}{2}) + \frac{s}{2}$, meaning that, by Corollary 2.8, the vertices of D can be covered with $s + 1$ subgraphs $D_1, \ldots, D_{s+1}$, which satisfy:

$$\begin{aligned} \bar{\Delta}(D_i) \leq t \quad &: \quad 1 \leq i \leq s \\ \bar{\Delta}(D_i) \leq r - \frac{1}{2} \quad &: \quad i = s + 1. \end{aligned}$$

Therefore, by Corollary 2.6,

$$\begin{aligned} \psi_m(D_i) \leq \lceil t/m \rceil \quad &: \quad 1 \leq i \leq s \\ \psi_m(D_i) \leq 1 + \lfloor (r - 1/2)/m \rfloor \quad &: \quad i = s + 1. \end{aligned}$$

We thus have

$$\begin{aligned} \psi_m(D) &\leq \sum_{i=1}^{s+1} \psi_m(D_i) \\ &\leq \lceil t/m \rceil \cdot s + \lfloor (r - 1/2)/m \rfloor + 1 \\ &\leq \lceil t/m \rceil \cdot \left\lfloor \frac{\bar{\Delta} + 1/2}{t + 1/2} \right\rfloor + \left\lfloor \frac{\bar{\Delta}(D) - (t + 1/2) \cdot \left\lfloor \frac{\bar{\Delta}+1/2}{t+1/2} \right\rfloor}{m} \right\rfloor + 1 \\ &\leq \left\lfloor \frac{\bar{\Delta}(D)}{m} - \left(\frac{t + 1/2}{m} - \left\lceil \frac{t}{m} \right\rceil \right) \left\lfloor \frac{\bar{\Delta} + 1/2}{t + 1/2} \right\rfloor \right\rfloor + 1. \end{aligned}$$

Because t/m is an integer,

$$\psi_m(D) \leq \left\lfloor \frac{\bar{\Delta}(D) - \left(\frac{1}{2}\right) \left\lfloor \frac{\bar{\Delta}(D)+1/2}{t+1/2} \right\rfloor}{m} \right\rfloor + 1.$$

□

Of particular interest is the case where $m = 1$, where we have $\psi_m(D) = \chi_A(D)$, and partitioning D into m-degenerate sets is equivalent to partitioning D into acyclic sets. Theorem 2.1 immediately implies the following Corollary 2.9:

Corollary 2.9. *For an oriented graph D with $\bar{\Delta}(D) \geq 2$,*

$$\chi_A(D) \leq \left\lfloor \frac{4}{5} \cdot \bar{\Delta}(D) + \frac{2}{5} \right\rfloor + 1. \tag{3}$$

The bound in (3) approaches $\frac{4}{5} \cdot \bar{\Delta}(D)$ as $\bar{\Delta}(D)$ approaches ∞. This bound is a significant improvement over the bound of $\chi_A(D) \leq (1-e^{-13}) \cdot \tilde{\Delta}(D)$ proved by Harutyunyan and Mohar.

3 Large acyclic subgraphs of planar digraphs

In this section, we use a corollary of the Lucchesi-Younger theorem to prove Theorem 3.5, which gives a new lower bound on the size of the minimal acyclic subgraph of a planar oriented graph.

We begin with some definitions. Given a directed graph D and a subset X of its vertices $V(D)$, we define $\bar{X} = V(D)\backslash X$. If every edge between X and $\bar{X}$ points from X to $\bar{X}$, then the set of such edges is called a *directed cut*. A *dijoin* is a set of edges that has a non-empty intersection with every directed cut.

The Lucchesi-Younger theorem [15] gives the minimum size of a dijoin of a digraph:

Theorem 3.1 (Lucchesi-Younger [15]). *The minimum cardinality of a dijoin in a directed graph D is equal to the maximum number of pairwise disjoint directed cuts of D.*

In the case that D is planar, the Lucchesi-Younger theorem has a useful corollary for the dual of D. Given an oriented planar graph D, the *dual* of D, denoted $D^\star$, is defined as follows. For a given planar embedding of D, construct a vertex of $D^\star$ within each face of D. For each edge uv of D separating faces f and g of D, a corresponding edge $f^\star g^\star \in E(D^\star)$ is drawn between vertices $f^\star$ and $g^\star$. The direction of edge $f^\star g^\star$ is defined so that as it crosses uv, v is on the left. It is simple to verify that the graph $D^\star$ does not depend on the planar embedding of D.

Given any directed graph, a *feedback arc set* is a set of edges of minimum cardinality, so that when removed, the resulting directed graph is acyclic. The following well-known result establishes a bijection between the directed cycles of a planar oriented graph and the directed cuts in its dual:

Proposition 3.2. *If D is a planar oriented graph, then the directed cycles of D are in one-to-one correspondence with the directed cuts in $D^\star$.*

Proof. Pick a planar embedding of D and embed $D^\star$ in the plane as defined above. Given a directed cycle C in D, notice that all edges of $D^\star$ crossing an edge of C must travel in the same direction: specifically, if C is oriented clockwise, then all edges of $D^\star$ crossing C point inwards, and if C is oriented counterclockwise, then the edges of $D^\star$ crossing C point outwards. Thus, let $X \subset V(D^\star)$ consist of the vertices of $D^\star$ corresponding to all faces of D inside C, and therefore the edges of $D^\star$ connecting X and $V(D^\star)\backslash X$ form a directed cut.

Conversely, given a directed cut of $V(D^\star)$, we reverse the method above to obtain a directed cycle of D. $\square$

Proposition 3.2 establishes the following corollary of the Lucchesi-Younger theorem:

Corollary 3.3 ([15]). *For a planar oriented graph, the minimum size of a feedback arc set is equal to the maximum number of arc-disjoint directed cycles.*

Proof. Given a planar oriented graph D, by the Lucchesi-Younger theorem, the minimum cardinality of a dijoin of $D^\star$ is equal to the maximum number of disjoint directed cuts of $D^\star$. Now, by Proposition 3.2, any dijoin of $D^\star$ corresponds to a feedback arc set of D, and any set of disjoint directed cuts of $D^\star$ corresponds to a set of arc-disjoint directed cycles of D. This completes the proof. $\square$

We also need one well-known lemma, which follows easily from Euler's formula for planar graphs.

Lemma 3.4. *Any planar graph G with n vertices and m edges satisfies $m \leq 3n - 6$.*

We now state and prove Theorem 3.5, which gives a lower bound on the minimum size of an acyclic set in a planar graph.

Theorem 3.5. *If D is a planar digraph with digirth g on n vertices, then $\alpha_A(D) \geq n - 3n/g$. Moreover, if $g = 4$, then $\alpha_A(D) \geq 5n/12$, and if $g = 5$, then $\alpha_A(D) \geq 7n/15$.*

Proof. Given a planar oriented graph D of digirth g, let H be a collection of arc-disjoint directed cycles, each of which must have length at least g. Thus, by Lemma 3.4, if we denote the number of edges of D as $e(D)$, then the number of cycles in H is at most $e(D)/g \leq 3n/g$. Because a feedback arc set must contain at least one arc from each cycle in H, there is a collection of edges in D such that each edge belongs to one cycle from H and which is a feedback arc set. Now, for each such edge, we randomly delete one endpoint from D, and the resulting digraph, which has at least $n - 3n/g$ vertices, is acyclic.

For the cases $g = 4, 5$, we now obtain a slightly better bound using the greedy algorithm. Suppose that there are initially f feedback arcs, where $f \leq 3n/g$.

Let d equal the number of feedback arcs minus half the number of vertices. Thus, initially $d = f - n/2$. At each step, we remove the vertex v that is incident to the most feedback arcs, together with all feedback arcs incident to v. As long as $d > 0$, each step removes one vertex and at least two feedback arcs. Such a step decreases d by at least $3/2$. Let m be the number of steps taken before $d \leq 0$, and let $d' \leq 0$ be the final value of d. We conclude that

$$m \leq \frac{f - n/2 - d'}{3/2} = \frac{2f - n - 2d'}{3}.$$

After m steps, the number of vertices remaining is $n - m$, so the number of feedback arcs remaining is $d' + (n-m)/2$. We remove one vertex from each of these feedback arcs to create an acyclic set. The total number of vertices removed is

$$\begin{aligned}
m + d' + \frac{n-m}{2} &= \frac{n}{2} + d' + \frac{m}{2} \\
&\leq \frac{n}{2} + d' + \frac{1}{2} \cdot \frac{2f - n - 2d'}{3} \\
&= \frac{n + f + 2d'}{3} \\
&\leq \frac{n+f}{3} \\
&\leq \frac{n}{3} + \frac{n}{g}.
\end{aligned}$$

The number of vertices remaining in our acyclic set is thus at least

$$\frac{2n}{3} - \frac{n}{g}.$$

This is equal to $5n/12$ for $g = 4$ and $7n/15$ for $g = 5$. $\square$

4 Large acyclic subgraphs of general digraphs

In this section, we improve lower bounds on $\alpha_A(D)$, the size of the maximal acyclic set of a digraph D, by constant factors and analyze when our new bounds are stronger than existing ones.

4.1 Weakly regular digraphs

Recall that a digraph is *weakly regular* if the sums of the out-degree and in-degree of each vertex are all equal. In Theorem 4.1 below, we improve the bound in (2) of Ben-Eliezer et al. by a factor of nearly 10 for weakly regular graphs.

Theorem 4.1. *Suppose D is a weakly regular digraph with n vertices and at most ϵn^2 edges, and $\epsilon \geq 1/n$. Then D contains an acyclic set of size $\frac{A \log n}{\epsilon \log 1/\epsilon}$, where A is any constant strictly less than 1.*

The proof of Theorem 4.1 is a refinement of that of Ben-Eliezer et al. [10] and is provided in Appendix A.

4.2 Bounds for general digraphs

In Theorem 4.2 below, we give a lower bound on the maximal acyclic set of vertices of any digraph. This theorem asymptotically improves upon the bound in (1) by a factor of 2.

Theorem 4.2. *If D is a digraph on n vertices and with average out-degree d, then $\alpha_A(D) \geq \frac{2n}{d+4}$.*

The proof of Theorem 4.2 is provided in Appendix B.

We now compare the lower bound for $\alpha_A(D)$ in Theorem 4.2 to bounds already found, for instance the inequalities (1) and (2). To do so, given a digraph D, we let n denote the

$\alpha_A(D) \geq (1+o(1))n \cdot \frac{\log d}{d}$	Conjecture (Aharoni et al., 2008, [2])
$\alpha_A(D) \geq \frac{1}{10} \cdot \frac{n}{d} \cdot \frac{\log n}{\log n - \log d}$	Ben-Eliezer et al., 2012, [10]
$\alpha_A(D) \geq n/d$	Aharoni et al., 2008, [2]

Table 1: Previous results on the size of the largest acyclic set

number of vertices of D and d denote the average out-degree of D. Using this notation, we summarize previous results in Table 1.

Notice that the bound of Aharoni et al. [2] in Table 1 is stronger than that of Ben-Eliezer et al. [10] when $d < n^{9/10}$. Our bound of $\alpha_A(D) \geq \frac{2n}{d+4}$ is clearly stronger than that of Aharoni et al. for $d \geq 4$ and is stronger than that of Ben-Eliezer et al. when $d < n^{19/20}$, assuming d is asymptotically large. Therefore, except for very dense digraphs, our bound is stronger than all previous bounds.

5 Concluding Remarks

In this paper, we found improved upper bounds on $\chi_A(D)$ and lower bounds on $\alpha_A(D)$ for general digraphs D. However, the bounds in both Theorems 2.1 and 4.2 differ from the conjectured bounds in Conjectures 1.1 and 1.5, respectively, by a factor of $\log d$, where d is linearly related to the average degree of the digraph. It seems that a new technique is necessary to obtain the additional factor of $\log d$. However, it may be possible to improve the bound in Theorem 2.1 by a constant factor simply by refining the methods used in this paper. In particular, Conjecture 1.2 may be within reach using the techniques we used.

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A Proof of Theorem 4.1

Below we give the proof of Theorem 4.1. We first cite the following Lemma A.1, which was proved by Ben-Eliezer et al. [10].

Lemma A.1. *If D is an oriented graph on n vertices, then $\alpha_A(D) \geq \log n$.*

The proof of Theorem 4.1 is modeled off a proof of Ben-Eliezer et al. [10]. In the proof below, given a digraph D, and two subsets of the vertices of D, denoted U and S, we let $N_U^+(S)$ denote the set of all vertices of U which are out-neighbors of at least one vertex of S.

Proof of Theorem 4.1. Let U be a maximum acyclic set in D, and let $T = V \backslash U$ and $|T| = t$. We assume for the purpose of contradiction that

$$t \leq n - \frac{A \log n}{\epsilon \log 1/\epsilon},$$

for some constant A, to be chosen later. Without loss of generality, we may assume that $\sum_{v \in U} d^-(v) \leq \sum_{v \in U} d^+(v)$. Since D is weakly regular, we have that

$$\sum_{v \in U} d^-(v) \leq \sum_{v \in U} \frac{d^+(v) + d^-(v)}{2} < \epsilon n |U|.$$

Now we pick a set of vertices of T such that each vertex in T has a small out-neighborhood into U. Specifically, for each vertex w of T, the out-neighborhood of w in U has size at most $c\epsilon|U|$, for some constant c to be chosen later. The greatest possible number N of vertices in T which can have an out-neighborhood in U of size greater than $\epsilon c|U|$ is given by $N \leq n/c$. So, the set of vertices of T with an out-degree in U of size at most $c\epsilon|U|$, which we denote by T', has size at least $t - N \geq t - n/c$. For each vertex $w \in V(T)$, we define a set of vertices in U, denoted U_w, of size exactly $c\epsilon|U|$ that contains the out-neighborhood of w in U.

The number of subsets of U of size exactly $\epsilon c|U|$ is

$$\begin{aligned}\binom{|U|}{\epsilon c|U|} &\leq \left(\frac{e|U|}{\epsilon c|U|}\right)^{\epsilon c|U|} \\ &= n^{\frac{\log \frac{e}{\epsilon c} cA}{\log 1/\epsilon}}.\end{aligned}$$

Thus, by the pigeonhole principle, there exists a set $S \in T'$ with out-neighborhood in U of size at most $\epsilon c|U|$ and such that

$$|S| \geq \frac{t - n/c}{n^{\frac{\log \frac{e}{\epsilon c} cA}{\log 1/\epsilon}}}.$$

Moreover, by Lemma A.1, there exists an acyclic set in S, which we denote by S', of size at least $\log |S|$. We now claim that $|S' \cup (U \backslash N_U^+(S))| - |U| > 0$, which would complete the proof of Theorem 4.1 since U was maximal. To show this, we note that we want to show that

$$|S' \cup (U \backslash N_U^+(S))| - |U| \geq \log \left(\frac{n(1 - 1/c) - \frac{A \log n}{\epsilon \log 1/\epsilon}}{n^{\frac{\log \frac{e}{\epsilon c} cA}{\log 1/\epsilon}}} \right) - c\epsilon \frac{A \log n}{\epsilon \log 1/\epsilon} > 0. \tag{4}$$

Inequality (4) simplifies to

$$\log \left(n(1 - 1/c) - \frac{A \log n}{\epsilon \log 1/\epsilon} \right) > \frac{cA \log n}{\log 1/\epsilon} \cdot \log \left(\frac{2e}{\epsilon c} \right). \tag{5}$$

Notice that for the logarithms in (5) to be defined, we need $c > 1$. For n large enough, it suffices to show that

$$\log n > \frac{cA \log n}{\log 1/\epsilon} \cdot \log \left(\frac{2e}{\epsilon c} \right). \tag{6}$$

Inequality (6) reduces to

$$1/\epsilon > \left(\frac{2e}{\epsilon c} \right)^{cA},$$

or

$$\epsilon > \left(\frac{2e}{c} \right)^{\frac{cA}{cA - 1}}. \tag{7}$$

At this point, we choose c very slightly larger than 1, and A very slightly less than 1, so that $cA < 1$, and so that (7) holds. Notice that for any ϵ, we can choose A arbitrarily close to 1. $\square$

B Proof of Theorem 4.2

Below we provide the proof of Theorem 4.2. Given a digraph D on n vertices and m edges, we define the *density* of D to be m/n^2.

Proof. Let D be a digraph on n_1 vertices and with average out-degree d_1. We will use a recursive process to build up an acyclic set of vertices: at the ith step, for $i \geq 1$, we will begin with an induced subgraph D_i of D, and an acyclic set H_i. Notice that $D_1 = D$ and $H_1 = \emptyset$. We then add to H_i the vertex $v_i \in V(D_i)$ with smallest in-degree or out-degree in D_i and let D_{i+1} be the subgraph of D_i induced by the vertex set $V(D_i)\backslash(v_i \cup N^-(v_i))$ or $V(D_i)\backslash(v_i \cup N^+(v_i))$, whichever is smaller. We then repeat the process, until the number of vertices in D_i is near 0.

As we begin the ith step, let $n_i = |V(D_i)|$ and the average out-degree of D_i be d_i. We pick the vertex $v_i \in V(D_i)$ with smallest in-degree or out-degree in D_i, whichever is smaller. Suppose without loss of generality that $d^-(v_i) \leq d^+(v_i)$, and let $d^-(v_i) = \delta$. Clearly, $\delta \leq d_i$. Notice that all in-neighbors of v_i must have both in-degree and out-degree greater than or equal to d_i. Since the subgraph induced by $N^-(v_i)$ has at most $\delta^2/2$ edges, the number of edges of D_i incident to $V(D_i)\backslash N^-(v_i)$ is at least

$$2\delta d_i - \delta^2/2 = \delta(2d_i - \delta/2).$$

Thus, since the number of vertices in $V(G)\backslash N^-(v_i)$ is $n_i - \delta - 1$, the density of the subgraph

D_{i+1} induced by $V(G)\backslash N^-(v_i)$, is at most

$$\frac{d_i n_i - \delta(2d_i - \delta/2)}{(n_i - \delta - 1)^2} \leq \frac{d_i(n_i - \frac{3d_i}{2})}{(n_i - d_i - 1)^2},$$

so its average out-degree is at most

$$\frac{d_i(n_i - \frac{3d_i}{2})}{n_i - d_i - 1}.$$

Given any positive integers n, d, we define $f(n,d) = \frac{2n}{d+4}$; we want to show that given a digraph on n vertices and average out-degree d, there exists an acyclic set of size $f(n,d)$. To show that this is true, at each step of the process described above, given v_i, notice that the subgraph D_{i+1} of D_i induced by $V(D_i)\backslash N^-(v_i)$ has at least $n_i - d_i - 1$ vertices and average out-degree at most $\frac{d_i(n_i - \frac{3d_i}{2})}{n_i - d_i - 1}$. Thus, by induction, D_{i+1} has an acyclic set of size at least $f\left(n_i - d_i - 1, \frac{d_i(n_i - \frac{3d_i}{2})}{n_i - d_i - 1}\right)$. So, if f satisfies

$$1 + f\left(n - d - 1, \frac{d(n - \frac{3d}{2})}{n - d - 1}\right) \geq f(n,d), \tag{8}$$

for all n, d, then we can find an acyclic set of size $f(n_i, d_i)$ at the ith step of the process for all i, which would complete the proof of Theorem 4.2. We now prove the validity of (8). To simplify calculations, we make the substitution $A = \frac{2d}{d+4}$ and therefore $f(n,d) = \frac{nA}{d}$. Notice that we must have

$$1 + \frac{A(n-d-1)^2}{d\left(n - \frac{3d}{2}\right)} \geq \frac{nA}{d},$$

which reduces to

$$d\left(n - \frac{3d}{2}\right) \geq A\left(n\left(n - \frac{3d}{2}\right) - (n-d-1)^2\right). \tag{9}$$

Inequality (9) above is equivalent to

$$A\left(\frac{dn}{2}-d^2+2n-2d+1\right)\leq nd-\frac{3d^2}{2},$$

or

$$A\leq\frac{nd-\frac{3d^2}{2}}{\frac{(d+4)n}{2}-d^2-2d+1}. \tag{10}$$

We now substitute $A=\frac{2d}{d+4}$, so (10) becomes

$$\frac{nd-\frac{3d^2}{2}}{\frac{(d+4)n}{2}-d^2-2d+1}\geq 2\cdot\frac{d}{d+4}. \tag{11}$$

Inequality (11) above reduces to

$$-\frac{3d^2}{2}+\frac{2d^3}{d+4}+\frac{4d^2}{d+4}+\frac{2d}{d+4}=\frac{d^2}{2}-\frac{4d^2}{d+4}+\frac{2d}{d+4}\geq 0,$$

which is true as long as $d\geq 0$. Notice that with each step of the process, as long as $d\geq 2$, d decreases but stays positive at each step, completing the proof of the vailidity of (9).

At the end of the process, it simply remains to verify the base case. Since d decreases but stays positive as long as $n>4$ and $d\geq 2$, the base case occurs when $n\leq 4$ or $d<2$. If the base case occurs when $n\leq 4$, it is clearly possible to find an acyclic set of size $2\geq\frac{2n}{d+4}$. Otherwise, the base case occurs when $n>4$ and $d<2$, and we want to find an acyclic set of size $\frac{2n}{d+4}$. Now, note that by the bound in (1) by Aharoni et al. [2], we can find an acyclic set of size $\frac{n}{d+1}$. We have that $\frac{n}{d+1}\geq\frac{2n}{d+4}$ if $d\leq 2$, thus completing the proof. □