Characterization of the Line Complexity of Cellular Automata Generated by Polynomial Transition Rules

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Abstract

Cellular automata are discrete dynamical systems which consist of changing patterns of symbols on a grid. The changes are specified in such a way that the symbol in a given position is determined by the symbols surrounding that position in the previous state. Despite the simplicity of their definition, cellular automata have been applied in the simulation of complex phenomena as disparate as biological systems and universal computers. In this paper, we investigate the line complexity $a_T(k)$, or number of accessible coefficient blocks of length k, for cellular automata arising from a polynomial transition rule T. We first derive recursion formulas for the sequence $a_T(k)$ associated to polynomials of the form $1+x+x^n$ where $n \geq 3$ and the coefficients are taken modulo 2. We then derive functional relations for the generating functions associated to these polynomials. Extending to a more general setting, we investigate the asymptotics of $a_T(k)$ by considering a generating function $\phi(z) = \sum_{k=1}^{\infty} \alpha(k) z^k$ which satisfies a certain general functional equation relating $\phi(z)$ and $\phi(z^p)$ for some prime p. We show that for positive integral sequences s_k which are dependent upon a real number $x \in [1/p, 1]$ and for which $\lim_{k\to\infty} \left(\log_p s_k - \lfloor \log_p s_k \rfloor\right) = \log_p \frac{1}{x}$, the ratio $\alpha(s_k)/s_k^2$ tends to a piecewise quadratic function of x.

Summary

Cellular automata are discrete systems which consist of changing patterns of symbols on a grid. Despite the simplicity of their definition, cellular automata have been employed in applications as disparate as the simulation of biological systems and the modelling of computers. The purpose of this project is to investigate the line complexity, or the number of accessible coefficient strings of a given length, for cellular automata generated by iteratively multiplying polynomials of a specific class and reducing the coefficients modulo 2. We consider the recursive properties of the line complexity sequence, and consider the asymptotic properties of a related sequence in a more general setting.

1 Introduction

A *cellular automaton* is a system which consists of patterns of symbols on a grid. These patterns change at discrete time intervals. A given rule specifies the next state in such a manner that each symbol in the next state is determined by the surrounding symbols in the current state. Von Neumann, who initiated the study of cellular automata, investigated their connections to the modelling of biological systems [1]. Wolfram observes in [2] that although cellular automata are often constructed from identical simple components, they can nonetheless exhibit complex behavior. As Willson notes in [3], cellular automata can be used to model chaotic phenomena because their discrete structure facilitates exact computation. An example of a simple cellular automaton, Pascal's triangle modulo 2, is illustrated in Figure 1.

```
1                1                1                1
                1 1              1 1              1 1
                                1 0 1            1 0 1
                                                1 1 1 1
i=0             i=1              i=2              i=3

        1                1                1
       1 1              1 1              1 1
      1 0 1            1 0 1            1 0 1
     1 1 1 1          1 1 1 1          1 1 1 1
    1 0 0 0 1        1 0 0 0 1        1 0 0 0 1
                    1 1 0 0 1 1      1 1 0 0 1 1
                                    1 0 1 0 1 0 1
        i=4              i=5              i=6
```

Figure 1: Pascal's Triangle modulo 2

A particular state for a cellular automaton is called a *configuration*. A *d-dimensional cellular automaton* is one whose configurations are graphed on a d-dimensional grid. A configuration ω for a one-dimensional cellular automaton can be expressed by its Laurent series

$$\sum_{-\infty}^{\infty} a_i x^i,$$

where the superscripts correspond to the locations of the values a_i. For example, the expression

$1+3x+2x^3+x^7$ represents the string 13020001. A configuration is *finite* if a_i is nonzero for only finitely many integers i.

Given a configuration ω, the *transition rule* R for a cellular automaton determines a new configuration $R\omega$ in such a way that the value at a given index i in $R\omega$ is determined by values near i in ω. An *additive* transition rule is specified by a Laurent polynomial and it acts upon a configuration by multiplication, where the coefficients are taken modulo some prime p. For example, $(1+x)^2 = 1+x^2 \pmod 2$. Applying the transition rule $R = 1+x$ iteratively to the initial state $\omega_0 = 1$ and taking the coefficients modulo 2 produces Pascal's Triangle (see Figure 2).

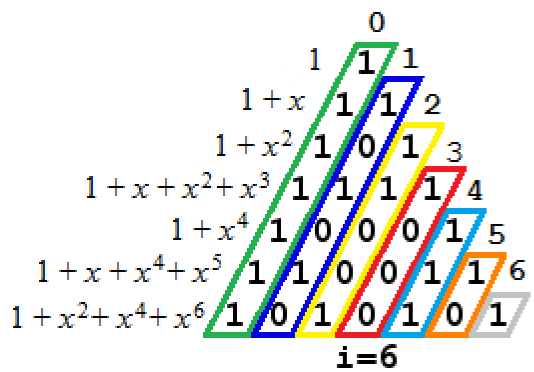

Figure 2: Revisiting Pascal's Triangle modulo 2

Sequences of length k which appear in some configuration are called *k-accessible blocks*. For example, the block 110011 appears in line 5 of the automaton shown in Figure 1, and is thus accessible. Denote by $a_R(k)$ the number of k-accessible blocks for a given transition rule R. This sequence is called the *line complexity* of the automaton. In this paper, we investigate the recursive and asymptotic properties of the sequence $a_T(k)$ for polynomials of the form $T = 1+x+x^n$ with $n \geq 3$ and the coefficients taken modulo 2.

Garbe [4] considered the sequence $a_R(k)$ for $R = 1+x$ with the coefficients taken modulo primes p, as well as $R = 1+x+x^2$ with the coefficients taken modulo small primes p, and investigated the asymptotic behavior of $a_R(k)/k^2$. We consider the case $p = 2$ and explore the more general polynomials described above. For certain sequences $s_k(x)$, where $x \in [1/p, 1]$, we will show that $\lim_{k\to\infty} a_T(s_k(x))/s_k(x)^2$ is a concatenation of quadratic functions of x.

In Section 2 we derive recursion formulas for $a_T(k)$, where T is as above. We consider the even and odd cases separately. In Section 3 we consider the generating functions associated to the sequences $a_T(k)$ for even and odd n, and derive functional relations which will be used later. In Section 4 we develop the asymptotic properties of $a_T(k)/k^2$ in significant generality.

2 Recursion Formulas for $a_T(k)$

Let $a_T(k)$ denote the line complexity of the cellular automaton whose initial state is $\omega_0 = 1$ and whose transition rule is given by $T = 1 + x + x^n$, where $n \geq 3$ and the coefficients are taken modulo 2. In this section, we will derive recursion formulas for the sequence $a_T(k)$.

We will denote by $\mathscr{A}$ the set of accessible blocks. We will use exponents to denote repetitions in strings of coefficients; for example, $10^3 1 = 10001$.

We first consider the case where $n \geq 4$ is even. We will assume that $k \geq n^2/2 + n$. For the purposes of this discussion, we will write $a(k)$ for $a_T(k)$.

We observe that if line r of the automaton (i. e., the configuration which results from applying the transition rule r times to the initial state) is given by $\omega_r = x_0 x_1 \cdots x_k$, where $k = nr$, then line $r+1$ is given by $T\omega_r = x_0(x_0 + x_1)(x_1 + x_2)\cdots(x_{n-2} + x_{n-1})(x_0 + x_{n-1} + x_n)(x_1 + x_n + x_{n+1})\cdots(x_{k-n} + x_{k-1} + x_k)(x_{k-n+1} + x_k)x_{k+2-n}x_{k+3-n}\cdots x_{k-1}x_k$. In view of the identity $T(s^2) = T(s)^2$ for additive transition rules T modulo 2, line $2r$ has the form $x_0 0 x_1 0 \cdots 0 x_k$. It follows that line $2r+1$ has the form $x_0 x_0 x_1 x_1 \cdots x_{n/2-1} x_{n/2-1}(x_0 + x_{n/2})x_{n/2}(x_1 + x_{n/2+1})\cdots(x_{k-n/2} + x_k)x_k x_{k-n/2+1} 0 x_{k-n/2+2} 0 \cdots 0 x_k$.

Let

$$
\begin{aligned}
A &= \{x_0 0 x_1 0 \cdots 0 x_{k-1} 0 \mid x_0 x_1 \cdots x_{k-1} \in \mathscr{A}\} \\
B &= \{0 x_0 0 x_1 0 \cdots 0 x_{k-1} \mid x_0 x_1 \cdots x_{k-1} \in \mathscr{A}\} \\
C &= \{(x_0 + x_{n/2})x_{n/2}(x_1 + x_{n/2+1})\cdots x_{n/2+k-1} \mid x_0 \cdots x_{n/2+k-1} \in \mathscr{A}\} \\
D &= \{x_{n/2}(x_1 + x_{n/2+1})\cdots x_{n/2+k-1}(x_k + x_{n/2+k}) \mid x_1 \cdots x_{n/2+k} \in \mathscr{A}\}.
\end{aligned}
$$

Then the accessible $2k$-blocks, since they must appear either on odd or even rows, belong to the sets A, B, C, or D. In order to determine the number of accessible $2k$-blocks, it is necessary to account for the cardinalities of the sets A, B, C, D, and their intersections. We have

$$
\begin{aligned}
a(2k) = |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\
&- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
&+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|.
\end{aligned}
$$

We observe that, by the definitions given above, $|A| = a(k)$, $|B| = a(k)$, $|C| = a(n/2+k)$, and $|D| = a(n/2+k)$. We observe that all the intersections are nonempty (they contain 0^{2k}).

For convenience of notation, we define $A_i = \{x_0 x_1 \cdots x_{i-1}\} \cap A$, and define B_i, C_i, and D_i similarly. The following lemmas will be used later.

Lemma 1. *We have* $|A \cap B| = |A \cap B \cap C| = |A \cap B \cap D| = |A \cap B \cap C \cap D| = 1$.

Proof. We observe that $|A \cap B| = |\{0^{2k}\}| = 1$. The lemma follows. $\square$

We will now prove two lemmas which will be used in the proof of Lemma 4.

Lemma 2. *The block* $a = 10^d 10^n$ *is not accessible for even* $d \le n/2 - 2$.

Proof. Since d is even, $a \notin A_{n+d+2}, B_{n+d+2}$. Suppose $a \in C_{n+d+2}$ (the case $a \in D_{n+d+2}$ is similar). If $w_0 w_1 \cdots w_{n+d/2} \in \mathscr{A}$, we have $1 = w_0 + w_{n/2}$, $\ldots$, $1 = w_{n/2+d/2}$ and $w_{n/2+d/2+i} = 0$, $w_{d/2+i} + w_{n/2+d/2+i} = 0$ for $1 \le i \le n/2$. Take $i = n/2$. Then $w_{n+d/2} = 0$ and $w_{n/2+d/2} + w_{n+d/2} = 0$, so that $w_{n/2+d/2} = 0$. This contradicts the assumption that $w_{n/2+d/2} = 1$. $\square$

Lemma 3. *Let* $d, m \ge 0$ *be integers. If* $b = 10^{2d+1} 10^{2m+1} \in \mathscr{A}$, *then* $10^d 10^m \in \mathscr{A}$.

Proof. If b appears on an even row r, then $10^d 10^m$ appears on row $r/2$. If, on the other hand, b appears on an odd row, we can write $1 = z_0 + z_{n/2}$, $0 = z_{n/2}$, $\ldots$, $0 = z_d + z_{n/2+d}$, $0 = z_{n/2+d}$, $1 = z_{d+1} + z_{n/2+d+1}$, $0 = z_{n/2+d+1}$, $\ldots$, $0 = z_{s+m+1} + z_{n/2+s+m+1}$, $0 = z_{n/2+s+m+1}$. This implies that $z_0 = z_{d+1} = 1$ and $z_i = 0$ for all other i. Thus, $z_0 z_1 \cdots z_{n/2+s+m+1} = 10^d 10^{m+1}$.
Since $z_0 z_1 \cdots z_{n/2+s+m+1} \in \mathscr{A}$ by hypothesis, $10^d 10^m \in \mathscr{A}$. $\square$

Lemma 4. *Suppose $k \geq n^2/2+n$. Then $|A \cap C| = n/2+1$.*

Proof. Strings in $A \cap C$ consist of numbers which satisfy the equations $x_0 = y_0 + y_{n/2}$, $0 = y_{n/2}$, $x_1 = y_1 + y_{n/2+1}$, $0 = y_{n/2+1}, \ldots, x_{k-1} = y_{k-1} + y_{n/2+k-1}$, $0 = y_{n/2+k-1}$. It follows that $x_i = y_i$ for $n/2 \leq i \leq k-1$. We will show that it is impossible to have $x_i = x_j = 1$ for $1 \leq i, j \leq n/2-1$ and $i \neq j$.

We now claim that the block described above can be expressed in the form

$$c = x_0 x_1 \cdots x_{i-1} x_i 0^{2^p(d+1)-1} x_j 0^{2^p(n+1)-1},$$

where x_i and x_j are the last two occurrences of the digit 1, $p \geq 0$ is an integer, and d is even. For this we must have a sufficient number of terminal zeros, and hence k must be sufficiently large. In particular, we must have $|c| \leq n/2+k$. We have $2^p \leq 2^p(d+1) \leq n/2$ and we must have $k \geq 2^p(n+1)-1$. Combining these inequalities gives $k \geq \frac{n}{2}(n+1)-1$. It is true by hypothesis that $k \geq n^2/2+n$, which is sufficient. Given the form of the block c, we now may iteratively construct the inaccessible block a from Lemma 2, leading to a contradiction in view of Lemma 3. Thus, the digit 1 must appear at most once in $x_0 x_1 \cdots x_{n/2-1}$. There are $n/2+1$ possible ways this could occur. Thus, $|A \cap C| = n/2+1$. The lemma follows. $\square$

Lemma 5. *Suppose $k \geq n^2/2+n$. Then $|B \cap D| = n/2+1$.*

Proof. We have the equations $0 = y_{n/2}$, $x_0 = y_1 + y_{n/2}$, $0 = y_{n/2+1}$, $x_1 = y_2 + y_{n/2+2}$, $0 = y_{n/2+2}, \ldots,$ $0 = y_{n/2+k-1}$, $x_{k-1} = y_k + y_{n/2+k}$, which imply that $y_i = 0$ for $n/2 \leq i \leq n/2+k-1$. As in the case of $A \cap C$, we know that the number 1 can appear at most once in $y_1 \cdots y_{n/2-1}$. We must consider the case of $y_{n/2+k}$. We therefore consider the accessibility of the blocks $y_1 y_2 \cdots y_{n/2-1} 0 \cdots 0 y_{n/2+k}$ and $x_0 x_1 \cdots x_{n/2-2} 0 \cdots 0 x_{k-1}$. We note that there are k and $k-n/2$ zeros, respectively. If $x_{k-1} = y_{n/2+k} = 1$ and two of the other digits (e.g., $x_0 = y_1$) are also 1, then there are $k+L$ and $k+L-n/2$ zeros, where $0 \leq L \leq n/2-2$. If $n/2$ is odd, then either $k+L$ or $k+L-n/2$ is even, leading to a contradiction as in the case of $A \cap C$. If $n/2$ is even, then either $k+L$ and $k+L-n/2$ are both even, leading to a contradiction, or both odd. In this case, we note that these blocks arise from blocks with $\frac{k+L-1}{2}$ and $\frac{k+L-1}{2}-\frac{n}{4}$ zeros, respectively. We iterate this process until the number of zeros is even. We thus have $n/2-1$ cases as in the case of $A\cap C$, as well as the trivial case and the case $0\cdots 01$. It follows that $|B\cap D| = n/2+1$. □

Lemma 6. *Suppose $k \geq n^2/2+n$. Then $|A\cap D| = |B\cap C| = |A\cap C\cap D| = |B\cap C\cap D| = 1$.*

Proof. Consider the set $A\cap D$. We have $x_0 = y_{n/2}$, $0 = y_1 + y_{n/2+1}$, $x_1 = y_{n/2+1}$, $\ldots$, $x_{k-1} = y_{n/2+k-1}$, $0 = y_k + y_{n/2+k}$, so that $y_i = y_{n/2+i} = x_i$ for $1 \leq i \leq k-1$. By the periodicity relations, the elements of $A\cap D$ arise from strings of the form $(x_0\cdots x_{n/2-1})^d$ for some $d \geq 0$. Suppose $(x_0\cdots x_{n/2-1})^d \in A'$, where the lengths of elements in A' are $dn/2$ (the case of B' is similar). If $n/2$ is odd, then $x_1 = x_3 = \cdots = x_{n/2-2} = x_0 = 0$, so that $x_0 = x_{n/2} = 0$, $x_2 = x_{n/2+2} = 0$, etc. Thus, $x_i = 0$ for all i. If $n/2$ is even, we apply the same argument to the block $(x_0x_2\cdots x_{n/2-2})^d$. If $(x_0\cdots x_{n/2-1})^d \in C'$ (defined similarly), then we have $x_0 = z_0 + z_{n/2}$, $x_1 = z_{n/2}$, $x_2 = z_1 + z_{n/2+1}$, etc. for some $z_0, z_1, \ldots$. We have $z_i = x_{2i} + x_{2i+1}$ for all i. Thus,

$$z_{n/2+i} = x_{n+2i} + x_{n+2i+1} = x_{2i} = x_{2i+1} = z_i,$$

so that $x_{2i} = z_i + z_{n/2} + i = 0$. This essentially reduces to the previous case. The case of D' is similar. We require $d \geq n+2$. Since $dn/2 = k$, $k \geq n^2/2+n$ is sufficient. Thus, $|A\cap D| = 1$. It follows by similar reasoning that $|B\cap C| = 1$, and hence that $|A\cap C\cap D| = |B\cap C\cap D| = 1$. □

Lemma 7. *We have $|C\cap D| = a(n)$.*

Proof. We have the systems of equations

$$\begin{cases} x_i + x_{n/2+i} = y_{n/2+i} \\ x_{n/2+i} = y_{i+1} + y_{n/2+i+1}, \end{cases}$$

where $0 \leq i \leq k-1$, as well as $x_k + x_{n/2+k} = y_{n/2+k}$. It follows by adding the pairs of equations that $x_i = y_{i+1} + y_{n/2+i} + y_{n/2+i+1}$. In particular, if $1 \leq i \leq k-1$, we also have $x_i + x_{n/2+i} + x_{n/2+i-1} = y_{n/2+i} + y_i + y_{n/2+i} = y_i$, which implies that $y_{n/2+i} = x_{n/2+i} + x_{n-1+i} + x_{n+i}$. It follows that $x_{n/2+i} + x_{n-1+i} + x_{n+i} = x_i + x_{n/2+i}$. Hence $x_{n+i} = x_{n+i-1} + x_i$. The numbers x_i are thus all determined by $x_0, \ldots, x_{n-1}$. It follows that $|C\cap D| \leq a(n)$.

We claim that $|C \cap D| \geq a(n)$. We will show this in the case of n even; the proof should be similar if n is odd.

We must show that for any $m \geq n-1$, there are at least $a(n)$ blocks of the form $x_0 x_1 \cdots x_m$, where $x_0 x_1 \cdots x_{n-1} \in \mathscr{A}$ and

$$x_k = x_{k-1} + x_{k-n} \tag{1}$$

for $n \leq k \leq m$. This is clearly true for $m = n-1$. Now suppose this is the case for some $m \geq n-1$. We will proceed by induction.

Let $x_0 x_1 \cdots x_m$ be an accessible block in row r, say, that satisfies equation (1). Then there is a block of the form $0x_0 0x_1 0 \ldots 0x_m 0$ in row $2r$. It follows that line $2r+1$ contains a block of the form

$$x_{n/2-1}(x_0 + x_{n/2}) x_{n/2} (x_1 + x_{n/2+1}) \cdots (x_{m-n/2} + x_m) x_m.$$

Write $y_0 y_1 \cdots y_{2m-n+2}$ for this block, that is,

$$y_{2i} = x_{i-1+n/2} \text{ for } i = 0, 1, \ldots, m - n/2 + 1,$$
$$y_{2i+1} = x_i + x_{i+n/2} \text{ for } i = 0, 1, \ldots, m - n/2.$$

We claim that $y_0 y_1 \cdots y_{2m-n+2}$ satisfies (1). For, suppose that $k \geq n/2$. Then $y_{2k} = x_{k-1+n/2}$, $y_{2k-1} = x_{k-1} + x_{k-1+n/2}$, and $y_{2k-n} = x_{k-1}$, so that $y_{2k} = y_{2k-1} + y_{2k-n}$. Moreover, we have $y_{2k+1} = x_k + x_{k+n/2}$, $y_{2k} = x_{k-1+n/2}$, $y_{2k-n+1} = x_{k-n/2} + x_k$, so that $y_{2k+1} - y_{2k} - y_{2k-n+1} = x_{k+n/2} - x_{k+n/2-1} - x_{k-n/2}$, and the last expression is 0, by (1).

It follows that $y_0 y_1 \cdots y_{2m-n+2}$ is determined by $y_0 y_1 \cdots y_{n-1}$. Since the map that takes $x_0 \cdots x_{n-1}$ to $y_0 \cdots y_{n-1}$ is injective, the $x_0 \cdots x_m$ are mapped to different $y_0 \cdots y_{2m-n+2}$, so that the number of accessible blocks of the form $y_0 \cdots y_{2m-n+2}$ is at least $a(n)$ by the induction hypothesis. Since $2m - n + 2 \geq m + 1$, the claim holds for $m+1$.

It follows that $|C \cap D| = a(n)$. $\square$

Theorem 1. *Suppose $n \geq 4$ is even. Then, given $k \geq n^2/2+n$ and the base cases $a_T(1)$, $a_T(2)$, $\ldots$, $a_T(n^2/2+n-1)$, we have*

$$a_T(2k) = 2a_T(k) + 2a_T(n/2+k) - a_T(n) - n - 2$$

and

$$a_T(2k+1) = a_T(k) + a_T(k+1) + a_T(n/2+k) + a_T(n/2+k+1) - a_T(n) - n - 2.$$

Proof. We recall that $a(2k) = a(k) + a(k) + a(n/2+k) + a(n/2+k) - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$ from the argument described above, and apply the lemmas. The formula for $a(2k)$ follows. The formula for $a(2k+1)$ follows by similar reasoning. $\square$

The following theorem for odd n is obtained by similar reasoning.

Theorem 2. *Suppose $n \geq 3$ is odd. Then, given $k \geq \frac{n(n+3)}{2} = n^2/2 + 3n/2$ and the base cases $a_T(1)$, $a_T(2)$, $\ldots$, $a_T(n^2/2+3n/2-1)$, we have*

$$a_T(2k) = 2a_T(k) + a_T\left(\frac{n-1}{2}+k\right) + a_T\left(\frac{n+1}{2}+k\right) - a_T(n) - n - 2$$

and

$$a_T(2k+1) = a_T(k) + a_T(k+1) + 2a_T\left(\frac{n+1}{2}+k\right) - a_T(n) - n - 2.$$

3 Generating Functions

In this section we will investigate generating functions for the sequences $a_T(k)$. We will derive functional equations which will be useful in the next section. The odd and even cases will be considered separately, as in the previous section.

For complex z with $|z| < \frac{1}{2}$ and for $n \geq 4$ even, write $N = n^2/2 + n$ and define

$$f_n(z) = \sum_{k=2N}^{\infty} a_T(k) z^k.$$

Note that $a_T(k) \leq 2^k$, so that the right-hand expression is defined.

Theorem 3. *Let $0 < |z| < \frac{1}{2}$ and let $n \geq 4$ be even. Then*

$$f_n(z) = P_n(z) - \frac{z^{2N}(a_T(n)+n+2)}{1-z} + \frac{1}{z^{n+1}}(1+z^n)(1+z)^2 f_n(z^2),$$

where $P_n(z)$ is a polynomial.

Proof. We will again write $a(k)$ for $a_T(k)$ in the proof. We have

$$f_n(z) = \sum_{k=2N}^{\infty} a(k) z^k = \sum_{k=N}^{\infty} a(2k) z^{2k} + z \sum_{k=N}^{\infty} a(2k+1) z^{2k}.$$

Applying Theorem 1 gives

$$\begin{aligned} f_n(z) = & \sum_{k=N}^{\infty} (2a(k) + 2a(n/2+k)) z^{2k} \\ & + z \sum_{k=N}^{\infty} (a(k+1) + a(k) + a(n/2+k) + a(n/2+k+1)) z^{2k} \\ & - (1+z) \sum_{k=N}^{\infty} (a(n)+n+2) z^{2k}. \end{aligned}$$

Therefore,

$$\begin{aligned} f_n(z) = & 2\sum_{k=N}^{\infty} a(k) z^{2k} + \frac{2}{z^n} \sum_{k=N+n/2}^{\infty} a(k) z^{2k} + \frac{1}{z} \sum_{k=N+1}^{\infty} a(k) z^{2k} \\ & + z \sum_{k=N}^{\infty} a(k) z^{2k} + \frac{1}{z^{n-1}} \sum_{k=N+n/2}^{\infty} a(k) z^{2k} + \frac{1}{z^{n+1}} \sum_{k=N+n/2+1}^{\infty} a(k) z^{2k} \end{aligned}$$

$$-(1+z)(a(n)+n+2)\left(\frac{1}{1-z^2}-\sum_{k=0}^{N-1} z^{2k}\right).$$

Collecting terms, we have

$$\begin{aligned}
f_n(z) =&2\sum_{k=N}^{2N-1} a(k)z^{2k}+\frac{2}{z^n}\sum_{k=N+n/2}^{2N-1} a(k)z^{2k}+\frac{1}{z}\sum_{k=N+1}^{2N-1} a(k)z^{2k}\\
&+z\sum_{k=N}^{2N-1} a(k)z^{2k}+\frac{1}{z^{n-1}}\sum_{k=N+n/2}^{2N-1} a(k)z^{2k}+\frac{1}{z^{n+1}}\sum_{k=N+n/2+1}^{2N-1} a(k)z^{2k}\\
&+\left(2+\frac{2}{z^n}+\frac{1}{z}+z+\frac{1}{z^{n-1}}+\frac{1}{z^{n+1}}\right)f_n(z^2)-\frac{(1+z)(a(n)+n+2)}{1-z^2}\\
&+(a(n)+n+2)\sum_{k=0}^{2N-1} z^k\\
=&P_n(z)-\frac{z^{2N}(a(n)+n+2)}{1-z}+\frac{1}{z^{n+1}}(1+z^n)(1+z)^2 f_n(z^2),
\end{aligned}$$

where

$$\begin{aligned}
P_n(z) =&2\sum_{k=N}^{N+n/2-1} a_T(k)z^{2k}+\left(2+\frac{2}{z^n}\right)\sum_{k=N+n/2}^{2N-1} a_T(k)z^{2k}\\
&+\left(z+\frac{1}{z}\right)\sum_{k=N+1}^{2N-1} a_T(k)z^{2k}+\left(\frac{1}{z^{n-1}}+\frac{1}{z^{n+1}}\right)\sum_{k=N+n/2+1}^{2N-1} a_T(k)z^{2k}\\
&+(a_T(N)+a_T(N+n/2))z^{2N+1}.
\end{aligned}$$

□

For complex z with $|z|<\frac{1}{2}$ and for $n\geq 3$ odd, write $M=\frac{n(n+3)}{2}$ and define

$$f_n(z)=\sum_{k=2M}^{\infty} a_T(k)z^k.$$

By reasoning similar to that above, we obtain the following theorem.

Theorem 4. *Let $0 < |z| < \frac{1}{2}$ and let $n \geq 3$ be odd. Then*

$$f_n(z) = Q_n(z) - \frac{z^{2M}(a_T(n)+n+2)}{1-z} + \frac{1}{z^{n+1}}(1+z^n)(1+z)^2 f_n(z^2),$$

where $Q_n(z)$ is a polynomial.

4 Asymptotic Behavior of $a_T(k)/k^2$

For the discussion of asymptotic behavior, we will work in a much more general framework. We will consider a generating function ϕ which is assumed to satisfy a general functional equation. The main result we derive in this section will include the case of $a_T(k)$.

We shall require the following facts about power series.

(a) If R is the radius of convergence of the power series, then in $|z| < R$ the sum of the series is analytic and its derivative has the same radius of convergence (see [5], Ch. 2, §2.4, Theorem 2(iii)).

(b) If f has a power series development in a disk, then the coefficients are uniquely determined (see [5], p. 40).

Let p be prime and let $D = \{z \in \mathbb{C} \mid |z| < 1/p\}$. For $|z| < 1$, let $\frac{1}{\lambda(z)} = \sum_{k=0}^{\infty} \gamma(k) z^k$, where $\gamma(0) = 1$ and $\gamma(k) = Ck^2 + f(k)$, where $C > 0$ is constant and $\lim_{k\to\infty} \frac{f(k)\log_p k}{k^2} = 0$. Let $\phi : D \to \mathbb{C}$ be a function given by the power series expression

$$\phi(z) = \sum_{k=1}^{\infty} \alpha_k z^k,$$

where $\alpha_k \leq p^k$, and assume that ϕ satisfies

$$\lambda(z)\phi(z) = R(z) + \lambda(z^p)\phi(z^p),$$

where $R : \mathbb{C} \to \mathbb{C}$ is a polynomial with $R(1) = 0$. Note that $R(0) = 0$.

Proposition 1. *We have*

$$\phi(z) = \frac{1}{\lambda(z)} \sum_{k=0}^{\infty} R\left(z^{p^k}\right).$$

Proof. We have that $\lambda(z)\phi(z) = R(z) + \lambda(z^p)\phi(z^p)$. Iterating this equation gives $\lambda(z)\phi(z) = R(z) + R(z^p) + \cdots + R\left(z^{p^k}\right) + \lambda\left(z^{p^{k+1}}\right)\phi\left(z^{p^{k+1}}\right)$. We now note that since λ and ϕ are analytic and hence continuous in D, we have

$$\lim_{k\to\infty} \lambda\left(z^{p^{k+1}}\right)\phi\left(z^{p^{k+1}}\right) = r(0)\phi(0) = 0.$$

Thus, $\lambda(z)\phi(z) = \sum_{k=0}^{\infty} R\left(z^{p^k}\right)$. The result follows. $\square$

We now use the above proposition to develop an explicit formula for the coefficients α_k for sufficiently large k.

Theorem 5. *Set $m = \deg R$, and write $R(z) = \sum_{j=1}^{m} c_j z^j$. Then for $k \geq m$,*

$$\alpha_k = \sum_{j=1}^{m} \sum_{t=0}^{\left\lfloor \log_p \frac{k}{j} \right\rfloor} c_j \gamma(k - jp^t).$$

Proof. We have

$$\phi(z) = \sum_{q=0}^{\infty} \gamma(q) z^q \sum_{k=0}^{\infty} R\left(z^{p^k}\right) = \sum_{j=1}^{m} \sum_{q=0}^{\infty} \gamma(q) z^q \sum_{k=0}^{\infty} z^{j\cdot p^k}.$$

We now note that $|f(q)| < q^2$, for otherwise we would not have $f(q)(\log_p q)/q^2 \to 0$. Thus

$$\sum_{q=0}^{\infty} |\gamma(q)||z|^q \leq \sum_{q=0}^{\infty} \left(\frac{Cq^2}{p^q} + \frac{|f(q)|}{p^q}\right) \leq (C+1) \sum_{q=1}^{\infty} \frac{q^2}{p^q}.$$

It follows that the series $\sum_{q=0}^{\infty} \gamma(q) z^q$ is absolutely convergent. We thus may form the Cauchy product of the series $\sum_{q=0}^{\infty} \gamma(q) z^q$ and $\sum_{k=0}^{\infty} z^{jp^k}$ as follows (see [6], Theorem 3.50):

$$\phi(z) = \sum_{j=1}^{m} c_j \sum_{k=1}^{\infty} \sum_{i=0}^{k} \gamma(k-i) b_{i,j} z^k,$$

where

$$b_{i,j} = \begin{cases} 1 \text{ if } i = jp^t \text{ for some integer } t \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Write $S = \{t \in \mathbb{N} \cup \{0\} | jp^t \leq k\}$. It follows that

$$\sum_{k=1}^{\infty} \alpha_k z^k = \sum_{k=1}^{m-1} \left(\sum_{j=1}^{m} \sum_{i=0}^{k} c_j \gamma(k-i) b_{i,j} \right) z^k + \sum_{k=m}^{\infty} \left(\sum_{j=1}^{m} \sum_{t \in S} c_j \gamma(k-i) b_{i,j} \right) z^k.$$

Thus for $k \geq m$, $\alpha_k = \sum_{j=1}^{m} \sum_{t \in S} c_j \gamma(k - jp^t) = \sum_{j=1}^{m} \sum_{t=0}^{\left\lfloor \log_p \frac{k}{j} \right\rfloor} c_j \gamma(k - jp^t)$ (if $k \geq m$, then $S \neq \emptyset$). This completes the proof. $\square$

Remark 1. *Define $r_n(z) = (1-z^n)(1-z)^2$ for all $z \in \mathbb{C}$, $n \in \mathbb{N}$. We note that in the case of the cellular automaton with transition rule $1 + x + x^n$ for even $n \geq 4$, we may take $p = 2$, $\phi_n(z) = f_n(z)/z^{n+1}$, and $R_n(z) = \frac{r_n(z)}{z^{n+1}} \left(P_n(z) - \frac{z^{2N}(a_T(n)+n+2)}{1-z} \right)$.*

Proposition 2. *Write $\frac{1}{r(z)} = \frac{1}{r_n(z)} = \sum_{k=0}^{\infty} \eta(k) z^k$. Then*

$$\eta(k) = \left(1 + \left\lfloor \frac{k}{n} \right\rfloor \right) \left(k + 1 - \frac{n}{2} \left\lfloor \frac{k}{n} \right\rfloor \right).$$

Proof. We observe that

$$\frac{1}{r(z)} = \frac{1}{1-z^n} \frac{1}{(1-z)^2} = \sum_{k=0}^{\infty} \beta_k z^k \sum_{q=0}^{\infty} (q+1) z^q,$$

where

$$\beta_k = \begin{cases} 1 \text{ if } k = nv \text{ for some integer } v \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

We note that the first series is dominated by the geometric series and is hence absolutely convergent. We again form the Cauchy product, obtaining

$$\frac{1}{r(z)} = \sum_{k=0}^{\infty}\sum_{q=0}^{k}(k-q+1)\beta_q z^k = \sum_{k=0}^{\infty}\sum_{v=0}^{\lfloor k/n\rfloor}(k-nv+1)z^k.$$

Thus, $\eta(k) = \sum_{v=0}^{\lfloor k/n\rfloor}(k-nv+1) = (1+\lfloor k/n\rfloor)(k+1-\frac{n}{2}\lfloor k/n\rfloor)$. This completes the proof. $\square$

Remark 2. *If ϕ is the generating function for a cellular automaton with line complexity $a_R(k)$, then $a_R(k+M) = \alpha(k) \equiv \alpha_k$ for some M. We may thus consider the asymptotic behavior of $\alpha(k)/k^2$ to determine that of $a_R(k)/k^2$. For example, if T is as in Theorem 1, then $a_T(k+n+1) = \alpha(k)$.*

Remark 3. *Write $\delta(k) = k\left\lfloor\frac{k}{n}\right\rfloor - \frac{n}{2}\left\lfloor\frac{k}{n}\right\rfloor^2 - \frac{k^2}{2n}$. Observe that $\delta(k+n) = \delta(k)$, and if $0 \le k \le n$, then $-\frac{n}{2} \le \delta(k) \le 0$. Thus $\delta(k) = O(1)$. Since $\eta(k) = \frac{k^2}{2n} + \left(k+1-\frac{n}{2}\left\lfloor\frac{k}{n}\right\rfloor + \left\lfloor\frac{k}{n}\right\rfloor\right) + \delta(k)$, we have*

$$\eta(k) = \frac{k^2}{2n} + O(k).$$

We now turn to the main result regarding the asymptotic behavior of $\alpha(k)/k^2$. For $y \in \mathbb{R}$, we will denote the *fractional part* of y by $\langle y\rangle = y - \lfloor y\rfloor$.

Theorem 6. *For $\frac{1}{p} \le x \le 1$, let $\{s_k(x)\}$ be a sequence such that $s_k(x) \equiv s_k \in \mathbb{N}$, $s_k \to \infty$, and $\lim_{k\to\infty}\langle\log_p s_k\rangle = \left\langle\log_p \frac{1}{x}\right\rangle = \log_p \frac{1}{x}$. Then*

$$\lim_{k\to\infty}\frac{\alpha(s_k)}{s_k^2} = C\sum_{j=1}^{\deg R} c_j\left(\frac{\sigma_j(p)^2}{p^2-1}x^2 + \frac{2\sigma_j(p)}{1-p}x - \lfloor\log_p j\rfloor - \varepsilon_j\right),$$

where

$$\varepsilon_j = \begin{cases} 1 \text{ if } \log_p \frac{1}{x} < \langle\log_p j\rangle \\ 0 \text{ otherwise,}\end{cases}$$

and

$$\sigma_j(p) = p^{1+\langle\log_p j\rangle - \varepsilon_j}.$$

Proof. By Theorem 5, we have

$$\begin{aligned}\frac{\alpha(s_k)}{s_k^2} &= \sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \left(\frac{C(s_k - jp^t)^2}{s_k^2} + \frac{f(s_k - jp^t)}{s_k^2} \right) \\ &= \sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \frac{C(s_k - jp^t)^2}{s_k^2} + O\left(\frac{f(s_k)\log_p s_k}{s_k^2} \right).\end{aligned}$$

Thus,

$$\lim_{k\to\infty} \frac{\alpha(s_k)}{s_k^2} = \lim_{k\to\infty} \sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \frac{C(s_k - jp^t)^2}{s_k^2} = \lim_{k\to\infty} C \sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \left(1 - \frac{2jp^t}{s_k} + \frac{j^2 p^{2t}}{s_k^2} \right)$$

$$= \lim_{k\to\infty} \left(C \sum_{j=1}^{m} c_j \left(\left\lfloor \log_p \frac{s_k}{j} \right\rfloor + 1 \right) - C \sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \frac{2jp^t}{s_k} + C \sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \frac{j^2 p^{2t}}{s_k^2} \right).$$

We note that

$$C \sum_{j=1}^{m} c_j \left(\left\lfloor \log_p \frac{s_k}{j} \right\rfloor + 1 \right) = C \sum_{j=1}^{m} c_j \left(1 - \left\langle \log_p \frac{s_k}{j} \right\rangle + \log_p s_k - \log_p j \right).$$

Since $R(1) = 0$, we have $\sum_{j=1}^{m} c_j = 0$. The above expression reduces to

$$-C \sum_{j=1}^{m} c_j \left\langle \log_p \frac{s_k}{j} \right\rangle - C \sum_{j=1}^{m} c_j \log_p j.$$

We have

$$\left\langle \log_p \frac{s_k}{j} \right\rangle = \begin{cases} \langle \log_p s_k \rangle - \langle \log_p j \rangle & \text{if } \langle \log_p s_k \rangle \geq \langle \log_p j \rangle \\ \langle \log_p s_k \rangle - \langle \log_p j \rangle + 1 & \text{otherwise,} \end{cases}$$

so that for all sufficiently large k, $\left\langle \log_p \frac{s_k}{j} \right\rangle = \langle \log_p s_k \rangle - \langle \log_p j \rangle + \varepsilon_j$ (for example, to show that

$\log_p \frac{1}{x} < \langle \log_p j \rangle$ implies $\langle \log_p s_k \rangle < \langle \log_p j \rangle$ for sufficiently large k, take K to be so large that

$$\left| \langle \log_p s_k \rangle - \log_p \frac{1}{x} \right| < \langle \log_p j \rangle - \log_p \frac{1}{x}$$

for all $k > K$). Therefore,

$$\begin{aligned}
\lim_{k\to\infty} C \sum_{j=1}^{m} c_j \left(\left\lfloor \log_p \frac{s_k}{j} \right\rfloor + 1 \right) &= -\lim_{k\to\infty} C \sum_{j=1}^{m} c_j \langle \log_p s_k \rangle + C \sum_{j=1}^{m} c_j \left(\langle \log_p j \rangle - \varepsilon_j - \log_p j \right) \\
&= -C \sum_{j=1}^{m} c_j \log_p \frac{1}{x} + C \sum_{j=1}^{m} c_j \left(\langle \log_p j \rangle - \log_p j - \varepsilon_j \right) \\
&= -C \sum_{j=1}^{m} c_j \left(\lfloor \log_p j \rfloor + \varepsilon_j \right).
\end{aligned}$$

We have that

$$-\sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \frac{2jp^t}{s_k} = -2 \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \frac{1}{p^{\log_p \frac{s_k}{j}}} p^t = \frac{2}{p^{\log_p \frac{s_k}{j}}} \frac{p^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor + 1} - 1}{1-p}$$

$$= \frac{2}{1-p} \left(p^{-\left\langle \log_p \frac{s_k}{j} \right\rangle + 1} - p^{-\log_p \frac{s_k}{j}} \right) = \frac{2}{1-p} \left(p^{-\langle \log_p s_k \rangle} p^{\langle \log_p j \rangle} p^{1-\varepsilon_j} - \frac{j}{s_k} \right).$$

Thus,

$$-\lim_{k\to\infty} C \sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \frac{2jp^t}{s_k} = C \sum_{j=1}^{m} \frac{2c_j}{1-p} p^{-\log_p \frac{1}{x}} p^{1+\langle \log_p j \rangle - \varepsilon_j} = C \sum_{j=1}^{m} \frac{2c_j \sigma_j(p)}{1-p} x.$$

Similarly,

$$\sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \left(\frac{j}{s_k}\right)^2 \left(p^2\right)^t = \frac{1}{p^{2\log_p \frac{s_k}{j}}} \frac{1-p^{2\left\lfloor \log_p \frac{s_k}{j} \right\rfloor+2}}{1-p^2}$$
$$= \frac{1}{1-p^2}\left(p^{-2\log_p \frac{s_k}{j}} - p^{2-2\left\langle \log_p \frac{s_k}{j} \right\rangle}\right)$$
$$= \frac{1}{1-p^2}\left(\frac{j^2}{s_k^2} - p^{2-2\left\langle \log_p s_k \right\rangle+2\left\langle \log_p j \right\rangle-2\varepsilon_j}\right).$$

Thus,

$$\lim_{k\to\infty} C\sum_{j=1}^{m} c_j \sum_{t=0}^{\left\lfloor \log_p \frac{s_k}{j} \right\rfloor} \left(\frac{j}{s_k}\right)^2 p^{2t} = -C\sum_{j=1}^{m} \frac{c_j}{1-p^2} p^{2\log_p x} p^{2+2\left\langle \log_p j \right\rangle-2\varepsilon_j}$$
$$= C\sum_{j=1}^{m} \frac{c_j \sigma_j(p)^2}{p^2-1} x^2.$$

The theorem follows. □

Corollary 1. *If T is as in Theorem 1, $\frac{1}{2} \le x \le 1$, and $R_n(z) = \sum_{j=1}^{\deg R_n} v_j z^j$, then*

$$\lim_{k\to\infty} \frac{a_T(s_k)}{s_k^2} = \frac{1}{2n} \sum_{j=1}^{\deg R_n} v_j \left(\frac{\sigma_j(2)^2}{3} x^2 - 2\sigma_j(2)x - \lfloor \log_2 j \rfloor - \varepsilon_j\right).$$

Example 1. Let $T = 1 + x + x^3$. Then if we denote the limit function above by $f_3(x)$, we obtain the following representation (using a computer)

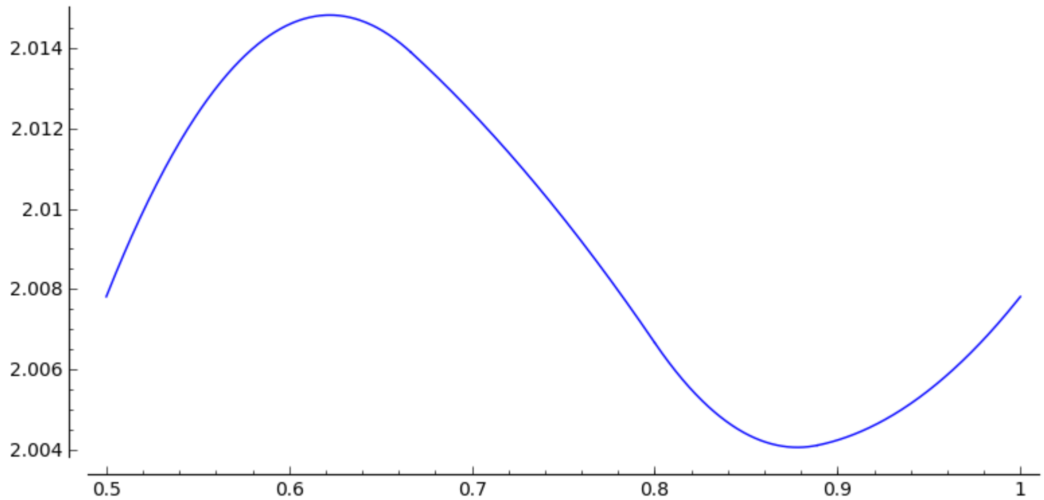

Figure 3: Plot of the limit function for the case $n = 3$

$$f_3(x) = \begin{cases} -\dfrac{15}{32}x^2 + \dfrac{7}{12}x + \dfrac{11}{6} & \text{if } \dfrac{1}{2} \le x < \dfrac{2}{3} \\[2ex] -\dfrac{3}{32}x^2 + \dfrac{1}{12}x + 2 & \text{if } \dfrac{2}{3} \le x < \dfrac{4}{5} \\[2ex] \dfrac{41}{96}x^2 - \dfrac{3}{4}x + \dfrac{7}{3} & \text{if } \dfrac{4}{5} \le x < \dfrac{8}{9} \\[2ex] \dfrac{83}{384}x^2 - \dfrac{3}{8}x + \dfrac{13}{6} & \text{if } \dfrac{8}{9} \le x \le 1. \end{cases}$$

We note that the maximum and minimum of f_3 are $\frac{272}{135}$ and $\frac{493}{246}$, respectively. Thus,

$$\limsup_{k\to\infty} \frac{a_T(k)}{k^2} = \frac{272}{135}$$

and

$$\liminf_{k\to\infty} \frac{a_T(k)}{k^2} = \frac{493}{246}.$$

See Figure 3 for a graph of f_3. See Figure 4 for an illustration of the convergence of $a_T(s_k)/s_k^2$ to f_3.

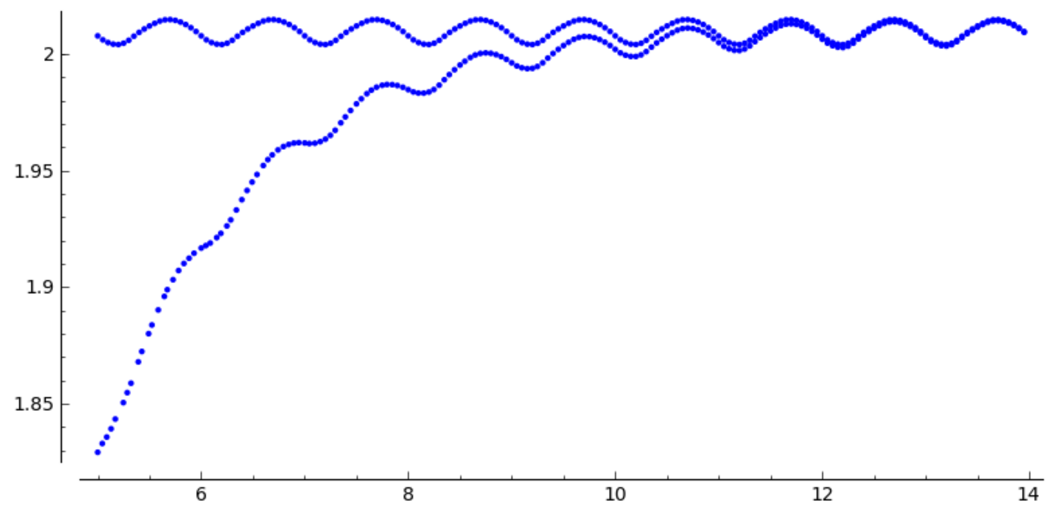

Figure 4: We plot $f_3(2^{-\langle y\rangle})$ (above) and $a_T(\lfloor y\rfloor)/\lfloor y\rfloor^2$ (below) versus $\log_2 y$ (horizontal) for various points $y \in [2^5, 2^{14})$.

Example 2. Let $T = 1 + x + x^4$. Then, denoting the limit function by $f_4(x)$, we obtain the following representation (again using a computer)

$$f_4(x) = \begin{cases} -\dfrac{235}{192}x^2 + \dfrac{11}{8}x + \dfrac{15}{8} & \text{if } \dfrac{1}{2} \le x < \dfrac{8}{15} \\[2ex] -\dfrac{1205}{1536}x^2 + \dfrac{29}{32}x + 2 & \text{if } \dfrac{8}{15} \le x < \dfrac{4}{7} \\[2ex] -\dfrac{617}{1536}x^2 + \dfrac{15}{32}x + \dfrac{17}{8} & \text{if } \dfrac{4}{7} \le x < \dfrac{8}{13} \\[2ex] -\dfrac{55}{768}x^2 + \dfrac{1}{16}x + \dfrac{9}{4} & \text{if } \dfrac{8}{13} \le x < \dfrac{4}{5} \\[2ex] \dfrac{245}{768}x^2 - \dfrac{9}{16}x + \dfrac{5}{2} & \text{if } \dfrac{4}{5} \le x \le 1. \end{cases}$$

We note that the maximum and minimum of f_4 are given by $\frac{2791}{1234}$ and $\frac{2207}{980}$, respectively. Thus,

$$\limsup_{k\to\infty} \frac{a_T(k)}{k^2} = \frac{2791}{1234}$$

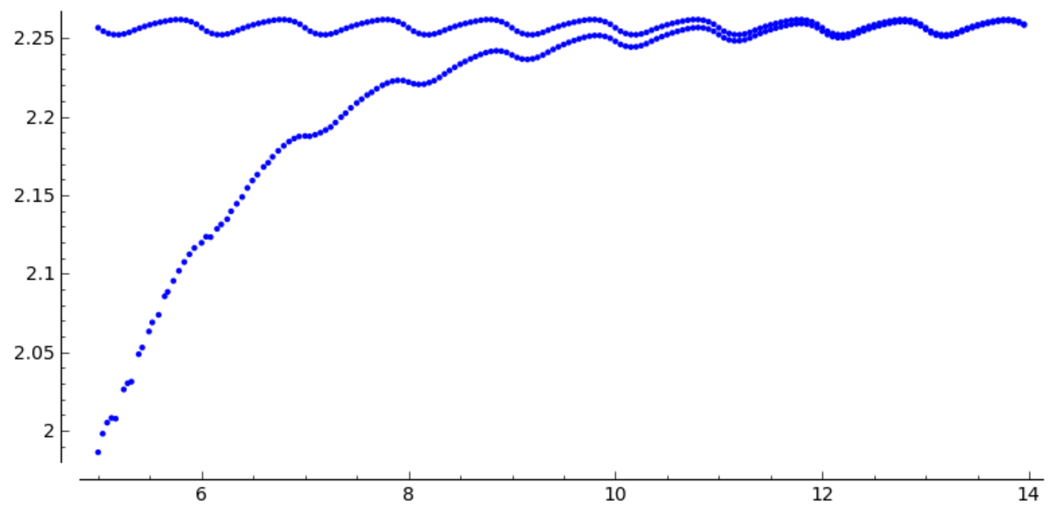

Figure 5: $f_4(2^{-\langle y\rangle})$ (above) and $a_T(\lfloor y\rfloor)/\lfloor y\rfloor^2$ (below) versus $\log_2 y$ (horizontal)

and

$$\liminf_{k\to\infty}\frac{a_T(k)}{k^2}=\frac{2207}{980}.$$

See Figure 5 for an illustration of the convergence of $a_T(s_k)/s_k^2$ to f_4.

5 Conclusion

We have found recursion relations for $a_T(k)$, where $T = 1+x+x^n$ for $n\geq 3$ and the coefficients are taken modulo 2. We have proven functional relations for the generating functions associated to $a_T(k)$ for even and odd n. Finally, we have proven that if $\phi(z)=\sum_{k=1}^{\infty}\alpha(k)z^k$ satisfies a certain functional equation relating $\phi(z)$ and $\phi(z^p)$, and if for $\frac{1}{p}\leq x\leq 1$, $s_k(x)\equiv s_k$ is a sequence divergent to infinity such that $\langle \log_p s_k\rangle$ converges to $\log_p\frac{1}{x}$, then the sequence $\alpha(s_k)/s_k^2$ tends to a piecewise quadratic function of x.

Possible directions of future research include investigating the following conjectures:

Conjecture 1. *Let T be given by a polynomial of degree n which cannot be expressed as a power of another polynomial, and take the coefficients modulo some prime p. Then*

$$a_T(k) = \sum_{j=0}^{p-1}\sum_{r=0}^{p-1} a_T\left(\left\lfloor \frac{k+jn+r}{p} \right\rfloor\right) - C,$$

where C is a constant depending upon T.

Conjecture 2. *If $a_T(k)$ arises from an irreducible polynomial transition rule of degree n with coefficients taken modulo a prime p, then if ϕ is a generating function for $a_T(k)$, we have $r_n(z)\phi(z) = R(z) + r_n(z^p)\phi(z^p)$ for some polynomial $R(z)$.*

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References

[1] J. von Neumann and A. W. Brooks. *Theory of Self-Reproducing Automata*. University of Illinois Press, Urbana, 1966.

[2] S. Wolfram. Cellular automata as models of complexity. *Nature*, 311:419–424, 1984.

[3] S. J. Willson. Computing fractal dimensions for additive cellular automata. *Physica 24D*, pages 190–206, 1987.

[4] K. Garbe. Patterns in the coefficients of powers of polynomials over a finite field. Preprint, 2012.

[5] L. V. Ahlfors. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable, Third Edition*. McGraw-Hill, New York, 1979.

[6] W. Rudin. *Principles of Mathematical Analysis, Third Edition*. McGraw-Hill, New York, 1976.