On the Number of Linear Extensions of Graphs

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Abstract

Given a bipartite graph, let us pick an acyclic orientation of its edges. Then, if we consider the partially ordered set (poset) induced by this orientation, the number of linear extensions of such a poset is maximal whenever the orientation is bipartite, or such that no directed path of length two exists. We define a sequence of automorphisms that injectively but non-bijectively map the set of linear extensions of a nonbipartite orientation to the set of linear extensions of a bipartite orientation. Additionally, we define such a sequence for simple odd cycle graphs and discuss extending mappings to apply to general nonbipartite graphs.

1 Introduction

The properties and applications of graphs are topics of high interest due to their ability to model problems in various fields. The unique structure of bipartite graphs in particular has led to many discussions of special properties of such graphs.

Atkinson [1] conjectured in 1988 that the maximum possible number of linear extensions of a bipartite graph occurs when the orientation of edges is bipartite. Later that year, Stachowiak [2] proved the result using the following recursive and inductive argument. Let $e(P)$ denote the number of linear extensions of a poset P. A minimal element of P is an element smaller than all elements it is connected to and a maximal element of P is an element larger than all elements it is connected to. Let $\mathrm{MIN}(P)$ denote the set of minimal elements of P, and $\mathrm{MAX}(P)$ denote the set of maximal elements of P. It is a known fact that, given a finite poset P, the number of linear extensions of the poset is equal to the sum of the number of linear extensions of each poset obtained by removing one minimal element from P [3]. Stachowiak used this result to derive the equation

$$2 \cdot e(P) = \sum_{x \in \mathrm{MIN}(P) \cup \mathrm{MAX}(P)} e(P - x). \tag{1}$$

In other words, the number of linear extensions of a poset is directly related to the sum of the number of linear extensions of each poset obtained by removing one minimal element or one maximal element from P. Induction on the number of vertices $n = |V|$ shows that the summation is maximized when P corresponds to a bipartite orientation, as we later explain.

Another method for proving Atkinson's conjecture is to count the number of linear extensions of the poset corresponding to a bipartite orientation and to compare that number to $n!$, the maximum number of linear extensions of a poset P with n elements [4]. In 1991, Brightwell and Winkler [5] showed that the problem of counting the number of linear extensions of an arbitrary poset is NP-complete. However, analyzing the relationship between number of possible linear extensions and orientation, without explicit counting, for a nonbipartite graph remains an open problem.

This paper develops a new approach to prove Atkinson's conjecture for linear extensions of bipartite graphs and begins discussing extensions of Atkinson's conjecture to cases involving linear extensions of nonbipartite graphs.

1.1 Definitions

A *graph* $G(V,E)$ is composed of a set of vertices V and edges E, where edges connect pairs of vertices. Edges are denoted $\{u,v\}$ if they are undirected and (u,v) if they are directed from u to v, where u and v are vertices in V. Furthermore, we call two vertices *adjacent* if they are connected by an edge. For this paper, we assume that all graphs are connected. All disconnected graphs can be handled as repeated cases of connected graphs.

A *bipartite graph* with n vertices is a graph such that

1. the set of vertices V is the union of two disjoint sets of vertices V_1 and V_2, denoted $V = V_1 \sqcup V_2$, and
2. if $\{u,v\}$ is an edge in E, then either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.

Figure 1 is an example of an undirected bipartite graph.

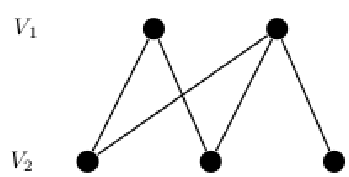

Figure 1: An undirected bipartite graph

Given an undirected graph $G(V,E)$, assign a direction to every edge in E. An *orientation* O of the edges includes either the ordered pair (u,v) or the ordered pair (v,u) for each edge with vertices $u,v \in V$. An *acyclic orientation* is an orientation such that if $(u,u_1),(u_1,u_2),\ldots,(u_p,v)$ are all in O, then $u \neq v$. In this paper, we only work with acyclic orientations. An orientation induces order relations between the vertices such that

- $u \leq v$ whenever (u,v) is a pair in E, and

- if $u \leq v$ and $v \leq w$, then $u \leq w$.

The order relations depend uniquely on the choice of graph G and orientation O. The set V with order relations forms a *partially ordered set* (poset) denoted $P_{G,O}$.

Let $[n] = \{1,2,\ldots,n\}$ and denote by $\mathbf{n}$ the set $[n]$ with its usual order [6]. A *linear extension* of $P_{G,O}$ is an assignment of the set $[n]$ to the n elements of V that respects the order relations of V given by $P_{G,O}$. More formally, a linear extension of $P_{G,O}$ is an order preserving bijection $f : P_{G,O} \to \mathbf{n}$ satisfying $u \leq v$ under $P_{G,O}$ if and only if $f(u) \leq f(v)$. For some partial orderings $P_{G,O}$, there are many possible linear extensions. For this paper, we consider only orientations of graphs with valid linear extensions.

Finally, define a *bipartite orientation* as an orientation that does not contain both pairs (u,v) and (v,w) for any triple of vertices $u,v,w \in V$. Since we are working with connected graphs, this is equivalent to the statement that all edges are oriented from V_1 to V_2 or vice versa. Figure 2 shows a linear extension of a bipartite orientation.

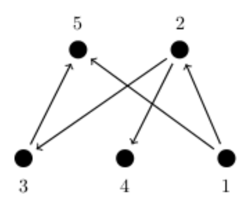

Figure 2: A bipartite orientation with an induced linear extension

1.2 Present Work

We consider the more general problem of comparing linear extensions of different posets of *nonbipartite graphs*. For nonbipartite graphs, we explore how to orient the edges to result in the maximum number of linear extensions. Extending Stachowiak's proof is difficult as his proof depends on each vertex being clearly defined as either minimal or maximal. With most nonbipartite graphs, however, at least one vertex is neither minimal nor maximal.

We thus introduce a different proof of Stachowiak's result for bipartite graphs, based on a well defined mapping from linear extensions of nonbipartite orientations to linear extensions of bipartite orientations. This proof may be easier to extend to nonbipartite graphs, as it depends only on the initial linear extension and not on the structure of the graph.

In Section 2, we present a different proof of Atkinson's conjecture for bipartite graphs. Our proof uses an injective mapping from linear extensions of nonbipartite orientations to linear extensions of bipartite orientations. In Section 2.1, we demonstrate a bijection between linear extensions of the two bipartite orientations of a bipartite graph, and in Section 2.2 we expand upon this result to show an injective mapping from linear extensions of nonbipartite orientations to linear extensions of bipartite orientations. In Section 3.1, we expand to nonbipartite graphs, defining an injective mapping from linear extensions of non-semibipartite orientations to linear extensions of semibipartite orientations of simple odd cycle graphs.

2 Linear Extensions of Bipartite Graphs

We begin with several fundamental definitions. An *injective function* is a function f such that for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$. A *surjective function* is a function f such that for all $b \in B$, there exists some $a \in A$ such that $f(a) = B$. A *bijective function* is a function f that is both injective and surjective.

Next, define an *automorphism* of a finite set as a bijection from the set to itself. Note that this is a permutation of the given set. Two automorphisms *commute* if the order in which they are applied to our finite set is irrelevant.

Let $G = G(V, E)$ be an n-vertex connected bipartite graph with fixed bipartition $V = V_1 \sqcup V_2$, where V_1 and V_2 are the disjoint sets of vertices on each side of the bipartition. We can visualize G as drawn in the plane, with all vertices in V_1 lying on the line $y = 1$ and all vertices in V_2 on the line $y = 0$. Two bipartite orientations exist for G. Let O_d denote the orientation with all edges directed from V_1 to V_2, and let O_u denote the orientation with all edges directed from V_2 to V_1. Figure 3 shows the two bipartite orientations for a bipartite graph.

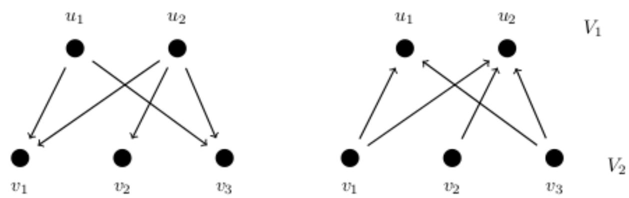

Figure 3: Two bipartite orientations for a bipartite graph

Our goal is to construct an injective map from the set $L(O)$ of all linear extensions of the poset $P_{G,O}$ induced by an arbitrary fixed orientation O of G to the set $L(O_u)$ of linear extensions of P_{G,O_u} induced by O_u. The choice of O_u instead of O_d is arbitrary, and we obtain identical results using O_d instead. We first examine the most extreme case where $O = O_d$, and the initial orientation is bipartite with all edges directed down from V_1 to V_2.

2.1 Mapping between Bipartite Orientations

Proposition 1. *There exists a bijection from the set of linear extensions $L(O_d)$ of P_{G,O_d} with edges oriented from V_1 to V_2, to the set of linear extensions $L(O_u)$ of P_{G,O_u} with edges oriented from V_2 to V_1.*

Proof. First, we define an automorphism $*$ of the set of all bijections from V to $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Given a bijection $f : V \to [n]$, define $f^* : V \to [n]$ to be a new function defined by

$$f^*(v) = n + 1 - f(v), \text{ for all } v \in V. \tag{2}$$

Figure 4 illustrates the action of the operator $*$ on a bipartite graph with $n = 5$.

Since $(f^*)^* = f^{*\circ*} = n + 1 - f^*(v) = f(v)$, $f^*(v)$ has an explicit inverse function, which happens to be itself. This means the number of elements in $f : V \to [n]$ is equal to the number of elements in $f^* : V \to [n]$. Thus, $*$ is bijective, and the map $*$ is an automorphism.

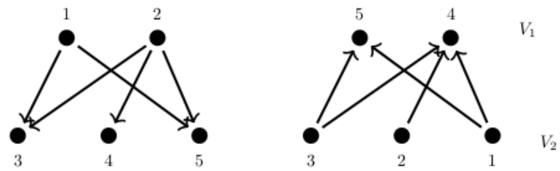

Figure 4: $f \to f^*$

We use the operator $*$ to transform linear extensions of P_{G,O_d} to linear extensions of P_{G,O_u}, and vice versa.

Given the bipartite orientation O_d directed from V_1 to V_2 of G, begin with a linear extension f of the poset P_{G,O_d}. At least one valid linear extension exists for any bipartite orientation O_d, since the extension with $f(V_1) : \{1,2,\dots,\lfloor \frac{n}{2} \rfloor\}$ and $f(V_2) : \{\lceil \frac{n}{2} \rceil, \dots, n-1, n\}$ always corresponds to a bipartite orientation directed from V_1 to V_2. By assumption, the map $f : V \to [n]$ is bijective, so we apply our automorphism $*$ to obtain the bijection $f^* : V \to [n]$. We claim that the map $f^* : V \to [n]$ is a linear extension of P_{G,O_u}.

We need to check that f^* respects the corresponding order induced on V by O_u. Suppose $u < v$ in P_{G,O_u}, where $u, v \in V$. This implies there is a directed path from u to v in O_u. By symmetry, there is a directed path from v to u in O_d, so $v < u$ in P_{G,O_d}. Thus, by the definition of a linear extension, we have $f(v) < f(u)$. This implies that $f^*(v) = n+1-f(v) > n+1-f(u) = f^*(u)$, so

$$f^*(v) > f^*(u),$$

as desired. This proves that f^* is a linear extension of O_u. Hence, the automorphism $*$ transforms every linear extension of P_{G,O_d} into a linear extension of P_{G,O_u}, so it induces a well defined map from $L(O_d)$ to $L(O_u)$. We also note that $L(O_d)$ and $L(O_u)$ are two subsets of the set of bijections $f : V \to [n]$.

By the same arguments, the operation $*$ applied to linear extensions of P_{G,O_u} maps to linear extensions of P_{G,O_d}. Additionally, if f is a linear extension of P_{G,O_d}, then f^* is a linear extension of P_{G,O_u}, and $f^{*\circ*} = f$ is again a linear extension of P_{G,O_d}. Therefore, the restriction of the automorphism $*$ to the set $L(O_d)$ or $L(O_u)$ is an invertible map from $L(O_d)$ to $L(O_u)$, or respectively from $L(O_u)$ to $L(O_d)$.

Thus, $*$ gives the desired bijection. $\square$

As a consequence of this bijection, $|L(O_d)| = |L(O_u)|$, and so our choice of O_u over O_d for this section, and the rest of the paper, does not affect our results.

2.2 Mapping from Nonbipartite to Bipartite Orientations

We now expand this approach to nonbipartite orientations. Because we are trying to show there are more linear extensions that correspond to bipartite orientations than nonbipartite orientations, we search for an *injective* rather than a bijective function from the set $L(O)$ of linear extensions of nonbipartite orientations to the set $L(O_u)$ of linear extensions of bipartite orientations.

For $u, v \in V$, we define an automorphism $*_{u,v}$ of the set of bijections from V to $[n]$. Given a bijection $f : V \to [n]$, applying $*_{u,v}$ to f yields the bijection $f^{*_{u,v}} : V \to [n]$ given by

$$f^{*_{u,v}}(w) = \begin{cases} f(v) & \text{if } w = u, \\ f(u) & \text{if } w = v, \\ f(w) & \text{otherwise.} \end{cases} \tag{3}$$

We now introduce a simple lemma which will be essential for our later results.

Lemma 2. *Consider a bijection $f : V \to [n]$. If for some $u, v, w_1, w_2 \in V$ with $f(u) < f(v)$, we have that $f^{*_{u,v}}(w_1) > f^{*_{u,v}}(w_2)$ but $f(w_1) < f(w_2)$, then $f(w_1)$ and $f(w_2)$ must both belong to the closed interval $[f(u), f(v)]$.*

Proof. From the definition of the operator $*_{u,v}$, at least one of $f(w_1)$ or $f(w_2)$ must be equal to one of $f(u)$ or $f(v)$, or else no transformation is performed. First suppose that $f(w_1) < f(u)$. Since this implies that $w_1 \neq u$ and $w_1 \neq v$, by definition $f^{*_{u,v}}(w_1) = f(w_1)$. Thus, either $w_2 = u$ or $w_2 = v$. If $w_2 = u$, then $f^{*_{u,v}}(w_2) = f(v) > f(u)$ and if $w_2 = v$, then

$f^{*_{u,v}}(w_2) = f(u)$. For both cases, we see that $f^{*_{u,v}}(w_1) = f(w_1) < f(u) \leq f^{*_{u,v}}(w_2)$. This contradicts our initial conditions, so we must have that $f(w_1) \geq f(u)$.

By a similar argument, we must have that $f(w_2) \leq f(v)$. Combining these two observations yields the relation

$$f(u) \leq f(w_1) < f(w_2) \leq f(v), \tag{4}$$

with either $f(u) = f(w_1), f(w_2) = f(v)$, or both. $\square$

We wish to find an *injective* mapping from a linear extension of a nonbipartite orientation to a linear extension of a bipartite orientation O_u. First we define an automorphism $*_k$ of the set of bijections from V to $[n]$, associated with a given orientation O_k of the edges of G, where k is an artificial indexing variable that we will use when we apply the operator. Let $W_k \subseteq V$ be the set of vertices of V which are incident to an edge whose orientation in O_k is directed from V_1 to V_2, or in other words, whose orientation in O_k and O_u are different. Less formally, W_k is the set of vertices incident to edges pointing down in our drawing of G with orientation O_k. Let $n_k = |W_k|$.

Suppose we have a given bijection $f_k : V \to [n]$. If $W_k = \emptyset$ (so $n_k = 0$), let $*_k = *_0$, where $*_0$ is the identity operator, so that $f_k^{*_k} = f$. If $n_k > 0$, consider the set $f_k(W_k)$ of images of W_k under f_k. We know that $f_k(W_k) \subseteq [n]$ and $|f_k(W_k)| = |W_k| = n_k$. Suppose that $f_k(W_k) = \{i_{1,k}, \ldots, i_{n_k,k}\}$ with $i_{1,k} < i_{2,k} < \cdots < i_{n_k,k}$. Then, define the operator $*_k$ applied to f_k to output the function $f_k^{*_k} : V \to [n]$ defined by

$$f_k^{*_k}(v) = \begin{cases} i_{(n_k+1-j),k} & \text{if } v \in W_k \text{ and } f_k(v) = i_{j,k}, \text{ with } j = 1 \text{ or } j = n_k, \\ f_k(v) & \text{otherwise.} \end{cases} \tag{5}$$

Since for every permutation of $[n]$ there is a corresponding ordering of V, the function $f_k^{*_k}$ is surjective. Furthermore, since $|V| = n$, $f_k^{*_k} : V \to [n]$ is bijective. Hence, the map $*_k$ is well-defined. As in the previous case, since the subset of vertices W_k is independent of the choice of bijection f_k, we see that $(f_k^{*_k})^{*_k} = f_k^{*_k \circ *_k} = f_k$. This shows the map $*_k$ is invertible, and is thus an automorphism of the set of bijections from V to $[n]$.

We now recursively and inductively apply our operator $*_k$ to linear extensions of posets $P_{G,O}$, making use of our artificial index k. Let $O_1 = O$ and $f_1 = f$, where f is a linear extension of $P_{G,O}$. Suppose, for induction, we already know the orientations $O_1, O_2, \ldots, O_k$, their associated linear extensions $f_1, f_2, \ldots, f_k$, and subsets of vertices $W_1, W_2, \ldots, W_k$, and that we wish to compute O_{k+1}, f_{k+1}, and W_{k+1}. We already have a linear extension $f_k : V \to [n]$ of $P_{G,O}$. Since f_k is bijective and we know W_k, we can apply the operator $*_k$ to f_k to obtain the bijection $f_{k+1} = f_k^{*_k} : V \to [n]$. This bijection f_{k+1} induces an orientation O_{k+1} of the edges of G such that f_{k+1} is a linear extension of $P_{G,O_{k+1}}$.

Figure 5 shows the result of $f_1^{*_1}$ for a sample linear extension of an orientation of a bipartite graph with $n = 7$. Note that, for this linear extension, W_1 is the set of vertices with labels $\{2, 6, 3, 7, 5, 4\}$, and so $n_1 = 6$.

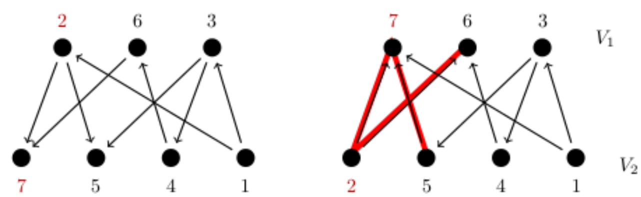

Figure 5: An example result of applying $f_1^{*_1}$

We now introduce several key properties of the defined mapping $*_k$.

Lemma 3. *Edges with the same orientation in O_k and O_u have the same orientation in O_{k+1} and O_u as well.*

Proof. If $W_k = \emptyset$ (so $O_k = O_u$), then we are done. If $W_k \neq \emptyset$, then because G is bipartite, there must exist some edges of the form (u, v) with $u \in V_1$ and $v \in V_2$. By the definition of $*_k$, which requires the labels $i_{1,k}$ and $i_{m_k,k}$ to be minimal and maximal respectively, if for $u, v \in V$ we have that $f_k(u) = i_{1,k}$ and $f_k(v) = i_{n_k,k}$, then u must be a vertex in V_1 and v must be a vertex in V_2. After performing $*_k$, we flip only the labels for u and v. Since $f_k(v) > f_k(u)$, any edges connected to u and directed from V_2 to V_1 in O_k will remain directed in the same direction. Similarly, any edges connected to v and directed up in O_k will remain in the same direction. Thus, edges with the same orientation in O_k and O_u will not be changed by $*_k$, so the statement is true. $\square$

Lemma 4. *The number of edges in G whose orientation in O_k and O_u differs is strictly greater than the number of edges whose orientation in O_{k+1} and O_u differs, unless $W_k = \emptyset$.*

Proof. Consider the vertex u such that $f_k(u) = i_{1,k}$. Since $i_{1,k}$ was chosen as minimal in the definition of $*_k$, then an edge incident to u whose orientation is different in O_k and O_u will have equal orientation in O_{k+1} and O_u. This edge necessarily exists by the definition of W_k, so the number of edges whose orientation in O_k and O_u differs strictly decreases as k increases. $\square$

The combination of Lemma 3 and Lemma 4 implies that

- there exists a minimal number m with $1 \leq m \leq \lfloor \frac{n_1}{2} \rfloor$ such that $O_m = O_u$. This follows directly from the observation that the number of edges with different orientations in O_k and O_u is strictly decreasing, and from the observation that there are at most $\lfloor \frac{n_1}{2} \rfloor$ pairs of numbers flipped by applying the composition $*_1 \circ *_2 \circ \cdots \circ *_k$, and

- for vertices $u, v \in V$ such that $f_k(u) = i_{1,k}$ and $f_k(v) = i_{n_k,k}$ with $k < m$, $u, v \in W_k$ but $u, v \notin W_{k+1}$.

Hence, we obtain $m-1$ pairwise disjoint sets of vertices $\{u_k, v_k\}_{1 \leq k < m}$ satisfying the following conditions

- $f(u_k) = i_{1,k}, f(v_k) = i_{n_k,k}$, and $u_k, v_k \in W_k$ but $u_k, v_k \notin W_{k+1}$, for all $k \in 1, 2, \ldots, m-1$.

- $i_{1,1} < i_{1,2} < \cdots < i_{1,m-1} < i_{n_{m-1},m-1} < \cdots < i_{n_2,2} < i_{n_1,1}$.

- $u_k \in V_1$ and $v_k \in V_2$ for all k.

- the operator $*_k$ is precisely $*_{u_k,v_k}$, using the notation of Lemma 2.

We now prove a theorem that will directly imply our main result.

Theorem 5. *For every nonbipartite orientation O of the bipartite graph $G = G(V,E)$, there is an injective but non-surjective map associating every possible linear extension of $P_{G,O}$ to a linear extension of P_{G,O_u}.*

Proof. We first prove that our defined map, which we will denote as $*_1^m := *_1 \circ \cdots \circ *_{m-1}$, is injective. An immediate consequence of having pairwise disjoint sets of vertices is that the automorphisms $*_k$ for all $k \in 1,2,\ldots,m-1$ commute. Additionally, by Lemma 2, the automorphisms $*_k^m := *_k \circ \cdots \circ *_{m-1}$ with $k \in 1,2,\ldots,m-1$ are such that if $f(w_1) < f(w_2)$ but $f^{*_k^m}(w_1) > f^{*_k^m}(w_2)$ for some $w_1, w_2 \in V$, then

$$i_{1,k} \leq f(w_1) < f(w_2) \leq i_{n_k,k}. \tag{6}$$

Informally, this operator can be thought of as applying the $*_k$'s in reverse order of the k's to f_m.

Denote $f_k^m := f^{*_k^m}$, and let O_k^m be the induced orientation of the edges of G by f_k^m. Using the inequalities (6), we see that if we let W_k^m be the set of vertices adjacent to an edge of G whose orientation in O_k^m and $O = O_1$ differs, then the maximal value of f_k^m on W_k^m is $i_{n_k,k}$ and its minimal value is $i_{1,k}$. This follows directly when $k = 1$ since $f_1^m = f_m$ and $W_1^m = W_1$. If $k > 1$, we can again use the inequalities (6) and the definition of the operators $*_k$ to find edges (possibly the same edge) $\{u_k, w\}$ and $\{v_k, x\}$ of G whose orientations in O_1 and O_u differ, but whose orientations induced by $f^{*_1^m \circ *_k}$ are the same as in O_1. This holds since, by definition, $f^{*_1 \circ \cdots \circ *_{k-1}}$ does not orient up all the edges of the form $\{u_k, w\}$ nor orients up all the edges of the form $\{v_k, x\}$, and by Lemma 2, $f^{*_{k+1}^m}$ does not affect the orientation of such edges either.

The composition of the operators $f^{*_1 \circ \cdots \circ *_{k-1}}$ and $f^{*_{k+1}^m}$ on f yields $f^{*_1^m \circ *_k}$ by commutativity, so the orientation induced by $f^{*_1^m \circ *_k}$ has at least one edge of the form $\{u_k, w\}$ pointing down and at least one edge of the form $\{v_k, x\}$ pointing down. However, this is not true for the orientation induced by $f^{*_1^m}$. Hence, we conclude that the change of orientation of some edges initially oriented down of the form $\{u_k, w\}$ and $\{v_k, x\}$ in the composition $*_1 \circ *_2 \circ \cdots \circ *_{m-1}$ is induced only by the operation $*_k$. Hence, by Lemma 2, those edges will be similarly oriented in O_k^m and O_u after applying $*_k^m$ to f, and oppositely oriented in O_k^m and O_1. This implies that W_k^m contains both $i_{1,k}$ and $i_{n_k,k}$.

Lastly, we note that by the commutativity of $*_1, *_2, \ldots, *_{m-1}$ and the property that for all $k \in 1,2,3,\ldots,m-1$ we have that $*_k \circ *_k = *_0$, then $f_k^m = f^{*_k^m} = (f_m)^{*_0 \circ *_1 \circ *_2 \circ \cdots \circ *_{k-1}}$. This shows that the numbers $i_{1,1} < i_{1,2} < \cdots < i_{1,m-1} < i_{m_{m-1},m-1} < \cdots < i_{m_2,2} < i_{m_1,1}$ are uniquely determined by f_m and O_u and can be recursively found from f_m. First compute $O_1^m = O_u$ and then compare with $O_1 = O$ to find $i_{1,1}$ and $i_{m_1,1}$ in f_m, which exhibits u_1 and v_1. Then apply $*_1 = *_{u_1,v_1}$ to f_m and compute O_2^m to compare with O_1. Thus, we find $i_{1,2}$ and $i_{m_2,2}$, which correspond to u_2 and v_2, which gives us $*_2$. Apply $*_2, *_3$, and so on.

Since the maximal and minimal elements of $f_k(W_k)$ are uniquely determined by f_m and O_u, O_1 and f_1 are uniquely determined by f_m and O_u. Thus, $*_1^m$ is a well defined injective map from a linear extension of any orientation O to a linear extension of a bipartite orientation O_u.

We now prove that $*_1^m$ is not a surjective map.

Given that O is a nonbipartite orientation of the bipartite graph G with linear extension f, there must exist a directed path of length two induced by O. We consider two cases:

1. In the first case, the path consists of edges (v_1, v_2) and (v_2, u_1) with $v_2 \in V_2$ and $v_1, u_1 \in V_1$. Since (v_1, v_2) is directed from V_1 to V_2, the vertices v_1 and v_2 belong to the set $W = W_1$. Among all possible linear extensions of the bipartite orientation O_u of G, there must exist some linear extension f_m with $f_m(v_2) = 1$ and $f_m(v_1) = n$. Let the linear extension f_m be our desired result of the mapping $*_1^m$. Since $v_1, v_2 \in W_1$, given the desired f_m, we have that $f(v_1) = i_{1,1} = 1$ and $f(v_2) = i_{n_1,1} = n$. However, then $f(u_1) < f(v_2)$, which contradicts our original orientation O.

2. In the second case, the path consists of edges (u_2, v_1) and (v_1, v_2) with $v_1 \in V_1$ and

$u_2, v_2 \in V_2$. Since (v_1, v_2) is directed from V_1 to V_2, the vertices v_1 and v_2 belong to the set $W = W_1$. Among all possible linear extensions of the bipartite orientation O_u of G, there must exist some linear extension f_m with $f_m(v_2) = 1$ and $f_m(v_1) = n$. Again, let f_m be our desired result of the mapping $*_1^m$. Since $v_1, v_2 \in W_1$, given the desired f_m, we have that $f(v_1) = i_{1,1} = 1$ and $f(v_2) = i_{n_1,1} = n$. However, then $f(v_1) < f(u_2)$, which contradicts our original orientation O.

Thus, for any given nonbipartite orientation O, there exists at least one bipartite orientation O_u such that applying the composition $*_1^m$ to f never maps to f_m. Hence, the mapping $*_1^m$ is not surjective.

□

2.3 General Result for Bipartite Graphs

Using Theorem 5, we can make a general statement about linear extensions of bipartite graphs, which is the same as Stachowiak's result.

Theorem 6. *Given a bipartite graph G, the number of linear extensions of $P_{G,O}$ is maximal whenever O is a bipartite orientation.*

Proof. We have defined a mapping $*_1^m$ from the set $L(O)$ of linear extensions of nonbipartite orientations to the set $L(O_u)$ of linear extensions of bipartite orientations of bipartite graphs. Since we proved $*_1^m$ is injective but not surjective, $|L(O_u)| > |L(O)|$ for any nonbipartite orientation O. The theorem follows directly. □

3 Extension to Nonbipartite Graphs

As a future study, we wish to extend our result to linear extensions of posets of nonbipartite graphs.

Since nonbipartite graphs cannot have a bipartite orientation, we define a *semibipartite orientation* as one that is as close to bipartite as possible. More formally, we label vertices that are the maximum of all their neighbors U, vertices that are the minimum of all their neighbors D, and vertices that are part of a path, or neither minimum nor maximum M; such a labeling corresponds to a *semibipartite orientation* if each vertex is labeled either U, D, or M such that adjacent vertices do not have the same label and $|M| \leq c$, where c is the number of distinct odd cycles in the graph.

3.1 Simple Odd Cycle Graphs

Simple odd cycle graphs can be visualized as convex polygons with an odd number of vertices. For an odd cycle graph, a semibipartite orientation O_S is a labeling of the vertices such that only one vertex is labeled M, with all other vertices labeled U or D such that adjacent vertices do not have the same label. The orientation depends solely on the arrangement and position of the one directed path of length two, which has vertices $D - M - U$. Figure 6 is an example of a labeling corresponding to a semibipartite orientation of a simple odd cycle.

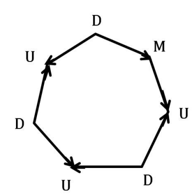

Figure 6: A simple odd cycle with a semibipartite labeling

Given a simple odd cycle $C = C(V, E)$ with $|V| = n$, label the vertices $v_1, v_2 ... v_n$ such that v_{j+1} is adjacent to v_j for every $j \in \{1, \ldots, n-1\}$ and $v_0 = v_n$. Begin with a random linear extension $f : V \to [n]$, inducing an orientation O. We will consider rotations of the same linear extension as indistinct.

Consider all directed paths of length two in O. At least one such path necessarily exists since, by definition, the vertex labeled M and its two adjacent vertices form a directed path of length two. Each path is determined by a unique set of three vertices $\{v_j, v_{j+1}, v_{j+2}\}$ such that $f(v_j) < f(v_{j+1}) < f(v_{j+2})$ and $j \in \{1, \ldots, n-1\}$. Let W_C denote all such sets of three vertices with $f(v_j) < n-2$ and $f(v_{j+2}) > 3$. Denote by v_i the vertex $v \in W_C$ with the smallest value $f(v)$, and define W_i as the set of vertices $\{v_i, v_{i+1}, v_{i+2}\}$. The set W_i is unique for each linear extension of $P_{C,O}$. Among all vertices $v \notin W_i$, denote by v_{max} the vertex with maximum value $f(v)$ and by v_{min} the vertex with minimum value $f(v)$.

For $v \in V$, we define a new automorphism $*_c$. Given a bijection $f : V \to [n]$, applying $*_c$ to f yields the bijection $f^{*_c} : V \to [n]$ given by

$$f^{*_c}(v) = \begin{cases} f(v_{max}) & \text{if } v = v_{i-1}, \\ f(v_{i-1}) & \text{if } v = v_{max}, \\ f(v_{min}) & \text{if } v = v_{i+3}, \\ f(v_{i+3}) & \text{if } v = v_{min}, \\ f(v) & \text{otherwise.} \end{cases} \tag{7}$$

The directed path with vertices v_i, v_{i+1}, v_{i+2} and $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ is unchanged by the operator. Given the initial conditions that $f(v_i) < n-2$ and $f(v_{i+2}) > 3$, there always exists v_{max}, v_{min} such that $f(v_{max}) > f(v_i)$ and $f(v_{min}) < f(v_{i+2})$. Applying $*_c$ yields $f(v_{i-1}) = f(v_{max}) > f(v_i)$. $f(v_i)$ is thus the minimum of its two neighbors, and has label D. Applying $*_c$ also yields $f(v_{i+3}) = f(v_{min}) < f(v_{i+2})$. $f(v_{i+2})$ is thus the maximum of its two neighbors, and has label U. $f(v_{i+1})$ has label M by definition, so the path (v_i, v_{i+1}, v_{i+2}) is of the form $D - M - U$.

We now consider the subgraph $C_i = C_i(V_i, E_i)$ of vertices $v \notin W_i$ excluding v_{max} and v_{min} and their connecting edges. Since $|V| = n$ is odd, $|V_i| = n-5$ is even by parity. Furthermore, C_i is a graph with no odd cycles, which is a well known property of bipartite graphs. [7] Thus, this remaining subgraph is a bipartite graph with induced orientation O_i. Applying the previously defined mapping $*_1^m$ from Section 2.2 to C_i results in a linear extension of P_{C_i,O_i}that corresponds to a bipartite orientation.

Denote $*_c^m : *_c \circ *_1^m$. We notice that:

1. $f(v_{max}) > f(v)$ and $f(v_{min}) < f(v)$ for all $v \in V_i$.

2. Applying $*_c^m$ assigns label U to v_{i-1}, label D to v_i, label U to v_{i+2}, and label D to v_{i+3}.

3. Applying $*_c^m$ induces a bipartite orientation of C_i, which corresponds to a labeling of the vertices V_i with D or U such that no adjacent vertices have the same label.

Thus, the operator $*_c^m$ induces an orientation with one directed path $D-M-U$ and all other vertices labeled D or U such that no adjacent vertices have the same label. This is our definition of a semibipartite orientation, so $*_c^m$ is a valid mapping from a linear extension f of a non-semibipartite orientation to a linear extension f_s of a semibipartite orientation.

Theorem 7. *For every non-semibipartite orientation of the simple odd cycle $C = C(V,E)$, there is an injective but non-surjective map associating every possible linear extension of $P_{C,O}$ to a linear extension of P_{C,O_S}.*

Proof. The initial chosen path $W_i : \{v_i, v_{i+1}, v_{i+2}\}$ directly determines the values $f(v_{max})$ and $f(v_{min})$. The remaining vertices and their corresponding edges form the bipartite subgraph C_i after applying $*_c$. This is equivalent to the statement that for every simple odd cycle C, the subgraph C_i is uniquely determined by the initial path W_i. We proved in Section 2.2 that $*_1^m$ generates a unique linear extension of a bipartite orientation for each linear extension of a nonbipartite orientation of a bipartite graph. Since W_i is unique for each original linear extension and each choice of W_i generates a unique bipartite subgraph C_i, $*_c^m$ injectively maps each linear extension of a non-semibipartite graph to a linear extension of a semibipartite graph.

We now prove that the mapping $*_c^m$ is non-surjective. After applying the operator $*_c^m$, necessarily either $f(v_i) = 1$ or $f(v_{i+3}) = 1$, since $f(v_{i+3})$ is defined as the minimum number not in the directed path. By similar logic, either $f(v_{i+2}) = n$ or $f(v_{i-1}) = n$. For $n > 5$, however, there must exist some linear extension of a semibipartite orientation where $1 \neq f(v_i)$ or $f(v_{i+3})$ and/or $n \neq f(v_{i+2})$ or $f(v_{i-1})$, which is impossible under the mapping $*_c^m$. Figure 7 is an example of such a linear extension.

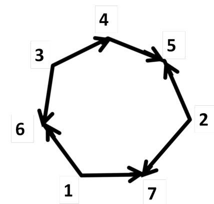

Figure 7: A linear extension that cannot result from applying $*_c^m$

Thus, for any non-semibipartite O, there exists at least one semibipartite O_S such that applying the operator $*_c^m$ never maps f to f_s. Hence, the mapping $*_c^m$ is not surjective. $\square$

3.2 General Result for Simple Odd Cycles

Theorem 8. *Given a simple odd cycle graph C, the number of linear extensions of $P_{C,O}$ is maximal whenever O is a semibipartite orientation.*

Proof. We have defined a mapping $*_c^m$ from the set $L(O)$ of linear extensions of non-semibipartite orientations to the set $L(O_S)$ of linear extensions of semibipartite orientations of simple odd cycle graphs. Since we proved $*_c^m$ is injective but not surjective, $|L(O_S)| > |L(O)|$ for any non-semibipartite orientation O. $\square$

4 Conclusion

We defined a mapping $*_1^m$, or a composition of the operators $*_k$, from linear extensions of nonbipartite orientations of a bipartite graph to linear extensions of bipartite orientations. The mapping $*_1^m$ is injective but not surjective, implying that the number of linear extensions corresponding to bipartite orientations is strictly greater than the number of linear extensions corresponding to nonbipartite orientations, so a bipartite orientation induces the maximal number of linear extensions for a bipartite graph. Using a similar approach, we also defined a mapping $*_c^m$ for simple odd cycle graphs, building on results from our bipartite mapping. This mapping $*_c^m$ is injective but not surjective, thus implying that a semibipartite orientation induces the maximal number of linear extensions for a simple odd cycle graph. Further studies will focus on investigating semibipartite orientations in increasingly complex graphs, such as combinations of odd cycles and trees or multiple connected odd cycles, in the hope of attaining results for general nonbipartite graphs.

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