On Successive Quotients of Lower Central Series Ideals for Finitely Generated Algebras

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Abstract

This paper examines the behavior of the successive quotients $N_i(A)$ of the lower central series ideals $M_i(A)$ of a finitely generated associative algebra A over $\mathbb{Z}$. We define the lower central series $L_i(A)$ by $L_1(A) = A$, $L_{i+1}(A) = [A, L_i(A)]$, $M_i(A) = A \cdot L_i(A) \cdot A$, and $N_i(A) = M_i(A)/M_{i+1}(A)$. We decompose the N_i into its free and torsion components using the structure theorem of finitely generated abelian groups, and we examine patterns in the ranks and torsion of N_i for algebras with various homogeneous relations, including x^2 in multiple variables, q-polynomial relation $yx - qxy$, and $x^m + y^m$. In order to do this, we create data tables with the ranks and torsion of various N_i, previously uncalculated, based on calculations done in the program *Magma*. This paper includes a complete description of N_i for the q-polynomial algebra, $\mathbb{Z}\langle x, y\rangle/(yx - qxy)$ and a proof for the ranks of N_2 for $A\langle x, y\rangle/(x^m + y^m)$, which provides insight into how changing the coefficient or degree of a relation affects rank and torsion, as well as general patterns for which primes appear in torsion.

1 Introduction

For the past several years, algebraists have been studying the lower central series $L_i(A)$, which are successive subspaces of an associative algebra A formed from the commutators of A. Thus, the lower central series and its related objects can be used to measure the noncommutativity of algebras. We consider the successive quotients of the two-sided ideals M_i generated by L_i, and we call these quotients N_i. We study the structure and properties of N_i to better understand the structure of associative algebras.

Lower central series quotients of free associative algebras were first studied by Feigin and Shoikhet [4]. They looked at the successive quotients $B_i = L_i/L_{i+1}$, and concluded that there was an isomorphism between the space $A/M_3(A)$ and the space of even differential forms. This isomorphism is essential in proving the pattern of $N_2(A_2/(x^m + y^m))$ found in this paper. The study of quotients continued with Etingof, Kim, and Ma [3], who completely described the quotient $A/M_i(A)$ for $i = 4$.

The study of B_i was continued in the work of Balagović and Balasubramanian [1], who looked at B_2 in the quotient of a free algebra. In particular, they provided a complete description of $B_2(A_2/(x^d + y^d))$, which is similar to the results found in this paper for $N_2(A_2/(x^m + y^m))$.

While the structure of B_i has been studied in multiple papers, the quotients N_i have been less studied. Kerchev [5] studied N_i for free algebras and computed $N_i(A_n)$ for several values of i and n. However, there is still much work to be done in studying the structure of N_i. In particular, torsion has never been calculated for N_i even for free algebras, and N_i have not been studied for algebras with relations.

In this paper, we study the behavior of N_i for an associative algebra $\mathbb{Z}\langle x, y\rangle$ with various relations. We also compute the ranks and torsion of various N_i using a computer program called *Magma*. The process of collecting data is explained in Section 3. Several patterns suggested by the data are contained in Section 4. In Section 5 we provide a complete description of N_i for algebras with the relation $yx - qxy = 0$, also known as q-polynomial algebras. The results concerning the structure of N_2 with the relation $x^m + y^m$ is proven in Section 6.

We begin with preliminary background to better understand the algebraic objects of study.

2 Preliminary Background

2.1 Associative Algebras and Their Lower Central Series

Definition 2.1. Let A be a vector space over a field k with a bilinear associative multiplication operation $(a, b) \mapsto a \cdot b$, which is also written as ab. If A also has a multiplicative identity, denoted by 1, then A is a **unital associative algebra**.

A free algebra is a unital associative algebra that is generated by a set of generators with no relation. In this paper, we are interested in associative algebras that are not necessarily free, more precisely, algebras with homogeneous relations. An algebra $A/\langle P\rangle$ will denote the quotient algebra of A by the ideal generated by the relation P. We define a bracket operation on A by $[a, b] = a \cdot b - b \cdot a$. This bracket operation satisfies $[a, a] = 0$ and the Jacobi identity:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

A vector space with a bilinear bracket operation $[a, b]$ such that the Jacobi identity and $[a, a] = 0$ are satisfied is called a **Lie algebra**, so any associative algebra is also a Lie algebra with $[a, b] = a \cdot b - b \cdot a$.

Definition 2.2. Let A be a Lie algebra. Define a series of Lie ideals inductively such that $L_1(A) = A$ and $L_{i+1}(A) = [A, L_i(A)]$, where the bracket of two subspaces C and D is defined as $[C, D] = \mathrm{span}([c, d])$ such that $c \in C, d \in D$. This series of Lie ideals is the **lower central series** of A. We abbreviate $L_i(A)$ as L_i.

We may make this definition over $\mathbb{Z}$ by using $\mathbb{Z}$-modules (i.e. Abelian groups) instead of vector spaces. Similarly, we can use this definition over any commutative ring R.

Definition 2.3. Denote the two-sided ideals generated by each L_i by M_i, i.e. $M_i = A \cdot L_i \cdot A$.

It is easy to see that $M_i = A \cdot L_i$. Using this definition, we can define the quotients N_i.

Definition 2.4. Define the N_i to be the successive quotients M_i/M_{i+1}.

Now we introduce the idea of grading, which is crucial to representing data effectively.

Definition 2.5. Let A be a module over a commutative ring k. The module A is **graded** if A has a direct sum decomposition into submodules $\bigoplus_{i \geq 0} A_i$. If A is an algebra such that $A_i \cdot A_j \subset A_{i+j}$, then A is a **graded algebra**.

Example 1. The simplest example of a graded algebra is a polynomial ring $k[x_1, \ldots, x_n]$, where the grading is by the degree of the polynomial. We observe that N_i is graded, as it inherits its grading from A. Our study is simplified if we look at N_i by its "degree," which is denoted by d. The part of N_i at degree d will be denoted as $N_i[d]$, which is a finitely generated k-module.

2.2 Torsion and Classfication of Finitely Generated Abelian Groups

Definition 2.6. An element a of an Abelian group G is a **torsion element** if $n \cdot a = 0$ for some positive integer n. Conventionally, 0 is also considered a torsion element. In this case, we say that a is an n-torsion element. All of the torsion elements in G form a subgroup of G.

To clarify the concept of torsion, we offer a simple example.

Example 2. Consider the group $G = \mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. This group has *2-torsion*, *3-torsion*, and *6-torsion*. 0, 2, and 4 are 3-torsion elements, 0, 3 are 2-torsion elements, and all elements are 6-torsion elements. All 2 and 3-torsion elements are also 6-torsion elements.

The idea of torsion becomes especially important due to the Structure Theorem of Finitely Generated Abelian groups, which states that groups can be separated into their free and torsion components.

Theorem 2.1 (Structure Theorem of Finitely Generated Abelian Groups)**.** *Every finitely generated Abelian group G is isomorphic to a finite direct sum of infinite cyclic groups and cyclic groups of order p^n, for various primes p. This decomposition is unique up to order of summands.*

The theorem can be restated as

$$G \cong F \oplus T,$$

where F is the free component, which is isomorphic to $\mathbb{Z}^r$ for some $r \in \mathbb{Z}$, and T is the torsion component, consisting of a finite sum of cyclic groups of order p^n for various primes p. In this case, r is called the **rank** of the free component, known simply as "rank."

The goal of the project is to determine the structure N_i for algebras over $\mathbb{Z}$ by studying patterns in the ranks and torsion of N_i.

2.3 Sample Calculations for $\mathbb{Z}\langle x, y\rangle$

We provide a set of sample calculations to illustrate how L_i, M_i, and N_i are constructed. We consider the free associative algebra $\mathbb{Z}\langle x, y\rangle$. Because it is not known whether torsion exists in

N_i for free algebras generated by 2 variables, we focus on calculating the ranks of N_i. First, we calculate the bases of the first few L_i in low degrees.

By definition, $L_1 = A$, so L_1 spanned by all the monomials in x, y. The next row L_2 is formed from the set of all $[A, L_1]$. As the minimum degree of a non-trivial part of L_1 is 1, the minimum degree of a non-trivial part of L_2 is 2. The only term in the basis of $L_2[2]$ is $[x, y]$, as $[y, x] = -[x, y]$. $L_2[3]$ is spanned by $[L_1[1], L_1[2]]$. Thus, the potential basis vectors are $[x, xy]$, $[y, xy]$, $[x, yx]$, $[x, y^2]$, $[y, x^2]$, $[y, yx]$, $[x, x^2]$, and $[y, y^2]$. However, both $[x, x^2]$ and $[y, y^2]$ are 0. In addition, $[x, xy] + [x, yx] = [y, x^2]$. Thus, $[y, x^2]$ is not linearly independent and can be removed from the basis. Similarly, $[y, xy] + [y, yx] = [x, y^2]$, and $[x, y^2]$ can be eliminated. Only 4 terms remain in the basis of $L_2[3]$. Calculating $L_3[3]$ is more straightforward, as $L_3[3] = [L_1[1], L_2[2]]$. The results of the basis of L_i can be found in Table 1, where the top row indicates the degree d.

$L_i[d]$	1	2	3
L_1	x, y	x^2, xy, yx, y^2	x^3, x^2y, xyx, yx^2, y^3, y^2x, yxy, xy^2
L_2	0	$[x, y]$	$[x, xy]$, $[y, xy]$, $[x, yx]$, $[y, yx]$
L_3	0	0	$[x, [x, y]]$, $[y, [x, y]]$

Table 1: Bases for L_i

The M_i can be constructed from L_i, as $M_i = A \cdot L_i \cdot A = A \cdot L_i$. By this definition, $L_i[i] = M_i[i]$, as $L_i[i]$ must be multiplied by scalars on both sides for the minimum non-trivial degree i. Thus, $M_1 = A$, and $M_2[2]$ is also easy to compute. Calculating $M_2[3]$ is slightly more complicated. $M_2[3] = L_1[0] \cdot L_2[3] + L_1[1] \cdot L_2[2]$. Therefore, the possible terms in the basis of $M_2[3]$ are $[x, xy]$, $[y, xy]$, $[x, yx]$, $[y, yx]$, $x[x, y]$, and $y[x, y]$. Eliminating linearly dependent terms, the basis of $M_2[3]$ contains $[x, xy]$, $[y, xy]$, $[x, yx]$, and $[y, yx]$. Then, $M_3[3] = L_3[3]$, so Table 2 is complete.

$M_i[d]$	1	2	3
M_1	x, y	x^2, xy, yx, y^2	x^3, x^2y, xyx, yx^2, y^3, y^2x, yxy, xy^2
M_2	0	$[x, y]$	$[x, xy]$, $[y, xy]$, $[x, yx]$, $[y, yx]$
M_3	0	0	$[x, [x, y]]$, $[y, [x, y]]$

Table 2: Bases for M_i

Now we calculate the ranks of $N_i[d]$, which are the cardinalities of the basis of each $N_i[d]$. As $N_i = M_i/M_{i+1}$, $\text{rank}(N_i[d]) = \text{rank}(M_i[d]) - \text{rank}(M_{i+1}[d])$. Thus, computing the ranks of each $N_i[d]$ becomes a simple subtraction problem. The ranks are shown in Table 3.

$N_i[d]$	1	2	3
N_1	2	3	4
N_2	0	1	2

Table 3: Free Ranks for N_i

3 Data Collection

We first compile data tables of the ranks and torsion of N_i for various relations. By changing the number of variables, coefficients, or degree of the relations, we can find patterns and form conjectures about the behavior of the ranks and torsion of N_i.

Data is collected by running computations in *Magma* [2]. This code was run for many relations over the integers, and the outputs were then organized into table form by grading. The left column displays the N_i, while the top row is organized by grading (degree). Each term in the data table includes the rank, which is displayed outside of the parentheses, and the torsion, which is displayed within the parentheses.

Here, we provide an example of how the data was processed and organized into tables. The example below shows how to format the output in *Magma* to a data table.

Example 3. The output code

```
$N_ 4 $ & 8 16(Abelian Group isomorphic to Z/2 + Z/2 + Z/4 + Z/4)
```

is expressed as $16(2^2 \cdot 4^2)$ for $N_4[8]$ in a table.

The expression has a rank of 16, which is the last numbered output before the parentheses. The torsion is slightly more difficult to express. The output data in the parentheses represents the direct sum of many cyclic groups. While prime power cyclic groups do not need to be further decomposed, other groups can be decomposed into coprime components. For example, $\mathbb{Z}_{60}$ can be decomposed into $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ by the result in group theory that states $\mathbb{Z}_{mn} = \mathbb{Z}_m \times \mathbb{Z}_n$ for m, n coprime. The data is decomposed into prime powers for the tables.

These components can then be combined through exponent rules (eg. Z$/2$ + Z$/2$ given in the output is expressed in the table as 2^2). Thus, the final form of the term in the data table for $N_4[8]$ is $16(2^2 \cdot 4^2)$.

We look for patterns within these data tables, then try to prove them.

4 Observations of Patterns in Data

After compiling tables of algebras with various relations, we find several patterns in the ranks and torsion of the N_i.

4.1 Patterns in $\mathbb{Z}\langle x_1, \ldots, x_n\rangle/(x_1^2)$

Interesting patterns arise for the algebra with the relation $x_1^2 = 0$ with a number of variables, shown in Table 4. With two variables, it seems that the torsion and ranks of N_i stabilize rather quickly—that is to say, the torsion and ranks do not chnage as d increases. Reading across the rows shows the stabilization of the ranks from $N_i[i+2]$ and a stabilizing torsion.

$N_i[d]$	2	3	4	5	6	7	8	9
N_2	1	1(2)	1(2)	1(2)	1(2)	1(2)	1(2)	1(2)
N_3	0	2	3(2)	$3(2^2)$	$3(2^2)$	$3(2^2)$	$3(2^2)$	$3(2^2)$
N_4	0	0	2	$3(2^2)$	$3(2^4)$	$3(2^5)$	$3(2^5)$	$3(2^5)$
N_5	0	0	0	4	$7(2^3)$	$7(2^7 \cdot 3)$	$7(2^9 \cdot 3)$	$7(2^{10} \cdot 3)$
N_6	0	0	0	0	5	$9(2^5)$	$9(2^{12} \cdot 3 \cdot 5)$	$9(2^{16} \cdot 3 \cdot 5)$
N_7	0	0	0	0	0	9	$18(2^7)$	$19(2^{19} \cdot 3^2 \cdot 5)$
N_8	0	0	0	0	0	0	12	$25(2^{12})$
N_9	0	0	0	0	0	0	0	20

Table 4: $x_1^2 = 0$, two variables

This yields a conjecture about the stabilization of the ranks and torsion of N_i.

Conjecture 4.1.1. *For $\mathbb{Z}\langle x,y\rangle/(x^2)$, $N_i[j] \cong N_i[j+1]$ for $j \geq 2i-1$.*

It would be interesting to know how soon the ranks stabilize for algebras with more generators, and which primes will ultimately appear in torsion.

4.2 Patterns in $\mathbb{Z}\langle x, y\rangle/(yx - qxy)$

We notice that the ranks are non-zero only for the diagonal $N_i[i]$, and these ranks are equal to $i-1$, as seen in Tables 9 through 11, found in Appendix A.1. Additionally, fixing a d, the torsion in all $N_i[d]$ for $i < d$ are the same: $(\mathbb{Z}_{q-1})^{d-1}$. Along the diagonal $i = d$, there is a more interesting pattern with more primes. We are able to completely describe N_i in this case, and we show that on the diagonal $i = d$, all primes will eventually appear, except those that divide q. A more detailed description and proof of the result on the torsion in $\mathbb{Z}\langle x, y\rangle/(yx - qxy)$ are provided in Section 5.

4.3 Patterns in $\mathbb{Z}\langle x,y\rangle/(x^m+y^m)$

By examining the algebras $\mathbb{Z}\langle x,y\rangle/(x^m+y^m)$, we discover several patterns occurring across tables in N_i for $i = 2, 3, 4$. The complete set of data can be found in Tables 12 to 14 in Appendix A.2. For N_2, the ranks follow a palindromic pattern similar to the one found in B_2 [1], which is shown in Table 5, where the left column indicates values of m.

$N_2[d]$	2	3	4	5	6	7	8
2	1						
3	1	2	1				
4	1	2	3	2	1		
5	1	2	3	4	3	2	1

Table 5: Ranks for $N_2[d]$ with the relation $x^m + y^m = 0$

Proposition 4.1. *In the algebra* $\mathbb{Z}\langle x, y\rangle/(x^m + y^m)$, $\text{rank}(N_2[k]) = k-1$ *for* $k < m$, $\text{rank}(N_2[k]) = 2m - k - 1$ *for* $m \leq k \leq 2m-2$*, and 0 for all other values of* k.

This proposition will be proven in Section 6.

The patterns developing in the ranks of N_3 and N_4 are almost palindromic, and we see pseudo-arithmetic sequences. The values for the ranks of N_3 are displayed in Table 6, with the left column showing values of m.

$N_3[d]$	3	4	5	6	7	8	9	10
2	2	1	0	0	0	0	0	0
3	2	5	4	1	0	0	0	0
4	2	5	8	7	4	1	0	0
5	2	5	8	11	10	7	4	1

Table 6: Ranks for $N_3[d]$ with the relation $x^m + y^m = 0$

Conjecture 4.3.1. *In the algebra* $\mathbb{Z}\langle x, y\rangle/(x^m + y^m)$, $\text{rank}(N_3[k]) = 3k - 7$ *for* $k \leq m+1$, $\text{rank}(N_3[k]) = 6m - 3d + 1$ *for* $m + 1 < k < 2m + 1$*, and 0 for all other values of* k.

While N_2 and N_3 seem to have easily generalizable patterns, N_4 is slightly more complicated. The bolded numbers in Table 7 are the ones that remain consistent as d increases.

$N_4[d]$	4	5	6	7	8	9	10
2	2						
3	**3**	7	**4**				
4	**3**	**8**	13	**10**	**4**		
5	**3**	**8**	**14**	19	**16**	**10**	**4**

Table 7: Ranks for $N_4[d]$ with the relation $x^m + y^m = 0$

Conjecture 4.3.2. *The ranks of* N_4 *will become stable outside of the diagonal* $d = m + 2$*, where* d *is the degree of the grading. We expect* $\text{rank}(N_4(A_2/(x^m + y^m))[2m - k])$ *to stabilize for large* m *and fixed* $k \geq 0$*. This rank vanishes for* $k < 0$.

5 Complete Description of $N_i\left(\mathbb{Z}\langle x,y\rangle/(yx-qxy)\right)$

We consider the specific algebra $\mathbb{Z}\langle x,y\rangle$ with the relation $yx-qxy$, also known as a q-polynomial algebra. A clear pattern emerges in the ranks and torsion of the N_i, as shown in Tables 9 through 10, which are located in Appendix A.1. The torsion in $N_i[d]$ for $i<d-1$ is $(\mathbb{Z}_{q-1})^{d-1}$. The torsion along the diagonal $i=d$ has a more interesting pattern. To understand this pattern, we use a finer grading on the degrees, defining x to have degree $\langle 1,0\rangle$ and y to have degree $\langle 0,1\rangle$ where for degree $\langle u,v\rangle$, $d=u+v$.

Theorem 5.1. *Let $A=\mathbb{Z}\langle x,y\rangle/(yx-qxy)$, where $q\in\mathbb{Z}$, $q\neq\pm1$. The* $\operatorname{rank}(N_i[j])=i-1$ *for $i=j$. Otherwise, the rank is 0. The torsion in $N_i[d]$, also written as* $\operatorname{Tor}(N_i[d])$, *for $i<d-1$ is $(\mathbb{Z}_{q-1})^{d-1}$. Along the diagonal $i=d-1$,* $\operatorname{Tor}(N_i[d])=\bigoplus_{u+v=d}\mathbb{Z}_{q^{(u,v)}-1}$.

Note. In Theorem 5.1 and its proof, the greatest common divisor of u and v is denoted as (u,v).

Now we give some preliminary information for the proof. First, we consider the bases of the spaces L_k and M_k. We first note that because $yx=qxy$ under the relation, we can express the result of any bracket operation as a sum of x^uy^v with some coefficients. We first construct a table (see Table 8 for L_k on the diagonal $k=u+v$, keeping in mind that $yx=qxy$). For $k>1$, the basis element of $L_k[\langle u,v\rangle]$ will be denoted by $S^k_{u,v}x^uy^v$.

Definition 5.1. $S^k_{u,v}$ is the largest possible integer such that $L_k[\langle u,v\rangle]\subset\operatorname{span}\left(S^k_{u,v}x^uy^v\right)$.

In the table for $k>1$, we include only the coefficients of the bases, $S^k_{u,v}$, and the terms in which $u<v$, as $\langle u,v\rangle$ and $\langle v,u\rangle$ are symmetric. There is no torsion if $u,v=0$. The rank of $N_i[j]=0$ if $j\neq i$ and $N_i[j]=i-1$ if $j=i$.

	$\langle 0,1\rangle$	$\langle 0,2\rangle$	$\langle 1,1\rangle$	$\langle 0,3\rangle$	$\langle 1,2\rangle$	$\langle 0,4\rangle$	$\langle 1,3\rangle$	$\langle 2,2\rangle$
L_1	x	y^2	xy	y^3	xy^2	y^4	xy^3	x^2y^2
L_2	0	0	$(q-1)$	0	$(q-1)$	0	$(q-1)$	(q^2-1)
L_3	0	0	0	0	$(q-1)^2$	0	$(q-1)^2$	$(q-1)\cdot(q^2-1)$
L_4	0	0	0	0	0	0	$(q-1)^3$	$(q-1)^2\cdot(q^2-1)$

Table 8: Construction of L_k for the relation $yx-qxy=0$

We consider the bases along the diagonal $k=u+v$ and find that there is a pattern (Lemma 5.1).

Lemma 5.1. *For the algebra $\mathbb{Z}\langle x, y\rangle/(yx - qxy)$, with $q \in \mathbb{Z}$ and $q \neq \pm 1$, and $S^k_{u,v}$ for $k = u + v$, is*

$$S^k_{u,v} = (q-1)^{u+v-2} \cdot (q^{(u,v)} - 1). \tag{5.1}$$

To prove this lemma, we include a few known facts in number theory:

Fact 1. $(\lambda^m - 1, \lambda^n - 1) = \lambda^{(m,n)} - 1$

Fact 2. $(a, b) = 1 \Longrightarrow (a, bc) = (a, c)$

Fact 3. $\left(\dfrac{i}{h}, \dfrac{j}{h}\right) = 1 \Longleftrightarrow (i, j) = h$

Proof. We prove Lemma 5.1 by induction on u, v. The base case is satisfied, as

$$S^2_{1,1} = (q-1)^{1+1-2}(q^{(1,1)} - 1) = q - 1.$$

Because $L_k[\langle u, v\rangle] = [x, L_k[\langle u-1, v\rangle]] + [y, L_{k-1}[\langle u, v-1\rangle]]$, $S^k_{u,v}$ satisfies the recursive equation

$$S^k_{u,v} = \left(S^{k-1}_{u-1,v}(q^v - 1), S^{k-1}_{u,v-1}(q^u - 1)\right). \tag{5.2}$$

Assuming that

$$S^{k-1}_{u-1,v} = (q-1)^{u+v-3}(q^{(u-1,v)} - 1),$$
$$S^{k-1}_{u,v-1} = (q-1)^{u+v-3}(q^{(u,v-1)} - 1),$$

we want to show that

$$\begin{aligned} S^k_{u,v} = \Big((q-1)^{u+v-3}(q^{(u-1),v} - 1)(q^v - 1), (q-1)^{u+v-3}(q^{(u,v-1)} - 1)(q^u - 1)\Big) \\ = (q-1)^{u+v-2}(q^{(u,v)} - 1). \end{aligned} \tag{5.3}$$

We can pull out the expression $(q-1)^{u+v-3}$, as it is common to both of the components of the greatest common divisor. Thus, equation (5.3) is reduced to

$$S^k_{u,v} = (q-1)^{u+v-3}\left((q^{(u-1,v)} - 1)(q^v - 1), (q^{(u,v-1)} - 1)(q^u - 1)\right). \tag{5.4}$$

We set the following:

$$q^{(u-1,v)} - 1 = \alpha,$$

$$q^v - 1 = \beta,$$

$$q^{(u,v-1)} - 1 = \gamma,$$

$$q^u - 1 = \delta.$$

Using Facts 3 and 1, we have

$$(\alpha, \gamma\delta) = q - 1 \Longleftrightarrow \left(\frac{\alpha}{q-1}, \frac{\gamma\delta}{q-1}\right) = 1 \tag{5.5}$$

and

$$\left(\frac{\alpha}{q-1}, \frac{\gamma\delta}{q-1}\right) = 1 \Longrightarrow \left(\frac{\gamma\delta}{q-1}, \beta\right) = \left(\frac{\gamma\delta}{q-1}, \frac{\alpha\beta}{q-1}\right) \tag{5.6}$$

We also note that

$$\left(\beta, \frac{\gamma}{q-1}\right) = 1 \Longrightarrow \left(\beta, \frac{\gamma\delta}{q-1}\right) = (\beta, \delta). \tag{5.7}$$

By Fact 1, $(\alpha, \gamma) = q-1$, as $((u-1, v), (u, v-1)) = 1$. Thus, $\left(\frac{\alpha}{q-1}, \frac{\gamma}{q-1}\right) = 1$. By equation (5.5), $\left(\frac{\gamma\delta}{q-1}, \beta\right) = \left(\frac{\gamma\delta}{q-1}, \frac{\alpha}{q-1}\beta\right)$ is true.

Because $(\beta, \gamma) = q-1$ by Fact 1, $\left(\beta, \frac{\gamma}{q-1}\right) = 1$. By Fact 1, $(\beta, \delta) = (q^{(u,v)}-1)$. Thus, by equation (5.7), $(\beta, \delta) = \left(\beta, \frac{\gamma}{q-1}\delta\right) = (q^{(u,v)} - 1)$. By equation (5.6), $\left(\frac{\gamma\delta}{q-1}, \frac{\alpha\beta}{q-1}\right) = \left(\beta, \frac{\gamma\delta}{q-1}\right) = q^{(u,v)} - 1$.

It follows by Fact 3 that $(\alpha\beta, \gamma\delta) = (q-1)(q^{(u,v)} - 1)$.

□

We can now prove Theorem 5.1.

Proof. We denote the basis of $M_k[\langle u, v\rangle]$ as $T^k_{u,v}x^u y^v$, and note that for $k = u + v$, $S^k_{u,v} = T^k_{u,v}$, as the terms for L_k on the diagonal are of the lowest possible degree. Thus, L_k must be multiplied by constants on either side to form $M_k = A \cdot L_k \cdot A$.

Now we consider $T^k_{u,v}$ for $u + v > k$. We note that

$$M_{u+v-1}[\langle u, v\rangle] = L_{u+v-1}[\langle u, v\rangle] + x \cdot L_{u+v-1}[\langle u-1, v\rangle] + y \cdot L_{u+v-1}[\langle u, v-1\rangle].$$

Set $T_1 x^u y^v$ to be the basis of $x \cdot L_{u+v-1}[\langle u-1, v\rangle]$ and $T_2 x^u y^v$ to be the basis of of $y \cdot L_{u+v-1}[\langle u, v-1\rangle]$, where $T_1 = S^{u+v-1}_{u-1,v}$ and $T_2 = S^{u+v-1}_{u,v-1}$. It is true that $T_1 x^u y^v$, $T_2 x^u y^v$ span $M_{u+v-1}[\langle u, v\rangle]$. Thus, $(T_1, T_2)x^u y^v$ spans $M_{u+v-1}[\langle u, v\rangle]$. It is known that $(T_1, T_2) = (q-1)^{u+v-2}$. Then, $(q-1)^{u+v-2}x^u y^v$ spans $M_{u+v-1}[\langle u, v\rangle]$ by Lemma 5.1 and $T^{u+v-1}_{u,v} = (q-1)^{u+v-2}$. This suggests that $T^k_{u,v} = (q-1)^{k-1}$ if $u + v > k$.

Using this information, we can calculate the torsion N_i. Using the bases of M_k, we divide to get

$\text{Tor}\,(N_{u+v-1}[\langle u,v\rangle]) = \dfrac{\mathbb{Z}\cdot T_{u,v}^{u+v}x^u y^v}{\mathbb{Z}\cdot T_{u,v}^{u,v-1}x^u y^v} = \mathbb{Z}_{q^{(u,v)}-1}$. Summing over all u, v yields

$$\text{Tor}\,(N_{k-1}[k]) = \bigoplus_{u+v=k} \mathbb{Z}_{q^{(u,v)}-1}.$$

The ranks are easy to verify given the bases. The space $M_k[k]$ is a free Abelian group with basis $S_{1,k-1}^k xy^{k-1}, \ldots, S_{k-1,1}^k x^{k-1}y$ with $q \neq \pm 1$, while $M_{k+1}[k] = 0$. So $N_k[k]$ is free of rank $k-1$. Below and above the diagonal $i = d$, the ranks of $M_k[k]$ and $M_k[k+1]$ are the same, so the rank of $N_k[k]$ is 0. $\square$

Corollary 5.1. *All primes except those that divide q appear in the torsion of $N_i[i+1]$ for some i.*

Proof. Given that the N_i has $\left(q^{(u,v)} - 1\right)$-torsion, by Fermat's little theorem, all primes except those that divide q will appear in the torsion of N_i. $\square$

With Theorem 5.1 and Corollary 5.1, we now have a clearer idea of how the coefficients of a relation affect the ranks and torsion of N_i.

6 Ranks of $N_2(\mathbb{Z}\langle x,y\rangle/(x^m + y^m))$

We wish to find the basis of $N_2(A_2/(x^m+y^m))$ in order to prove Proposition 4.1, where $A_2 = \mathbb{Q}\langle x,y\rangle$. To do so, we use the short exact sequence $0 \to N_2 \to A/M_3 \to A/M_2 \to 0$, where $A = {}^{A_2}/_{(x^m+y^m)}$. First, we find the generators of A/M_3. We then prove the linear independence of these generators by using the isomorphism $A_2/M_3 \cong \Omega^{even}(\mathbb{Q}^2)_*$ found in Feigin and Shoikhet's paper [4], thus proving the result for the ranks of $N_2(\mathbb{Z}\langle x,y\rangle/(x^m + y^m))$.

6.1 Generators of A/M_3

We consider $A = \mathbb{Q}\langle x,y\rangle/(x^m + y^m)$ with $u = [x,y]$. We want to show that $x^i y^j$ for $0 \leq i \leq m-1, 0 \leq j$ and $x^i y^j u$ for $0 \leq i,j \leq m-1$ span A/M_3. To do this, we show that the following relations are satisfied in A/M_3: $u^2 = 0$, $[u,x] = [u,y] = 0$, $x^m + y^m = 0$, and $x^{m-1}u = y^{m-1}u = 0$. Because $x^m + y^m = 0$ in A, the relation also holds in A/M_3.

Lemma 6.1. *In A/M_3, $u^2 = 0$.*

Proof.

$$u^2 = [x,y]\cdot[x,y] = [x,y]\cdot(xy - yx).$$

Because $[x,y]\cdot xy = [[x,y],x]y + x[x,y]y$,

$$u^2 = [[x,y],x]y + x[x,y]y - [x,y]yx.$$

Because $x[x,y]y - [x,y]yx = -[[x,y]y,x]$,

$$u^2 = [[x,y],x]y - [[x,y]y,x].$$

Because $[[x,y],x]y \in M_3$ and $-[[x,y]y,x] \in M_3$, the relation $u^2 = 0$ is satisfied in A/M_3. ☐

Lemma 6.2. *In A/M_3, the relations $[u,x] = [u,y] = 0$ hold.*

Proof. $[u,x] = [[x,y],x]$. Because $[[x,y],x] \in M_3$, $[u,x] = 0$ in A/M_3. Similarly, $[u,y] = 0$. ☐

Lemma 6.3. *In A/M_3, $x^{m-1}u = y^{m-1}u = 0$.*

Proof. We know that $0 = [x^m + y^m, x]$ because $x^m + y^m = 0$. Additionally, $[x^m + y^m, x] = [x, y^m]$. We will show that $[x, y^m] = my^{m-1}u$ in A_2/M_3 through induction. Because u and y commute with each other by $[u,y] = 0$, the base case is satisfied:

$$0 = [x, y^2] = xy^2 - y^2x = xy^2 - yxy + yxy - y^2x = uy + yu = 2yu,$$

since $[y,u] = 0$ in A_2/M_3. Now we assume that for some integer k, $[x, y^k] = ky^{k-1}u$. We can expand $[x, y^k]$ as follows:

$$0 = [x, y^k] = xy^k - y^kx = xy^k + \sum_{i=1}^{k-1}\left(-y^ixy^{k-i} + y^ixy^{k-i}\right) - y^kx.$$

Now we consider $[x, y^{k+1}] = xy^{k+1} - y^{k+1}$. We expand and factor:

$$0 = [x, y^{k+1}] = \left(xy^k + \sum_{i=1}^{k-1}\left(-y^ixy^{k-i} + y^ixy^{k-i}\right) - y^kx\right)y + y^kxy - y^{k+1}x.$$

Thus,

$$[x, y^{k+1}] = [x, y^k]y + y^ku = ky^ku + y^ku = (k+1)y^ku = 0$$

since $[y,u] = 0$ in A_2/M_3. We have proved through induction that $[x, y^m] = my^{m-1}u = 0$. Because m is some positive integer, $y^{m-1}u = 0$ is satisfied in A/M_3, since we are working over $\mathbb{Q}$. Similarly, $x^{m-1}u = 0$ is also satisfied. ☐

From Lemma 6.1, 6.2, and 6.3, we know that x^iy^j for $0 \le i < m$ and x^iy^ju for $i,j < m-1$ span A/M_3. Since u commutes with x and y, we can assume without loss of generality that u appears at the end of all expressions in A/M_3. Thus, the degree of u can either be 0 or 1, as for $a \ge 2$, $u^a = 0$.

Calculating the number of generators for each degree, we find that the dimension (from the data) for that degree equals the number of generators, predicted by Proposition 4.1.

We now show the linear independence of these generators.

6.2 The Basis of A/M_3

We consider the short exact sequence

$$0 \to M_2/M_3 \to A/M_3 \to A/M_2 \to 0,$$

and let f be the surjection $A/M_3 \to A/M_2$. A/M_2 is the abelianization of A, so $A/M_2 = \mathbb{Q}[x, y]/(x^m + y^m)$. Now we wish to prove that the generators $x^i y^j$ and $x^i y^j u$ are linearly independent.

Lemma 6.4. *The images of $x^i y^j$ in A/M_2 are linearly independent for $0 \leq i < m$.*

Proof. If $\sum \left(C_{ij} x^i y^j\right) = 0$ for C_{ij} constants that are not all zero, $x^m + y^m$ divides $\sum \left(C_{ij} x^i y^j\right)$. However, this is impossible, as $i < m$. Thus, if $\sum \left(C_{ij} x^i y^j\right) = 0$, all C_{ij} must be zero, proving that $x^i y^j$ are linearly independent in A/M_2. □

Lemma 6.5. *The spaces spanned by $x^i y^j$ for $0 \leq i < m$ and $x^i y^j u$ for $i, j < m - 1$ have 0 intersection.*

Proof. Let v be a common vector in the spaces spanned by the two sets of generators. Because v is a linear combination of some $x^i y^j u \in \ker f$, $f(v) = 0$. We know that v is also in the space spanned by $x^i y^j$, so it must be a linear combination of $x^i y^j$. However, we have proved by Lemma 6.4 that $x^i y^j$ is linearly independent in A/M_2, so in order for $f(v) = 0$, we must have $v = 0$. □

Lemma 6.6. *The generators $x^i y^j u$ for $i, j < m - 1$ are linearly independent in A/M_3.*

To prove this lemma, we work with $\Omega(\mathbb{Q}^2)$, the space of all **differential forms** over $\mathbb{Q}^2$. A **differential form** α is of the form $f_0 + f_1\, dx + f_2\, dy + f_3(dx \wedge dy)$, where f_i are polynomials of x and y. We assign dx and dy to be of degree one, so even differential forms are of the form $f_0 + f_3\, dx \wedge dy$. We write the space of differential forms of degree k as Ω^k, so the space of even differential forms is denoted by $\Omega^{even}(\mathbb{Q}^2)$. We define a distributive **wedge product**, $\wedge$, on $\Omega(\mathbb{Q}^2)$ such that $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy = -dy \wedge dx$. Functions commute with dx, dy, and $dx \wedge dy$. If f is a polynomial, we write $f \wedge \alpha$ as $f\alpha$ for any form α. The wedge product makes

$\Omega(\mathbb{Q}^2)$ a noncommutative ring. Now we define a linear map $d : \Omega^i \to \Omega^{i+1}$. If $f \in \mathbb{Q}[x, y]$, then $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy$, and df is of degree 1. We have the following properties:

$$d(f\,dx) = df \wedge dx = -(\partial f/\partial y)\,dx \wedge dy,$$

$$d(d\alpha) = 0,$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}(\alpha \wedge\, d\beta).$$

We also define an associative asterisk operation, $*$, such that $\alpha * \beta = \alpha \wedge \beta + (-1)^{\deg(\alpha)}(d\alpha \wedge\, d\beta)$. The even differential forms with this $*$ operation form a subring, denoted by $\Omega^{even}(\mathbb{Q}^2)_*$. Now we can prove Lemma 6.6.

Proof. We want to prove the linear independence of $x^i y^j u$ for $i, j < m-1$ in A/M_3, where $A = A_2/(x^m + y^m)$. We know that $A/M_3 = {}^{A_2/M_3(A_2)}/_{\langle P\rangle}$. If $x^i y^j u$ is independent in A/M_3, then $span(x^i y^j u) \cap \langle P\rangle = 0$, where $\langle P\rangle$ is the ideal generated by $x^m + y^m$ in $A_2/M_3(A_2)$.

We consider the isomorphism $\phi : A/M_3 \to \Omega^{even}(\mathbb{Q}^2)_*$ [4], where $\phi(x) = x$ and $\phi(y) = y$. Because $\Omega^{even}(\mathbb{Q}^2)_*$ is not a quotient, it is easier to study than A/M_3. We consider $\phi(x^i y^j u)$ and the ideal generated by $\phi(x^m + y^m)$, which is spanned by $\alpha * (x^m + y^m) * \beta$, where α and β are even differential forms. It is easy to calculate that $\phi(x^i y^j u) = 2x^i y^j dx \wedge dy$, so the images of $x^i y^j u$ in $\Omega^{even}(\mathbb{Q}^2)_*$ are forms of degree 2, with coefficient $2x^i y^j$, $i, j < m-1$. Thus, we only need to show that $\alpha * (x^m + y^m) * \beta = f + g\,dx \wedge dy$, where f and g are polynomials, and each term in g has either a power of x^{m-1} or y^{m-1}, ensuring no overlap with $\phi(x^i y^j u)$, where $i, j < m-1$, which will prove linear independence. Let $\alpha = f_0 + f_1\,dx \wedge dy$, $\beta = g_0 + g_1\,dx \wedge dy$, and $\gamma = (x^m + y^m)$. Thus, $d\alpha = df_o$, $d\beta = dg_0$, and $d\gamma = m(x^{m-1}\,dx + y^{m-1}\,dy)$. Then,

$$\alpha * \gamma = \alpha \wedge \gamma + d\alpha \wedge d\gamma = \gamma f_0 + df_0 \wedge d\gamma + \gamma f_1\,dx \wedge dy.$$

From this, we know that $d(\alpha * \gamma) = d(f_0\gamma) = \gamma df_0 + f_0\,d\gamma$, by the chain rule. Now we wish to find $(\alpha * \gamma) * \beta = (\alpha * \gamma) \wedge \beta + d(\alpha * \gamma) \wedge d\beta$. We calculate:

$$(\alpha * \gamma) \wedge \beta = \gamma f_0 g_0 + g_0\,df_0 \wedge d\gamma + \gamma g_0 f_1\,dx \wedge dy.$$

Now we calculate $d(\alpha * \gamma) \wedge d\beta$:

$$d(\alpha * \gamma) \wedge d\beta = \gamma df_0 \wedge dg_0 + f_0 d\gamma \wedge dg_0.$$

We calculate:

$$\alpha * \gamma * \beta = \gamma f_0 g_0 + \gamma(g_0 f_1\, dx \wedge dy + df_0 \wedge dg_0) + d\gamma \wedge (f_0\, dg_0 - g_0\, df_0).$$

It is easy to see that each term of $g\, dx \wedge dy = \gamma(g_0 f_1\, dx \wedge dy + df_0 \wedge dg_0) + d\gamma \wedge (f_0\, dg_0 - g_0\, df_0)$ has a power of x^{m-1} or y^{m-1}, as $\gamma = x^m + y^m$ and $d\gamma = m(x^{m-1}dx + y^{m-1}dy)$. Thus, $\phi(x^i y^j u)$ and the ideal generated by $\phi(x^m + y^m)$ have no intersection in $\Omega^{even}(\mathbb{Q}^2)_*$, so $x^i y^j u$ is independent in A/M_3. □

By Lemmas 6.4, 6.5, and 6.6, we have proved that $x^i y^j$ for $0 \leq i < m$ and $x^i y^j u$ for $i, j < m-1$ are the basis of A/M_3, and $x^i y^j u$, and using the short exact sequence, $x^i y^j u$ is the basis of $N_2(A)$. We now count the number of generators by degree in $N_2(A)$ to verify Proposition 4.1.

6.3 Counting $x^i y^j u$ in $N_2(A)$

We wish to count the number of generators $x^i y^j u$ for $i, j < m-1$ in N_2, where x and y are of degree 1, and u is of degree 2. The maximum degree of $x^i y^j u$ is $2m-2$, as $i, j < m-1$. Let k be the degree in which we are counting generators. We count by two cases and use the short exact sequence to find $\dim(N_2[k])$.

Case 1. $k \geq 2m-1$

Because the maximum degree of $x^i y^j u$ is $2m-2$, there will be no $x^i y^j u$ for $k \geq 2m-1$.

Case 2. $k \leq 2m-2$

For the maximum degree $2m-2$, $i = j = m-2$. Because u is of degree 2, we also know that $i + j = k - 2$. So we wish to solve for i and j for $0 \leq i, j \leq m-2$. Because $j \leq m-2$, $k-2-i \leq m-2$ and $i \geq k-m$. For $k < m$, $0 \leq i \leq k-2$ has $k-1$ solutions. For $m \leq k \leq 2m-2$, $k - m \leq i \leq m-2$ has $2m-k-1$ solutions, which confirms the results of Proposition 4.1.

7 Conclusion

We have carefully examined the ranks and torsion of N_i for several classes of relations. Using a program in *Magma*, we generated data which we examined for patterns in the structure of N_i. In particular, we looked at homogeneous relations in two variables $(x^m + y^m)$, q-polynomials $(yx - qxy)$, and x^2 in multiple variables. We discovered patterns in all of these relations and provided a full proof of the ranks of $N_2(A_2/(x^m + y^m))$ and a complete description of the behavior of N_i for the

q-polynomial algebra. These results illustrate how the degree and coefficients in a homogeneous relation can affect the ranks and torsion of N_i of the algebra with that relation, particularly with regard to which primes appear in torsion.

There remains much to study about the structure of N_i. For example, the pseudo-arithmetic pattern in the ranks of the N_3 and N_4 (Conjectures 4.3.1 and 4.3.2) with relation $x^m + y^m$ is yet to be proved, perhaps with previous results [3]. The pattern found in N_2 for that relation matches the one found in B_2, which suggests a natural isomorphism $B_2 \to N_2$, at least over $\mathbb{Q}$. We plan to show that this mapping is an isomorphism by using the description of A/M_3 by generators and relations.

In addition, the behavior of torsion can be studied for $x^2 = 0$ of multiple variables, and in particular, Conjecture 4.1.1 could be proved. More work could be done in discovering which primes appear in the torsion, and why they appear.

The study of N_i and its structure has applications in the study of commutativity in groups and rings, and it can be used to study maps between groups. It also has connections in cyclic cohomology. Studying N_i can help us build a better understanding of the structure of a general associative algebra. Outside of its mathematical implications, the lower central series has wider applications, as noncommutative algebras often appear in quantum theory, so studying lower central series ideas can help build our fundamental understanding of the universe, as well as bring about the technological advancements quantum theory promises.

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A Data Tables

The tables contain the free and torsion components of $N_i[d]$, with the relation used in the caption.

A.1 $\mathbb{Z}\langle x,y\rangle / (yx - qxy)$

$N_i[d]$	2	3	4	5	6	7	8	9	10	11
N_2	1	$0(3^2)$	$0(3^3)$	$0(3^4)$	$0(3^5)$	$0(3^6)$	$0(3^7)$	$0(3^8)$	$0(3^9)$	$0(3^{10})$
N_3	0	2	$0(3^3)$	$0(3^4)$	$0(3^5)$	$0(3^6)$	$0(3^7)$	$0(3^8)$	$0(3^9)$	$0(3^{10})$
N_4	0	0	3	$0(3^4)$	$0(3^5)$	$0(3^6)$	$0(3^7)$	$0(3^8)$	$0(3^9)$	$0(3^{10})$
N_5	0	0	0	4	$0(3^4 \cdot 9)$	$0(3^6)$	$0(3^7)$	$0(3^8)$	$0(3^9)$	$0(3^{10})$
N_6	0	0	0	0	5	$0(3^6)$	$0(3^7)$	$0(3^8)$	$0(3^9)$	$0(3^{10})$
N_7	0	0	0	0	0	6	$0(3^7 \cdot 5)$	$0(3^8)$	$0(3^9)$	$0(3^{10})$
N_8	0	0	0	0	0	0	7	$0(3^7 \cdot 9)$	$0(3^9)$	$0(3^{10})$
N_9	0	0	0	0	0	0	0	8	$0(3^9 \cdot 11)$	$0(3^{10})$
N_{10}	0	0	0	0	0	0	0	0	9	$0(3^{10})$

Table 9: $yx + 2xy$

$N_i[d]$	2	3	4	5	6	7	8	9	10
N_2	1	$0(4^2)$	$0(4^3)$	$0(4^4)$	$0(4^5)$	$0(4^6)$	$0(4^7)$	$0(4^8)$	$0(4^9)$
N_3	0	2	$0(4^2 \cdot 8)$	$0(4^4)$	$0(4^5)$	$0(4^6)$	$0(4^7)$	$0(4^8)$	$0(4^9)$
N_4	0	0	3	$0(4^4)$	$0(4^5)$	$0(4^6)$	$0(4^7)$	$0(4^8)$	$0(4^9)$
N_5	0	0	0	4	$0(4^3 \cdot 7 \cdot 8)$	$0(4^6)$	$0(4^7)$	$0(4^8)$	$0(4^9)$
N_6	0	0	0	0	5	$0(4^6)$	$0(4^7)$	$0(4^8)$	$0(4^9)$
N_7	0	0	0	0	0	6	$0(4^4 \cdot 5 \cdot 8^2 \cdot 16)$	$0(4^8)$	$0(4^9)$
N_8	0	0	0	0	0	0	7	$0(4^8 \cdot 7^2)$	$0(4^9)$
N_9	0	0	0	0	0	0	0	8	$0(4^5 \cdot 8^4 \cdot 61)$
N_{10}	0	0	0	0	0	0	0	0	9
N_{11}	0	0	0	0	0	0	0	0	0

Table 10: $yx + 3xy$

$N_i[d]$	2	3	4	5	6	7	8	9	10	11
N_2	1	$0(5^2)$	$0(5^3)$	$0(5^4)$	$0(5^5)$	$0(5^6)$	$0(5^7)$	$0(5^8)$	$0(5^9)$	$0(5^{10})$
N_3	0	2	$0(3 \cdot 5^3)$	$0(5^4)$	$0(5^5)$	$0(5^6)$	$0(5^7)$	$0(5^8)$	$0(5^9)$	$0(5^{10})$
N_4	0	0	3	$0(5^4)$	$0(5^5)$	$0(5^6)$	$0(5^7)$	$0(5^8)$	$0(5^9)$	$0(5^{10})$
N_5	0	0	0	4	$0(3^2 \cdot 5^5 \cdot 13)$	$0(5^6)$	$0(5^7)$	$0(5^8)$	$0(5^9)$	$0(5^{10})$
N_6	0	0	0	0	5	$0(5^6)$	$0(5^7)$	$0(5^8)$	$0(5^9)$	$0(5^{10})$
N_7	0	0	0	0	0	6	$0(3^3 \cdot 5^7 \cdot 17)$	$0(5^8)$	$0(5^9)$	$0(5^{10})$
N_8	0	0	0	0	0	0	7	$0(5^8 \cdot 13^2)$	$0(5^9)$	$0(5^{10})$

Table 11: $yx + 4xy$

A.2 $\mathbb{Z}\langle x, y\rangle/(x^m + y^m)$

$N_i[d]$	1	2	3	4	5	6	7	8
N_2	0	1	$0(2^2)$	$0(2^2)$	$0(2^2)$	$0(2^2)$	$0(2^2)$	$0\ (2^2)$
N_3	0	0	2	$1(2^2)$	$0(2^4)$	$0(2^4)$	$0(2^4)$	$0(2^4)$
N_4	0	0	0	2	$0(2^4)$	$0(2^5)$	$0(2^6)$	$0(2^6)$
N_5	0	0	0	0	4	$2(2^3)$	$0(2^6)$	$0(2^7)$
N_6	0	0	0	0	0	3	$0(2^6)$	$0(2^7)$
N_7	0	0	0	0	0	0	6	$3(2^4)$
N_8	0	0	0	0	0	0	0	4
N_9	0	0	0	0	0	0	0	0

Table 12: $x^2 + y^2$

$N_i[d]$	1	2	3	4	5	6	7	8	9
N_2	0	1	2	$1(3^2)$	$0(3^3)$	$0(3^3)$	$0(3^3)$	$0(3^3)$	$0(3^3)$
N_3	0	0	2	5	$4(3^4)$	$1(3^8)$	$0(3^9)$	$0(3^9)$	$0(3^9)$
N_4	0	0	0	3	$7(2)$	$4(2^4 \cdot 3^7)$	$0(2^4 \cdot 3^{14})$	$0(2 \cdot 3^{15})$	$0(3^{15})$
N_5	0	0	0	0	6	$16(2^2)$	$11(2^6 \cdot 3^{14})$	$2(2^7 \cdot 3^{30})$	$0(2^4 \cdot 3^{35})$
N_6	0	0	0	0	0	9	$22(2^5)$	$11(2^{17} \cdot 3^{22})$	$0(2^{17} \cdot 3^{43})$
N_7	0	0	0	0	0	0	18	$45(2^7)$	$30(2^{21} \cdot 3^{39})$
N_8	0	0	0	0	0	0	0	27	$3(2^{14})$

Table 13: $x^3 + y^3$

$N_i[d]$	1	2	3	4	5	6	7	8	9
N_2	0	1	2	3	$2(4^2)$	$1(4^3)$	$0(4^4)$	$0(4^4)$	$0(4^4$
N_3	0	0	2	5	8	$7(2 \cdot 4^3)$	$4(2^2 \cdot 4^6)$	$1(2^4 \cdot 4^7)$	$0(2^4 \cdot 4^8)$
N_4	0	0	0	3	8	$13(2 \cdot 5)$	$10(2^8 \cdot 3^2 \cdot 4^4 \cdot 5^2)$	$4(2^7 \cdot 3^4 \cdot 4^{12} \cdot 5^2)$	$0(2^10 \cdot 3^4 \cdot 4^{14})$
N_5	0	0	0	0	6	18	$30(2^2 \cdot 5^2)$	$26(2^{13} \cdot 3^4 \cdot 4^8 \cdot 5^5)$	$12(2^{26} \cdot 3^2 \cdot 4^{18} \cdot 5^6)$
N_6	0	0	0	0	0	9	30	$49(2^4 \cdot 4^3 \cdot 5^5)$	$38(2^{26} \cdot 3^{10} \cdot 4^{20} \cdot 5^{12})$
N_7	0	0	0	0	0	0	18	63	$106(2^{12} \cdot 4^3 \cdot 5^{10})$
N_8	0	0	0	0	0	0	0	30	$110(2^2)$

Table 14: $x^4 + y^4$

References

[1] Martina Balagovic and Anirudha Balasubramanian. On the lower central series of a graded associative algebra. *Journal of Algebra*, 328:287–300, 2011.

[2] Wieb Bosma, John Cannon, and Catherine Playoust. The magma algebra system. i. the user language. *J. Symbolic Output.*, 24(3-4):235–265, 1997.

[3] Pavel Etingof, John Kim, and Xiaoguang Ma. On universal lie nilpotent associative algebras. *Journal of Algebra*, 321:697–703, 2009.

[4] Boris Feigin and Boris Shoikhet. On $[A,a]/[a,[a,a]]$ and on a w_n-action on the consecutive commutators of free associative algebras. *Math. Res. Lett.*, 14(5):781–795, 2007.

[5] George Kerchev. On the filtration of a free algebra by its associative lower central series. *preprint*, arXiv(1101.5741v1), 2011.