PRIMES STEP 2016 Entrance Exam
Part 1. General Questions. Logic.

Your name:

Exercise 1. 1 point. Everything I said or will say before 2017 will become a lie on Jan 1, 2017. Was this statement true, a lie, or a paradox?

Answer:

Exercise 2. 1 point. I draw several triangles on a piece of paper. First I showed the paper to Lev and asked him how many triangles are there. Lev said 5 and he was right. Then I showed the paper to Sasha and asked him how many triangles are there. Sasha said 3 and he was right. How many triangles are there on the paper? Explain.

Answer:

Exercise 3. 1 point. It takes me 2 minutes to get up the stairs from the first floor to the third floor. Assuming that I am very fit and will not change my climbing speed, how much time does it take me to get from the first floor to the 8th floor?

Answer:

Exercise 4. 1 point. I was driving with the speed of 60 miles an hour. What should be my speed if I want to pass each mile one minute faster?

Answer:

Exercise 5. 1 point. On a fictional island, all inhabitants are either knights, who always tell the truth, or knaves, who always lie. John and Bill are residents of the island of knights and knaves. John says: We are both knaves. Who is what? Explain.

Solution:

Exercise 6. 2 points. On a fictional island, all inhabitants are either knights, who always tell the truth, or knaves, who always lie. You come across three strangers, one dressed in black, one in white, and one in red. The one in red remains silent, but the other two speak:
Black: All of us are knaves.
White: Exactly one of us is a knight.
What are these individuals? Knights? Knaves? Which is which? Explain.

Solution:

Exercise 7. 2 points. In the game of Inky Pinky, the first player offers a concise, clear definition and the second player must translate that definition into two words that rhyme. The first player also indicates the number of syllables in each word by saying "Ink Pink" for one-syllable words, "Inky Pinky" for two-syllable words and so on. For example: Question: a yearly handbook. Inkity Pinkity. Answer: an annual manual.
Solve the following questions:

- Inexpensive land vehicle. Ink Pink.
- A dumb little boy with a bow and arrow. Inky Pinky.
- Frozen bike. Inkity Pinkity.
- Royal cloth. Inkitity Pinkitity.

Answer:

PRIMES STEP 2016 Entrance Exam
Part 2. Number Theory. Combinatorics. Algebra. Geometry.

Exercise 8. 1 point. What can you say if the number of divisors of a number including 1 and itself is 2? 3?

Answer:

Exercise 9. 2 points. To get to your account at a bank in Wonderland you need to dial a 7-digit password. The bank disconnects the phone as soon as you dial a wrong digit. What is your strategy to get to an account? In how many tries in the worst case are you guaranteed to brake the password and login? Explain.

Solution:

Exercise 10. 1 point. I am buying five bagels. The store sells sesame, plain, and garlic bagels. In how many different ways can I buy my five bagels?

Answer:

Exercise 11. 1 point. I have 8 students in my math club. I want to choose 2 teams of 3 people each for a math battle. In how many ways can I do it? You do not need to calculate the number: a formula is enough.

Answer:

Exercise 12. 1 point. Find the remainder of the polynomial $P(x) = x^{81} + x^{27} + x^9 + x^3 + x$ when dividing by $x - 1$. Explain.

Answer:

Exercise 13. 1 point. Find all possible a and b such that the equation $x^3 + ax + b = 0$ has three distinct roots that form an arithmetic progression. Explain.

Answer:

Exercise 14. 2 points. A square $PQRS$ is inscribed into a triangle ABC, so that point P is on the side AB, point Q is on the side AC, and points R and S are on the side BC, Express the side of the square through the length a of the side BC and the altitude h_a, from the vertex A to the side BC.

Answer:

Exercise 15. 2 points. There is an infinite wall on the plane in the form of a straight line. You have the materials to build an extra piece of wall of given length M. The enclosure must be rectangular in shape. For some strange reason you want to build an enclosure of the maximal area and you can use the existing piece of wall. What shape should your enclosure be? Why?

Answer:

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Part 3. Advanced Topics.

Exercise 16. 1 point. On Monday the baby said A, on Tuesday AU, on Wednesday AUUA, on Thursday AUUAUAAU. What will she say on Saturday?

Answer:

You can see that this very gifted baby increases her talking capacity twice each day. If the baby continues indefinitely, her text converges to an infinite sequence that mathematicians call the Thue-Morse sequence. Of course, mathematicians use zeros and ones instead of A and U, so the sequence looks like 01101001100101101 00. . ..

Exercise 17. 2 points. Prove that every other term reproduces the whole sequence.

Solution:

Exercise 18. 2 points. Prove that the sequence doesn't contain substrings 000 and 111.

Solution: