

PRIMES STEP 2015 Entrance Exam. Solutions.
Part 1. General Questions. Logic.

Summary. The highest score was 10 points, the lowest score was 0 points. The cutoff was 7 points. Two people with 6 and 6.5 points will be on the waiting list.
Every problem was completely solved by at least one person.

Exercise 1. There were six bowls full of candy on the table. Mike ate three bowls of candy. How many bowls are there on the table now?

Solution. This was a trick question. Mike couldn't have eaten the bowls. There are still six bowls on the table.

Exercise 2. Mary's mom has four children. The first child is called April. The second May. The third June. What is the name of her fourth child?

Solution. This was a question on paying attention. It starts "Mary's mom." The fourth child is Mary, not July.

Exercise 3. John is five times older than his sister Kate, and in 6 years he will be twice as old as she is. How old is Kate?

Solution. Denote Kate's age by k . Then John is $5k$ now and $5k+6$ in six years. That means, $5k+6 = 2(k+6)$. Therefore, $k = 2$.

Exercise 4. Detective Radstein is investigating a robbery. He apprehends three suspects: Anne, Bill, and Caroline. The detective knows that no one else could have participated in the robbery. During the interrogation the suspects make the following statements:

- Anne: I didn't do it. Bill did it.
- Bill: I didn't do it. Caroline did it.
- Caroline: I didn't do it. Anne did it alone.

Detective Radstein also discovered that one of the suspects always tells the truth, and two others always lie. But he doesn't know who is the truth-teller.

Solution. (Note. I deleted the question by mistake. The question was, "Who committed the robbery?" Because of that, the full credit was given to students for both types of answer: who committed the robbery and who is the truth-teller.)

If Anne tells the truth, then Bill and Caroline didn't do it. In this case Caroline must also be a truth-teller. Contradiction. If Caroline is a truth-teller, then Anne is a liar and Bill participated in the robbery. This contradicts Caroline's statement. Thus, Bill is the truth-teller. Bill didn't participate in the robbery and Caroline did. But this means that Anne's statement that she didn't do it is a lie.

Answer: Bill is the truth-teller. Anne and Caroline committed the robbery together.

Exercise 5. “John has more than a hundred books,” said Pete.

“No, he has less than one hundred books,” said Ann.

“He, surely, has at least one book,” said Mary.

If only one statement is true, how many books does John have?

Solution. If Pete’s statement is true, then Mary’s statement is also true—contradiction. If Ann’s statement is true, then Mary’s statement must be a lie. Therefore, John has zero books. If Mary’s statement is true, then other statements are lies and John has 100 books. Answer: John has either 0 or 100 books.

Exercise 6. There are four coins lying on the table in a row. Some of the coins are fake (lighter) and other coins are real. It is known that there exist at least one fake coin and one real coin. Also, every real coin lies to the left of every fake coin. Using one weighing on a balance scale find the type of every coin on the table. Explain.

Solution. There are three possibilities for the row of coins from left to right: RRRF, RRFF, and RFFF. There are many solutions to the problem. The most common among the students was to weigh the first and the last coins against the middle two coins. If the scale balances, then RRFF. If the middle coins are heavier, then RRRF. If the middle coins are lighter then RFFF.

Exercise 7. There are 6 glasses on the table in a row. The first three are empty, and the last three are filled with water. How can you make it so that the empty and full glasses alternate, if you are allowed to touch only one of the glasses? (You can’t push one glass with another.)

Solution. The most common solution was: Pour the water from the fifth glass into the second glass.

Exercise 8. There are six numbers on the blackboard: 1, 2, 3, 4, 5, 6. In one move you are allowed to add 1 to any two of them. How many moves do you need to make all numbers equal each other? Explain.

Solution. If all the numbers are the same, the total is even. Currently the total is odd. It is impossible to change the total from odd to even by adding 2 at every move.

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Part 2. Number Theory. Combinatorics. Algebra. Geometry.

Exercise 9. Find the largest square of the form $1! + 2! + 3! + \dots + n!$. Explain.

Solution. The several first sums of factorials are: 1, 3, 9, 33, 153. Higher factorials end in zero. Therefore the sum will end in 3 for all the following values of n . No square ends in 3, therefore the largest square is 9.

Exercise 10. Let $SOD(n)$ be the sum of the digits of n . Suppose $f(n)$ is the result of iterating SOD many times until we get a single digit. That is, $f(n) = SOD(SOD(SOD \dots (n) \dots))$. Find $f(2^{2015})$.

Solution. The single digit that remains after applying this operation to 2^n is the same as the remainder of 2^n modulo 9. Let's look at the list of the remainders as a function of n : 2, 4, 8, 7, 5, 1, 2, 4, and so on. The list is periodic with period 6. Therefore, in the 2015th place we will see 5.

Exercise 11. I am buying six bagels. The store sells sesame, plain, and garlic bagels. In how many different ways can I buy my six bagels?

Solution. This is the standard combinations-with-repetitions-allowed problem. The answer is $\binom{8}{2} = 28$. It is possible to solve this without using the formula. For example, if I buy 4 sesame bagels, the other two bagels can be distributed in three ways: both plain, plain and garlic, and both garlic. Similarly, we see that there are 1 way if there are 6 sesame bagels, 2 ways if there are 5 sesame bagels, and so on. The total is $1 + 2 + 3 + \dots + 7 = 28$.

Exercise 12. I have 10 students in my math club. I want to choose 2 teams of 3 people each for a math battle. In how many ways can I do it?

Solution. I can choose the first team in $\binom{10}{3}$ ways. After that I can choose the second team in $\binom{6}{3}$. The total is the product divided by 2, because I can swap teams. The answer is 2100. I gave partial credit to students who didn't divide by 2.

Exercise 13. Find the sum of all the coefficients of the polynomial $(x^2 - 4x + 1)^{2015}$.

Solution. The sum of the coefficients of a polynomial is the value of this polynomial at $x = 1$. The answer is $(-2)^{2015}$.

Exercise 14. In the equation $3x^2 - 3x + 1 = 0$, what is the sum of the squares of the roots?

Solution. Denote roots as x and y . The sum of the roots is 1: $x + y = 1$. The product is $1/3$: $xy = 1/3$. The sum of the squares of the roots is $x^2 + y^2 = (x + y)^2 - 2xy = 1 - 2/3 = 1/3$.

Exercise 15. A triangle ABC is given. The median BX , the altitude BY , and the angle bisector BZ are drawn from the vertex B . They intersect the line AC in three different points X , Y , and Z . Which of these three points is in the middle? Explain.

Solution. Suppose BC is the shorter side. Then angle C is greater than angle A . This means the angle YBC is smaller than the angle ABY . Thus Y is closer to C than Z . In addition, Z divides AC proportionately to $|AB|/|BC|$. Thus, Z is closer to C , then the middle point. The answer is Z .

Exercise 16. There is an infinite wall on the plane in the form of a straight line. You have the materials to build an extra piece of wall of any shape of given length M . For some strange reason you want to build an enclosure of the maximal area and you can use the existing piece of wall. What shape should your enclosure be? Why?

Solution. Suppose we have an enclosure of some shape. Reflect it with respect to the infinite wall and remove the infinite wall. The resulting enclosure has the maximal area with the perimeter of length $2M$. It has to be a circle. That means, that the answer is a semi-circle.