Gabriel's Theorem and the Subspaces Problem

Sophia Hou, Henrick Rabinovitz, and Derek Zhao

MIT PRIMES
Mentor: Serina Hu

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Outline

The Pairs of Subspaces Problem

One-Subspace Problem

Note

All spaces will be assumed to be complex vector spaces.

One-Subspace Problem

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Idea

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Definition (Isomorphism)

We say a vector space V is *isomorphic* to V' if there exists an *isomorphism*, defined to be a bijective linear map, from V to V'. For pairs (V, W) where W is a subspace of V, we define an isomorphism from (V, W) to (V', W') to be an isomorphism from V to V' that takes W to W'.

One-Subspace Problem

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Refinement

Can we classify, up to isomorphism, all pairs of spaces (V, W) where W is a subspace of V and both spaces are finite dimensional?

Solution One-Subspace Problem

1. $\dim V \geq \dim W \geq 0$.
2. Let $\dim W = m$ and $\dim V = m + n$.
3. Extend a basis of m elements for W to a basis of $m + n$ elements for V; let the additional n elements generate W'.
4. Then $V \cong W \oplus W'$, $W \cong \mathbb{C}^m$, $W' \cong \mathbb{C}^n$.
5. Thus, our classification is pairs of the form $(\mathbb{C}^m \oplus \mathbb{C}^n, \mathbb{C}^n)$.

Pairs of Subspaces Problem

To make a harder problem, we consider two subspaces instead of just one:

Problem

Can we classify up to isomorphism all triples (V, W_1, W_2) of finite-dimensional vector spaces such that W_1 and W_2 are subspaces of V?

We will solve this in a series of steps.

Step 1: Remove Excess

Step 1

Consider the subpspace $W_1 + W_2$ of V. As we did for the one-subspace problem, get a complement of this subspace in V, W_3. Then $V = (W_1 + W_2) \oplus W_3$.

Step 2: Remove Intersection

Step 2

Let the intersection of W_1 and W_2 be W_0. Let W_4 and W_5 be complements of W_0 in W_1 and W_2 respectively. Then $W_1 = W_0 \oplus W_4$, $W_2 = W_0 \oplus W_5$, and $W_1 + W_2 = W_3 \oplus W_4 \oplus W_5$.

Step 3: Putting it Together

Step 3

$V = (W_1 + W_2) \oplus W_3 = W_0 \oplus W_3 \oplus W_4 \oplus W_5$, $W_1 = W_0 \oplus W_4$, and $W_2 = W_0 \oplus W_5$. Letting a, b, c, d be the dimensions of W_0, W_3, W_4, W_5 respectively, our classification is:

$$(\mathbb{C}^a \oplus \mathbb{C}^b \oplus \mathbb{C}^c \oplus \mathbb{C}^d, \mathbb{C}^a \oplus \mathbb{C}^c, \mathbb{C}^a \oplus \mathbb{C}^d)$$

Quiver Representations

Quivers

Definition

A quiver is a directed graph.

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Examples

$$\bullet \longrightarrow \bullet$$

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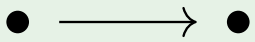

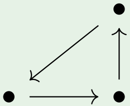

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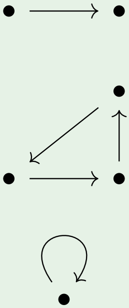

Dynkin Diagrams

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Let Γ denote a graph and R_Γ be the adjacency matrix. The Cartan matrix of Γ is $A_\Gamma = 2I - R_\Gamma$.

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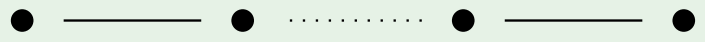

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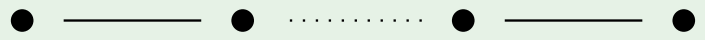
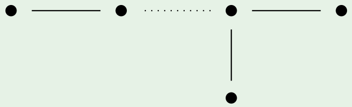


D_n :



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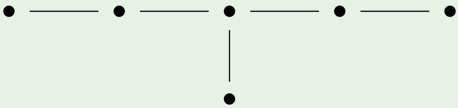
E_6 :



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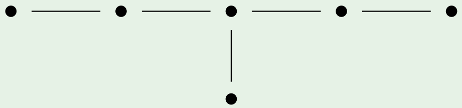
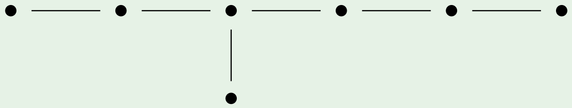


E_7 :



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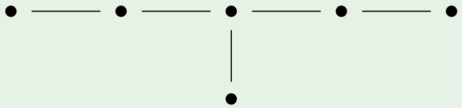

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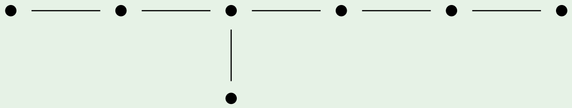
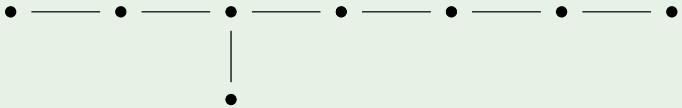


E_8 :



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2. Classifying representations of A_3 solves the pair of subspaces problem.
3. Classifying representations of D_4 solves the triples of subspaces problem.

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A representation of a quiver Q is an assignment of each vertex i to a vector space V_i and each edge h_{ij} to a linear map $x_{ij} \colon V_i \to V_j$.

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Definition

The direct sum of two representations (V, x) and (W, y) is $(V \oplus W, x \oplus y)$.

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A representation of a quiver is indecomposable if it cannot be written as the direct sum of subrepresentations.

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Remark

Considering indecomposable representations are helpful because they can be thought of building blocks for all representations.

Gabriel's Theorem

Roots

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Proposition

There are finitely many roots of B_Γ.

Roots

Definition

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Definition

A root $\alpha = \sum_{i=1}^{n} = k_i\alpha_i$ is called a positive root if $k_i \geq 0$ for all i.

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Theorem

A quiver Q of type A_n, D_n, E_6, E_7, E_8, has finitely many indecomposable representations. Furthermore, the dimension vector of an indecomposable representation corresponds with a positive root and every positive root corresponds with one indecomposable representation.

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Remark

The proof of this theorem involves looking at reflection functors, which preserves indecomposable representations and dimension.

The Triples of Subspaces Problem

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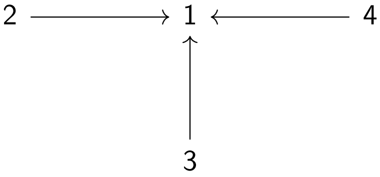

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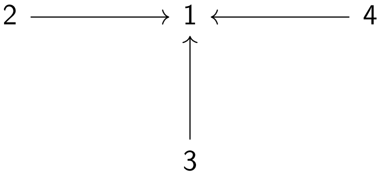

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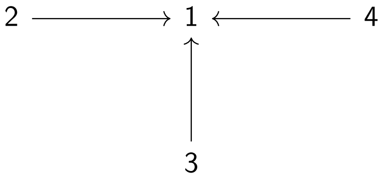

We want to find indecomposable representations of the above quiver. While we could do a similar process as with the pairs of subspaces problem, the process is much more complicated.

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Gabriel's Theorem states that the dimension vectors of the indecomposable representations and the positive roots of B_Γ have a 1-to-1 correspondence.

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Gabriel's Theorem states that the dimension vectors of the indecomposable representations and the positive roots of B_Γ have a 1-to-1 correspondence.

If we can find the the positive roots of B_{D_4}, we can match these with indecomposable representations of D_4.

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$$A_{D_4} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

Solving the Problem

Solution

Let $B(x,x) = x^T A_{D_4} x = 2$ where x is a root and let x be some vector in $\mathbb{Z}^4$ such that $x = (a\ b\ c\ d)$ for some $a, b, c, d \in \mathbb{Z}$. Then in order for x to be a positive root, we want $a, b, c, d \geq 0$.

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$$B(x,x) = (a\ b\ c\ d)\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$
$$= 2a^2 + 2b^2 + 2c^2 + 2d^2 - 2ab - 2ac - 2ad = 2$$

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It turns out that there are only 12 solutions to this equation where $a, b, c, d \geq 0$. These solutions are:

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$$\begin{matrix} (1\ 0\ 0\ 0) & (0\ 1\ 0\ 0) & (0\ 0\ 1\ 0) & (0\ 0\ 0\ 1) \\ (1\ 1\ 0\ 0) & (1\ 0\ 1\ 0) & (1\ 0\ 0\ 1) & (1\ 1\ 1\ 0) \\ (1\ 1\ 0\ 1) & (1\ 0\ 1\ 1) & (1\ 1\ 1\ 1) & (2\ 1\ 1\ 1) \end{matrix}$$

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Note that 3 of these solutions are not injective and thus, cannot contribute to our triples of subspaces problem. Specifically, these are the following indecomposable representations:

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Relating Back to the Triples of Subspaces Problem

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Can we classify up to isomorphism all quadruples (V, W_1, W_2, W_3) of finite-dimensional vector spaces such that W_1, W_2, W_3 are subspaces of V?

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We can relate this to the quivers we found by letting the numbers at each vertex represent the dimensions of V, W_1, W_2, and W_3.

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Using a method similar to the beginning of the presentation, paired with the indecomposable representations, we can deduce the following.

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$$\begin{array}{ccccc} W_1 & \longrightarrow & V & \longleftarrow & W_3 \\ & & \uparrow & & \\ & & W_2 & & \end{array}$$

is $\bigoplus_I m_I \cdot I$ where the I are the indecomposable representations and m_I is its multiplicity.

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For this representation, the multiplicity of it is equal to the dimension of the complement of $W_1 + W_2 + W_3$ in V.

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For this case, if we let W_4 be the intersection of $W_1, W_2, \text{and} W_3$ and W_5 be the complement of W_4 in W_1, then the multiplicity is the dimension of the direct sum of the complement of $W_1 + W_2 + W_3$ in V and W_5.

Acknowledgements

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Also a big thank you to Prof. Etingof for his book *Introduction to Representation Theory* in which this presentation is based upon.