Vertex functions of type D Nakajima quiver varieties

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Abstract

We study the quasimap vertex functions of type D Nakajima quiver varieties. When the quiver varieties have isolated torus fixed points, we compute the coefficients of the vertex functions in the K-theoretic fixed point basis. We also give an explicit combinatorial description of zero-dimensional type D quiver varieties and their vertex functions using the combinatorics of minuscule posets. Using Macdonald polynomials, we prove that these vertex functions can be expressed as products of q-binomial functions, which proves a degeneration of the conjectured 3d mirror symmetry of vertex functions. We provide an interpretation of type D spin vertex functions as the partition functions of the half-space Macdonald processes of Barraquand, Borodin, and Corwin. This hints that the geometry of quiver varieties may provide new examples of integrable probabilistic models.

Contents

1 Introduction

1.1 Quiver varieties and 3d mirror symmetry

This paper is about vertex functions of type D Nakajima quiver varieties. Nakajima quiver varieties are one of the central objects in geometric representation theory, defined by Nakajima in [Nak94] to give a geometric construction of representations of Kac–Moody algebras. They include as special cases cotangent bundles of partial flag varieties and moduli spaces of sheaves on resolutions of Kleinian singularities. They are also one of the main ways of constructing examples of symplectic resolutions, and hence fit into the programs of symplectic duality and 3d mirror symmetry studied, for example, in [AO21, BLPW14, Din20, Din22b, Kam22, KS22, IS96, RSVZ21].

Vertex functions, defined by Okounkov in [Oko17], are generating functions for certain counts of quasimaps to quiver varieties. They are K-theoretic analogs of I-functions from Gromov–Witten theory. They provide interesting examples of q-series, containing so-called basic hypergeometric series [GR04], and are central in 3d mirror symmetry.

1.2 Vertex functions in type D

In the existing literature, all computations of vertex functions are done only for (finite or affine) type A quiver varieties. The main goal of this paper is to move beyond the type A setting to type D quiver varieties. One immediate complication is that the set of torus fixed points on these varieties need not be zero-dimensional, as it is in the type A setting. Indeed, the geometry of the fixed components of type D quiver varieties need not be trivial, and their Betti numbers were described by Nakajima in [Nak04].

To deal with this, we will restrict ourselves to quiver varieties with framing vectors which are nonzero only at minuscule vertices, for which this issue does not arise. On such quiver varieties, we will give an explicit combinatorial description of the torus fixed points in terms of the so-called minuscule posets (see [Pro99, Pro86, Ste94]); see Theorems 4.7 and 4.12. For the case of a one-dimensional framing where the quiver varieties parameterize certain modules over the preprojective algebra of the quiver, this relationship was noticed before in [DEKMF24]. From this description, we can immediately read off the weights of the restrictions of the tautological bundles to fixed points; see Equation (11). This poset description also turns out to be enough to allow us to use the localization theorem to write explicit formulas for these type D vertex functions.

Theorem 1.1 (Theorem 5.6)**.** *There is an explicit combinatorial formula for vertex functions of type D quiver varieties with framings at minuscule vertices.*

1.3 Zero-dimensional type D varieties

As proven in [Oko17], when one sends equivariant parameters to some infinity in the vertex function of a quiver variety, one obtains the vertex function of the fixed point set. Such vertex functions are nontrivial even if the fixed points are isolated. From a sufficiently deep understanding of them, one can read off formulas for the characters of the tangent space of the 3d mirror dual variety; see [DS20a] for more details. Vertex functions of zero-dimensional type A quiver varieties were studied in [DS20b], where it was shown that such varieties are in natural bijection with Young diagrams. Certain product formulas, which are the type A version of our Theorem 1.2 below, were also proven.

For our goals in this paper, it is thus important to study the (nontrivial) vertex functions for zero-dimensional type D quiver varieties. We turn our attention to this in Section 6, proving a formula expressing the vertex functions of these zero-dimensional type D quiver varieties as products of q-binomial series.

To state the formula, we need some notation. We fix a type D_n quiver and denote by Φ the set of roots. Let α_i for $1 \leq i \leq n$ be a choice of simple roots, and let ω_i be the fundamental weights. Let v and w be the dimension and framing vectors, respectively, such that $\mathsf{w}_i = 1$ for a single minuscule vertex i and is 0 otherwise. Let $\mathcal{M}(\mathsf{v}, \mathsf{w})$ be the Nakajima quiver variety associated to this data. As usual, it should be thought of as encoding the space of weight $\mu := \sum_i \mathsf{w}_i \omega_i - \sum_i \mathsf{v}_i \alpha_i$ inside a representation with highest weight $\lambda := \sum_i \mathsf{w}_i \omega_i$. The vertex function $V(\boldsymbol{z})$ of $\mathcal{M}(\mathsf{v}, \mathsf{w})$ is an element of the ring $\mathbb{Q}(q, \hbar)[[z_i]]_{1 \leq i \leq n}$.

Theorem 1.2 (Theorem 6.1)**.** *Let $\Phi' = \{\alpha \in \Phi \mid (\mu, \alpha) < 0\}$. The vertex function $V(\boldsymbol{z})$ of $\mathcal{M}(\mathsf{v}, \mathsf{w})$ factorizes into the following product of q-binomial series:*

$$V(\boldsymbol{z}) = \prod_{\alpha \in \Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha),$$

where $z_\alpha := \prod_i z_i^{n_i}$ and $\mathsf{a}_\alpha := \sum_i n_i \mathsf{a}_i$ for $\alpha = \sum_i n_i \alpha_i$, for some $\mathsf{a} \in \mathbb{Z}^{Q_0}$.

In the previous Theorem, the series F is the q-binomial series, defined by

$$F(z) := \sum_{d \geq 0} \frac{(\hbar)_d}{(q)_d} z^d, \quad \text{where} \quad (x)_d := \prod_{i=0}^{d-1} (1 - x q^i).$$

Our method of proof is the following. First, we start with the localization formula for the vertex function, Theorem 5.6, which works for type D quiver varieties that are not necessarily single points. For quiver varieties which are single points, and for these only, we are able to rewrite the localization formula as a certain sum of skew-Macdonald polynomials. Finally, we compute this sum by repeated applications of the skew-Cauchy identity.

1.4 Consequences

We mention a few applications of Theorem 1.2. As discussed above, one is supposed to be able to read off from this the character of the tangent space of the 3d mirror dual variety. By comparing with the formulas of [KP21], Theorem 1.2 shows immediately that this is the case. This fact is used in the proof of the quantum Hikita conjecture for ADE quiver varieties in [DKK]. Vertex functions for arbitrary quiver varieties are also conjectured to have a certain relation to the vertex functions of their 3d mirror dual varieties [AO21, Din20, Din22b, SZ22]. All of the cases in which this has been proven use formulas like Theorem 1.2 as an ingredient. We hope that this will one day be the case for Theorem 1.2 as well.

The techniques used here also allow us to study vertex functions with descendants. As in [DS22], descendants given by exterior powers of tautological vector bundles can be inserted by applying a Macdonald difference operator to the vertex with trivial descendant. More generally, any q-difference operator acting diagonally on Macdonald polynomials with eigenvalues given as a function of $t^{n-i}q^{\lambda_i}$ (for example, Noumi's q-difference operator from [NS21]), can be used to insert descendants. As in [DS22], one can interpret this as giving an explicit rational formula for the *capped* descendant vertex, an object significantly harder to compute than the (bare) vertex.

Theorem 6.2 also has a probabilistic interpretation in terms of Macdonald processes. Macdonald processes were first introduced by A. Borodin and I. Corwin in [BC14], which are probability measures on sequences of partitions defined in terms of nonnegative specializations of the Macdonald symmetric functions and two Macdonald parameters $q, t \in [0, 1)$, which, in our paper, correspond to $q, \hbar$ respectively. Half-space Macdonald processes were first introduced by Borodin et al in [BBC20], and have connections to half-space systems, such as the Kardar–Parisi–Zhang (KPZ) stochastic PDE (see [KPZ86, Cor12]) and the log-gamma directed polymer in a half-quadrant (see [OSZ13]).

In the case of type D_n quiver varieties with framing $\mathsf{w} = \delta_n$, Theorem 1.2 shows that vertex functions are exactly partition functions for half-space Macdonald processes. The type A version of this statement, relating vertex functions to ordinary Macdonald processes is a consequence of [DS20b], though it was not noticed there. From this perspective, vertex functions with descendants compute the expectation values of certain observables of (ordinary or half-space) Macdonald processes.

1.5 Future directions

The results of this paper point to several future directions.

One such direction is to complete the study of simply laced Dynkin diagrams by studying vertex functions of type E quiver varieties where the framing is a sum of minuscule framings. Indeed, the description of fixed points in terms of minuscule posets and the subsequent formula for vertex functions, applies immediately to this setting. We believe that the symmetric function techniques of this paper should be applicable to that setting as well, leading to product formulas for the vertex functions of zero-dimensional type E quiver varieties.

There are also quiver varieties which are not Dynkin type which still have some zero-dimensional torus fixed components. We believe the methods of this paper can be extended to cover some of these situations as well, namely, those in which the torus fixed points can be described as slant sums, see [Pro99], of minuscule posets.

Finally, given the correspondence between the partition function of Macdonald processes and half-space Macdonald processes with vertex functions of type A and D quiver varieties, we hope that the geometry of quiver varieties will give rise to new and interesting probabilistic processes. According to this dream, this paper already provides something new: vertex functions of type D quiver varieties with framing $\mathsf{w} = \delta_1$ should be interpreted as the partition function of a new probabilistic process. What should be the definition of this process and even the space on which this process is defined is at present unknown to us. We hope to revisit this in the future.

1.6 Outline

This paper is structured as follows.

In Section 2, we give background on Nakajima quiver varieties.

In Section 3, we review the properties of Macdonald polynomials that we require.

In Section 4, we give an explicit description of zero-dimensional type D quiver varieties with a single framing at a minuscule vertex. We formulate this in terms of the combinatorics of minuscule posets and explain how to read off weights of tautological vector bundles from this description.

In Section 5, we review the definition of vertex functions. Then we spell out the localization formula in the type D setting.

In Section 6, we state our product formula for zero-dimensional type D quiver varieties and explain some applications.

In Sections 7 and 8 we prove the product formula using the localization theorem and properties of Macdonald polynomials.

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2 Background

The goal of this section is to review the definition of Nakajima quiver varieties, defined in [Nak98, Nak94], which are moduli spaces of certain quiver representations. The following material is standard, and can be found in [Gin09], for example.

2.1 Definition of Nakajima quiver varieties

Let $Q = (Q_0, Q_1)$ be a quiver with vertex set Q_0 and arrow set Q_1. Let $o(e)$ and $i(e) \in Q_0$ denote the outgoing (source) and incoming (target) vertices of the arrow $e \in Q_1$, respectively, so that e is an arrow from $o(e)$ to $i(e)$. Equivalently, this is written as $o(e) \xrightarrow{e} i(e) \in Q_1$.

Choose $\mathsf{v}, \mathsf{w} \in \mathbb{N}^{Q_0}$. Let V_i and W_i be complex vector spaces such that $\dim V_i = \mathsf{v}_i$ and $\dim W_i = \mathsf{w}_i$. Let

$$\mathrm{Rep}_Q(\mathsf{v}, \mathsf{w}) := \bigoplus_{e \in Q_1} \mathrm{Hom}\left(V_{o(e)}, V_{i(e)}\right) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}\left(W_i, V_i\right).$$

This is known as the space of framed representations of Q. The vector v is called the dimension, and w is called the framing dimension. The trace pairing identifies $\mathrm{Hom}(V_i, V_j)^*$ with $\mathrm{Hom}(V_j, V_i)$ and thus induces an isomorphism

$$\begin{aligned} T^* \mathrm{Rep}_Q(\mathsf{v}, \mathsf{w}) \cong & \bigoplus_{e \in Q_1} \mathrm{Hom}\left(V_{o(e)}, V_{i(e)}\right) \oplus \mathrm{Hom}\left(V_{i(e)}, V_{o(e)}\right) \\ & \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}\left(W_i, V_i\right) \oplus \mathrm{Hom}\left(V_i, W_i\right). \end{aligned} \tag{1}$$

The group $G_{\mathsf{v}} := \prod_{i \in Q_0} GL(V_i)$ acts on $\mathrm{Rep}_Q(\mathsf{v}, \mathsf{w})$, inducing a Hamiltonian action on $T^* \mathrm{Rep}_Q(\mathsf{v}, \mathsf{w})$ with associated moment map

$$\mu : T^* \mathrm{Rep}_Q(\mathsf{v}, \mathsf{w}) \to \mathrm{Lie}(G_{\mathsf{v}})^* \cong \prod_{i \in Q_0} \mathrm{End}(V_i),$$

where the isomorphism is by the trace pairing. Let θ be a character of G_{v}, which we will view as a vector $\theta \in \mathbb{Z}^{Q_0}$ with associated character

$$\begin{aligned} \chi_\theta : G_{\mathsf{v}} &\to \mathbb{C}^\times, \\ (g_i)_{i \in Q_0} &\mapsto \prod_{i \in Q_0} \det(g_i)^{\theta_i}. \end{aligned}$$

Definition 2.1 ([Nak94])**.** The Nakajima quiver variety associated to the data $Q, \mathsf{v}, \mathsf{w}$, and θ is the algebraic symplectic reduction

$$\mathcal{M}_{Q,\theta}(\mathsf{v},\mathsf{w}) := T^* \operatorname{Rep}_Q(\mathsf{v},\mathsf{w}) /\!/\!/\!/_\theta\, G_{\mathsf{v}} = \mu^{-1}(0) /\!/_\theta\, G_{\mathsf{v}}.$$

The notation $\mu^{-1}(0) /\!/_\theta\, G_{\mathsf{v}}$ refers to the GIT quotient of $\mu^{-1}(0)$ by G_{v} with respect to the stability parameter θ.

In this paper, we will only consider the positive stability parameter $\theta = (1, 1, \ldots, 1)$. So from now on, we will simplify notation and write $\mathcal{M}_Q(\mathsf{v},\mathsf{w})$ with this choice of stability understood.

2.2 More explicit description

We will now make Definition 2.1 more explicit.

Points of $\mathcal{M}_Q(\mathsf{v},\mathsf{w})$ are represented by elements of $T^* \operatorname{Rep}_Q(\mathsf{v},\mathsf{w})$. Using (1), we will denote elements of this latter space by a tuple

$$\left((X_e)_{e\in Q_1}, (Y_e)_{e\in Q_1}, (A_i)_{i\in Q_0}, (B_i)_{i\in Q_0} \right),$$

where $X_e \in \operatorname{Hom}(V_{o(e)}, V_{i(e)})$, $Y_e \in \operatorname{Hom}(V_{i(e)}, V_{o(e)})$, $A_i \in \operatorname{Hom}(W_i, V_i)$, and $B_i \in \operatorname{Hom}(V_i, W_i)$. We will sometimes abuse notation and denote such a tuple simply by (X, Y, A, B). In this notation, the moment map is given explicitly by

$$\begin{aligned} &\mu\left(\left((X_e)_{e\in Q_1}, (Y_e)_{e\in Q_1}, (A_i)_{i\in Q_0}, (B_i)_{i\in Q_0} \right) \right) \\ &= \left(\sum_{\substack{e\in Q_1 \\ i(e)=i}} X_e Y_e - \sum_{\substack{e\in Q_1 \\ o(e)=i}} Y_e X_e + A_i B_i \right)_{i\in Q_0} . \end{aligned} \tag{2}$$

For generic choices of θ, including the positive stability, the notions of θ-stability and θ-semistability from geometric invariant theory coincide; see Section 3 of [Gin09]. Furthermore, it is known that G_{v} acts freely on the set of stable points. Thus $\mathcal{M}_Q(\mathsf{v},\mathsf{w})$ is a smooth quasiprojective variety which as a set, is equal to

$$\mu^{-1}(0)^s / G_{\mathsf{v}},$$

where $\mu^{-1}(0)^s$ denotes the set of stable points in $\mu^{-1}(0)$. The set of stable points in $T^* \operatorname{Rep}_Q(\mathsf{v},\mathsf{w})$ has the following explicit description.

Proposition 2.2 (Proposition 5.1.5 of [Gin09])**.** *A point* $(X, Y, A, B) \in T^* \operatorname{Rep}_Q(\mathsf{v},\mathsf{w})$ *is stable, if for any collection* $(S_i)_{i\in Q_0}$*, the conditions*

- S_i *is a subspace of* V_i*,*
- $\operatorname{Im}(A_i) \subset S_i$*,*
- *for all* $i \xrightarrow{e} j \in Q_1$*, we have* $X_e(S_i) \subset S_j$ *and* $Y_e(S_j) \subset S_i$*,*

imply that $S_i = V_i$*.*

2.3 Torus action and fixed points

The action of $G_{\mathsf{w}} := \prod_{i\in Q_0} GL(W_i)$ on $T^*\operatorname{Rep}_Q(\mathsf{v},\mathsf{w})$ descends to an action on $\mathcal{M}_Q(\mathsf{v},\mathsf{w})$. In addition, the dilation of cotangent fibres also descends to an action of a torus which we will write as $\mathbb{C}^\times_\hbar$. Choose a decomposition $W_i \cong W_i' \oplus W_i''$ for each $i \in Q_0$. Let $\mathsf{w} = \mathsf{w}' + \mathsf{w}''$ be the corresponding decomposition of the framing vector. Let $\mathsf{A} \subset G_{\mathsf{w}}$ be the rank 1 torus which acts on each W_i' with weight 1 and on each W_i'' with weight 0. Then

$$\mathcal{M}_Q(\mathsf{v},\mathsf{w})^{\mathsf{A}} = \bigsqcup_{\substack{\mathsf{v}',\mathsf{v}''\in\mathbb{N}^{Q_0}\\ \mathsf{v}'+\mathsf{v}''=\mathsf{v}}} \mathcal{M}_Q(\mathsf{v}',\mathsf{w}') \times \mathcal{M}_Q(\mathsf{v}'',\mathsf{w}''). \tag{3}$$

This is commonly referred to as the tensor product property of Nakajima varieties [Nak01]. Iterating the tensor product property, one sees that fixed points of a maximal torus of G_{w} are given by products of Nakajima varieties with one dimensional framing. For $i \in Q_0$, let $\delta_i \in \mathbb{N}^{Q_0}$ be the framing vector with 1 in position i and 0 elsewhere.

Proposition 2.3 ([Nak04])**.** *If Q is an ADE quiver, then*

$$\bigsqcup_{\mathsf{v}\in\mathbb{N}^{Q_0}} \mathcal{M}_Q(\mathsf{v},\delta_i)$$

consists of isolated points if and only if i is a minuscule vertex.

2.4 Polarization

The vector spaces V_i descend to vector bundles $\mathscr{V}_i$ on $\mathcal{M}$ which are called the tautological vector bundles. Similarly, the spaces W_i descend to topologically trivial bundles $\mathscr{W}_i$. These bundles carry T-equivariant structures. Furthermore, the K-theory class of the tangent bundle of $\mathcal{M}$ decomposes as

$$T\mathcal{M} = T^{1/2} + \hbar^{-1}(T^{1/2})^\vee \in K_{\mathsf{T}}(\mathcal{M}),$$

where

$$T^{1/2} = \sum_{e\in Q_1} \operatorname{Hom}\left(\mathscr{V}_{o(e)}, \mathscr{V}_{i(e)}\right) + \sum_{i\in Q_0} \operatorname{Hom}\left(\mathscr{W}_i, \mathscr{V}_i\right) - \sum_{i\in Q_0} \operatorname{Hom}\left(\mathscr{V}_i, \mathscr{V}_i\right). \tag{4}$$

3 Macdonald polynomials

In this section, we review some properties of Macdonald polynomials that we will need later on. The following is standard and can be found in [Mac95, Gas95]. The standard notation in the literature is related to ours by $\hbar = t$.

3.1 Symmetric functions

Let $\mathcal{F} = \mathbb{C}(q,\hbar) \otimes_{\mathbb{C}} \mathbb{C}[p_1, p_2, \dots]$ be the ring of symmetric functions in infinitely many variables $x = (x_1, x_2, \dots)$, where p_i denotes the i-th power-sum symmetric function. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. The Macdonald inner product on $\mathcal{F}$ is given by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \prod_{n\geq 1} n^{m_n(\lambda)} m_n(\lambda)! \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-\hbar^{\lambda_i}},$$

where $m_n(\lambda)$ is the multiplicity of n in λ and $\ell(\lambda)$ is the number of nonzero parts of λ.

3.2 Macdonald polynomials

Macdonald P-polynomials $P_\lambda(x; q, \hbar)$ are uniquely determined by two properties. First, they expand upper-triangularly in the monomial basis m_μ as

$$P_\lambda = \sum_{\mu \trianglelefteq \lambda} u_{\lambda\mu} m_\mu, \quad u_{\lambda\mu} \in \mathbb{Q}(q, \hbar),$$

where $u_{\lambda\lambda} = 1$ and $\trianglelefteq$ denotes the dominance order on partitions. Second, they are orthogonal with respect to the Macdonald inner product, meaning that $\langle P_\lambda, P_\mu \rangle = 0$ for $\lambda \neq \mu$. The dual basis $Q_\lambda(x; q, \hbar)$ is uniquely defined by the $(q,\hbar)$-Hall pairing $\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda\mu}$. Specifically, $Q_\lambda = b_\lambda^{-1} P_\lambda$, where

$$b_\lambda = \prod_{\square \in \lambda} \frac{1 - q^{a(\square)} \hbar^{\ell(\square)+1}}{1 - q^{a(\square)+1} \hbar^{\ell(\square)}}, \tag{5}$$

and $a(\square)$ and $\ell(\square)$ are the arm and leg lengths of the box $\square$ in the Young diagram of λ, defined by

$$a(\square) = \lambda_i - j,$$
$$\ell(\square) = \lambda'_j - i.$$

Let x and y be two independent variable sets. The Macdonald polynomials satisfy the Cauchy identity, which reads

$$\sum_\lambda P_\lambda(x) Q_\lambda(y) = \prod_{i,j} \frac{(\hbar x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} =: \Pi(x, y),$$

where the product runs over all pairs of indices i, j.

3.3 Eigenfunctions

Macdonald polynomials can also be characterized as simultaneous eigenfunctions of a commuting family of Macdonald difference operators. For each variable x_i, define the q-shift operator T_{q,x_i} acting on a function $f(x_1, x_2, \dots)$ by

$$T_{q,x_i} f(x_1, \dots, x_i, \dots) = f(x_1, \dots, qx_i, \dots).$$

One defines the Macdonald difference operators D_r (also called "Ruijsenaars operators" in type A) as follows:

$$D_r = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{k=1}^{r} \prod_{\substack{1 \le j \le n \\ j \ne i_k}} \frac{x_{i_k} - \hbar x_j}{x_{i_k} - x_j} T_{q,x_{i_k}}.$$

The operator D_r is a finite sum of products of the shift operators T_{q,x_i}, each multiplied by rational coefficients in the x_j. In particular, for $r = 1$,

$$D_1 = \sum_{i=1}^{n} \prod_{\substack{1 \le j \le n \\ j \ne i}} \frac{x_i - \hbar x_j}{x_i - x_j} T_{q,x_i},$$

which is the first Macdonald difference operator in n variables. Macdonald polynomials $P_\lambda(x; q, \hbar)$ satisfy the joint eigenfunction property:

$$D_r P_\lambda(x) = e_r(\hbar^{n-1} q^{\lambda_1}, \dots, \hbar^{n-l(\lambda)} q^{\lambda_{l(\lambda)}})(q, \hbar) P_\lambda(x)$$

for each $r = 1, 2, \dots, n$ where $e_r(\hbar^{n-1} q^{\lambda_1}, \dots, \hbar^{n-l(\lambda)} q^{\lambda_{l(\lambda)}})$ is the r-th elementary symmetric function of $\hbar^{n-1} q^{\lambda_1}, \dots, \hbar^{n-l(\lambda)}$.

For instance, when $r = 1$, the eigenvalue is

$$\sum_{i=1}^{n} \hbar^{n-i} q^{\lambda_i},$$

and when $r = n$, it is

$$\prod_{i=1}^{n} \left(\hbar^{n-i} q^{\lambda_i} \right) = \hbar^{\frac{n(n-1)}{2}} q^{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.$$

3.4 Skew-Macdonald polynomials

The skew-Macdonald polynomials $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ extend the P and Q-polynomials to skew shapes λ/μ. They are characterized by

$$\langle P_{\lambda/\mu}, Q_\nu \rangle = \langle P_\lambda, Q_\mu Q_\nu \rangle,$$

$$\langle Q_{\lambda/\mu}, P_\nu \rangle = \langle Q_\lambda, P_\mu P_\nu \rangle,$$

for all partitions ν. If $\mu \not\trianglelefteq \lambda$, then $P_{\lambda/\mu} = Q_{\lambda/\mu} = 0$. The even-leg weight is defined as

$$b_\lambda^{\mathrm{el}} = \prod_{\substack{\square\in\lambda \\ \ell(\square)\ \text{even}}} b_\lambda(\square).$$

Let $\varphi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ be defined by

$$\varphi_{\lambda/\mu} = \prod_{s\in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)},$$
$$\psi_{\lambda/\mu} = \prod_{s\in R_{\lambda/\mu}-C_{\lambda/\mu}} \frac{b_\mu(s)}{b_\lambda(s)}, \tag{6}$$

where $C_{\lambda/\mu}$ and $R_{\lambda/\mu}$ denote the sets of columns and rows of λ that contain boxes in the skew shape λ/μ, respectively.

3.5 Pieri rule

Pieri rule for Macdonald polynomials states that

$$P_\mu Q_{(a)} = \sum_\lambda \varphi_{\lambda/\mu} P_\lambda, \tag{7}$$

where the sum is over partitions λ such that $\lambda \succ \mu$ and $|\lambda| = |\mu| + a$. Here, $\varphi_{\lambda/\mu}$ are coefficients determined by the skew shape λ/μ, and the notation $\lambda \succ \mu$ means that λ interlaces μ from above, meaning that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots .$$

3.6 One variable specialization

The specialization of Macdonald polynomials to a single variable x with single-part partition (r),

$$P_{(r)}(x; q, \hbar) = x^r,$$
$$Q_{(r)}(x; q, \hbar) = \frac{(\hbar)_r}{(q)_r} x^r,$$

where $(a)_r = \prod_{k=0}^{r-1}(1 - aq^k)$. Moreover, P and Q vanish when the number of variables is less than the length of the partition:

$$P_\lambda(x_1, \dots, x_n; q, \hbar) = 0 \quad \text{if} \quad n < \ell(\lambda),$$

$$Q_\lambda(x_1, \dots, x_n; q, \hbar) = 0 \quad \text{if} \quad n < \ell(\lambda).$$

3.7

The even part function $\mathcal{E}_\lambda(x)$ is defined (see for example [Mac95, VI.7, Ex. 4(i)]) by

$$\mathcal{E}_\lambda(x) = \Phi(x)^{-1} \sum_{\substack{\mu \text{ even} \\ \lambda \prec \mu}} b_\mu^{\mathrm{el}} P_{\lambda/\mu}(x),$$

where

$$\Phi(x) := \sum_{\nu' \text{ even}} b_\nu^{\mathrm{el}} P_\nu(x).$$

When specialized to a single variable $x = (x_1)$,

$$\mathcal{E}_\lambda(x)\Big|_{x=(x_1)} = b_{e(\lambda)}^{\mathrm{el}} \psi_{e(\lambda)/\lambda} x_1^{|e(\lambda)|-|\lambda|},$$

where $e(\lambda)$ is the unique partition such that $e(\lambda)'$ is even and $\lambda \prec e(\lambda)$.

3.8

We use several properties of Macdonald polynomials in the proof of Theorem 6.1. The inversion symmetry states that

$$P_{(a)}(x_1^{-1}, \ldots, x_n^{-1}; q, \hbar) = (x_1 \cdots x_n)^{-a} P_{(a^{n-1})}(x_1, \ldots, x_n; q, \hbar). \tag{8}$$

We give a short proof of this formula below.

Proof. For $\lambda = (a)$, the Macdonald polynomial $P_{(a)}(x_1, \ldots, x_n; q, \hbar)$ is given by

$$P_{(a)}(x_1, \ldots, x_n; q, \hbar) = \sum_{i=1}^n x_i^a \prod_{\substack{j=1 \\ j \neq i}}^n \frac{x_i - \hbar x_j}{x_i - x_j}.$$

so we have

$$P_{(a)}(x_1^{-1}, \ldots, x_n^{-1}; q, \hbar) = \sum_{i=1}^n x_i^{-a} \prod_{\substack{j=1 \\ j \neq i}}^n \frac{x_i^{-1} - \hbar x_j^{-1}}{x_i^{-1} - x_j^{-1}}.$$

Note that

$$\frac{x_i^{-1} - \hbar x_j^{-1}}{x_i^{-1} - x_j^{-1}} = \frac{\frac{1}{x_i} - \frac{\hbar}{x_j}}{\frac{1}{x_i} - \frac{1}{x_j}} = \frac{1 - \hbar \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}} = \frac{1 - \hbar \cdot \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}},$$

so we get

$$P_{(a)}(x_1^{-1}, \ldots, x_n^{-1}; q, \hbar) = \sum_{i=1}^n x_i^{-a} \prod_{\substack{j=1 \\ j \neq i}}^n \frac{1 - \hbar \cdot \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}}.$$

Observe that each term x_i^{-a} can be rewritten as

$$x_i^{-a} = (x_1 x_2 \cdots x_n)^{-a} \cdot \frac{x_1^a x_2^a \cdots x_n^a}{x_i^a},$$

and we can factor out $(x_1 x_2 \cdots x_n)^{-a}$, so we get

$$P_{(a)}(x_1^{-1}, \ldots, x_n^{-1}; q, \hbar) = (x_1 x_2 \cdots x_n)^{-a} \sum_{i=1}^{n} \left(\frac{x_1^a \cdots x_n^a}{x_i^a} \right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1 - \hbar \cdot \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}},$$

so

$$\begin{aligned} &P_{(a)}(x_1^{-1}, \ldots, x_n^{-1}; q, \hbar) \\ &\quad = (x_1 x_2 \cdots x_n)^{-a} \sum_{i=1}^{n} \left(x_1^a \cdots x_{i-1}^a x_{i+1}^a \cdots x_n^a \right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1 - \hbar \cdot \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}}, \end{aligned}$$

and since for the partition (a^{n-1}) the Macdonald polynomial is given by

$$P_{(a^{n-1})}(x_1, \ldots, x_n; q, t) = \sum_{i=1}^{n} x_1^a \cdots x_{i-1}^a x_{i+1}^a \cdots x_n^a \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1 - t \cdot \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}},$$

this gives the formula. □

The relation between skew-Macdonald P and Q polynomials is given by

$$Q_{\lambda/\mu} = b_\lambda b_\mu^{-1} P_{\lambda/\mu}.$$

The branching property allows us to express Macdonald polynomials evaluated on a union of variables as a sum of products of skew polynomials:

$$\begin{aligned} Q_\lambda(x, z) &= \sum_\mu Q_{\lambda/\mu}(x) Q_\mu(z), \\ P_\lambda(x, z) &= \sum_\mu P_{\lambda/\mu}(x) P_\mu(z). \end{aligned} \tag{9}$$

For single-variable specializations and fixed parameters, the polynomials behave as monomials scaled by appropriate coefficients:

$$P_{(r)}(x; q, \hbar) = x^r, \quad Q_{(r)}(x; q, \hbar) = \frac{(\hbar)_r}{(q)_r} x^r.$$

The q-shifted factorial for negative indices is related to positive indices by

$$(a; q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n}. \tag{10}$$

4 Stable type D quiver representations

In this section, we study zero-dimensional type D quiver varieties. In other words, we explicitly characterize all stable type D quiver representations satisfying the moment map equations when there is a single framing at a minuscule vertex. Recall the definition of the D_n quiver:

Definition 4.1. The quiver $D_n = (Q_0, Q_1)$ is defined as follows:

- $Q_0 = \{1, 2, \ldots, n\}$,
- $i \to i+1 \in Q_1$ for all $i \in [1, n-2]$,
- $n-2 \to n \in Q_1$.

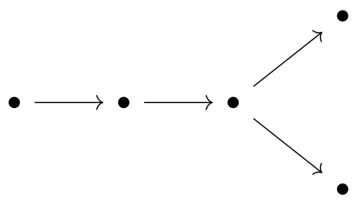

Figure 1: Dynkin diagram for D_5.

4.1 Case of $\mathsf{w} = (0^{n-1}, 1)$

We start with the case of a single framing at a spin node, which in our notation is either vertex $n-1$ or vertex n. Due to the symmetry of the D_n quiver, it does not matter which we consider. So for definiteness, assume that $\mathsf{w} = \delta_n$.

Let $\mathcal{M} = \mathcal{M}_{D_n}(\mathsf{v}, \delta_n)$ be a quiver variety. Recall from Proposition 2.3 that $\mathcal{M}$ is nonempty if and only if it is a single point. By [Nak94], such v correspond to weight spaces of the spin representation with highest weight ω_n. We require an explicit description of the v for which $\mathcal{M}$ is a point, along with a special representative of the point as a quiver representation.

We recall the formula for the dimension of an arbitrary quiver variety.

Proposition 4.2 (Dimension of $\mathcal{M}_Q(\mathsf{v}, \mathsf{w})$, [Nak94])**.** *The dimension of the space $\mathcal{M}_Q(\mathsf{v}, \mathsf{w})$ is given as follows:*

$$\frac{1}{2}\dim_{\mathbb{C}} \mathcal{M}_Q(\mathsf{v}, \mathsf{w}) = \sum_{e \in Q_1} \mathsf{v}_{o(e)}\mathsf{v}_{i(e)} + \sum_{i \in Q_0} \mathsf{v}_i \mathsf{w}_i - \sum_{i \in Q_0} \mathsf{v}_i^2.$$

In the case of $\mathsf{w} = \delta_n$, the dimension formula is

$$\dim_{\mathbb{C}} \mathcal{M} = 2 \left(\sum_{e \in Q_1} \mathsf{v}_{o(e)} \mathsf{v}_{i(e)} + \mathsf{v}_n - \sum_{i \in Q_0} \mathsf{v}_i^2 \right).$$

The possible v for which $\mathcal{M}$ is nonempty are explicitly described by the following lemma.

Lemma 4.3. *The quiver variety $\mathcal{M}_{D_n}(\mathsf{v}, \delta_n)$ is nonempty if and only if*

- $\mathsf{v}_{i+1} = \mathsf{v}_i$ *or* $\mathsf{v}_{i+1} = \mathsf{v}_i + 1$ *for all* $i \in [1, n-3]$,
- $\mathsf{v}_{n-2} = \mathsf{v}_{n-1} + \mathsf{v}_n$ *or* $\mathsf{v}_{n-2} + 1 = \mathsf{v}_{n-1} + \mathsf{v}_n$,
- $\mathsf{v}_n = \mathsf{v}_{n-1}$ *or* $\mathsf{v}_n = \mathsf{v}_{n-1} + 1$.

There are exactly 2^{n-1} such v.

Proof. That such v give rise to a zero dimensional quiver variety follows from the dimension formula. It is straightforward to see that there are 2^{n-1} such v satisfying the conditions stated in the lemma. By [Nak94], the number of v such that $\mathcal{M}_{D_n}(\mathsf{v}, \delta_n)$ is nonempty is the dimension of the spin representation with highest weight ω_n, which is known to be 2^{n-1}. We have thus exhibited all such v. □

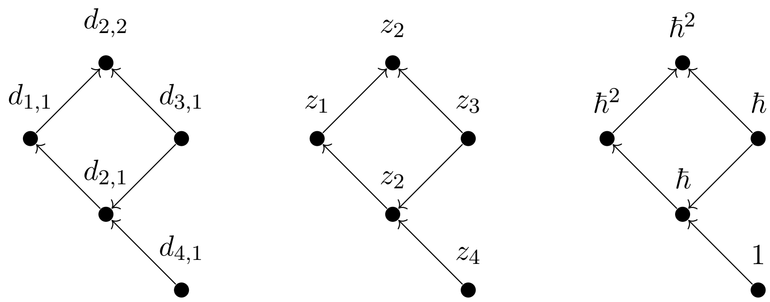

Figure 2: A D_4 poset of type ω_4, with $\mathsf{v} = (1, 2, 1, 1)$.

We now exhibit explicit quiver representations which are representatives of the unique point in $\mathcal{M}_{D_n}(\mathsf{v}, \delta_n)$ for the v described above. We will build such quiver representations from posets.

Definition 4.4. The type D_n spin poset $\mathcal{P} = (P, \leq)$ is defined by

$$P = \{d_{i,j}\}_{(i,j) \in I},$$

where $I = \{(i,j) : i \in [1, n-2], j \in [1, n]\} \cup \{(n-1, j) : j \in [1, \lfloor \frac{n+1}{2} \rfloor]\} \cup \{(n, j) : j \in [1, \lfloor \frac{n+2}{2} \rfloor]\}$. The partial order $\leq$ is defined by the following relations:

$$d_{i,j} \geq d_{i+1,j} \quad \text{for all } i \in [1, n-3],\ j \in [1, n],$$

$$d_{i,j} \leq d_{i+1,j+1} \quad \text{for all } i \in [1, n-3],\ j \in [1, n],$$

$$d_{n-2,2j} \geq d_{n-1,j}, \quad d_{n-2,2j-1} \leq d_{n-1,j} \quad \text{for all } j \in \left[1, \frac{n+1}{2}\right],$$

$$d_{n-2,2j-1} \geq d_{n,j} \quad \text{for all } j \in \left[1, \frac{n+2}{2}\right],$$

$$d_{n-2,2j-2} \leq d_{n,j} \quad \text{for all } j \in \left[2, \frac{n+2}{2}\right].$$

Moreover, the coloring function satisfies $\mathrm{c}(d_{i,j}) = i$ for all $i \in [1, n]$ and for all j, that is, each element $d_{i,j}$ is colored by vertex i. The weight function satisfies

$$\mathrm{wt}(d_{i,j}) = \hbar^{n-i+j-2} \quad \text{for all } i \in [1, n-2], j \in [1, n],$$

$$\mathrm{wt}(d_{n-1,j}) = \hbar^{2j-1} \quad \text{for all } j \in \left[1, \frac{n+1}{2}\right],$$

and

$$\mathrm{wt}(d_{n,j}) = \hbar^{2j-2} \quad \text{for all } j \in \left[1, \frac{n+2}{2}\right].$$

Definition 4.5. An order ideal of the type D_n spin poset, along with the induced coloring and weight function, is called a D_n poset of type ω_n.

Given a D_n spin poset $\mathcal{P} = (P, \leq)$ of type ω_n, we construct the corresponding quiver representation as follows. For each vertex $i \in Q_0$, assign a vector space V_i over $\mathbb{C}$ whose dimension is equal to the number of elements in P colored by i, i.e., $\dim V_i = \left|\mathrm{c}^{-1}(i)\right|$. For each element $d_{i,j} \in P$, assign a basis vector $e_{i,j} \in V_i$. We define linear maps $X_e : V_{o(e)} \to V_{i(e)}$ and $Y_e : V_{i(e)} \to V_{o(e)}$ for each edge $e \in Q_1$ in terms of the vectors $e_{i,j}$ by asserting that

- X_e maps $e_{o(e),j}$ to $e_{i(e),k}$ whenever there is a covering relation $d_{o(e),j} \leq d_{i(e),k}$,
- Y_e maps $e_{i(e),j}$ to $e_{o(e),k}$ whenever there is a covering relation $d_{i(e),j} \leq d_{o(e),k}$,
- the remaining matrix coefficients are zero.

We also define $A_n : \mathbb{C} \to V_n$ by $A_n(1) = d_{n,1}$ and $A_i = 0$ for $i < n$. We set $B_i = 0$ for all i. Overall, we obtain a collection of linear maps (X, Y, A, B) which is the corresponding quiver representation of the D_n poset $\mathcal{P}$ of type ω_n.

Proposition 4.6. *The quiver representation (X, Y, A, B) associated to a D_n poset of type ω_n is stable and satisfies the moment map equations.*

Proof. Recall Proposition 2.2 which gives a characterization of stability. Combinatorially, this means that one is able to reach any basis element from the framing lowest basis element by traversing through arrows. This follows from the fact that each nonempty D_n poset of type ω_n has a minimal element, namely $d_{n,1}$. It remains to show that the quiver representation associated to D_n poset of type ω_n satisfies $\mu = 0$ at each vertex. We need the right-hand side of Equation 2 to vanish identically:

$$\sum_{\substack{e \in Q_1 \\ i(e)=i}} X_e Y_e - \sum_{\substack{e \in Q_1 \\ o(e)=i}} Y_e X_e + A_i B_i = 0 \quad \text{for all } i \in Q_0.$$

For the D_n spin poset, this follows directly from the definitions, and is inherited by its order ideals. □

Putting together the results of this section, we obtain the following.

Theorem 4.7. *The nonempty (and hence zero-dimensional) quiver varieties $\mathcal{M}_{D_n}(\mathsf{v}, \delta_n)$ are in canonical bijection with D_n posets of type ω_n.*

Remark. Considering framing δ_{n-1} gives rise to a theorem analogous to the previous one, giving a canonical bijection between the nonempty quiver varieties $\mathcal{M}_{D_n}(\mathsf{v}, \delta_{n-1})$ and D_n posets of type ω_{n-1}. The latter are identical as posets to D_n posets of type ω_n. The weight function is identical and the coloring function differs by swapping colors $n-1$ and n.

4.2 Case of $\mathsf{w} = (1, 0^{n-1})$

Now we consider the case where $\mathsf{w} = \delta_1$, which we call the fundamental node. As in the previous subsection, Proposition 2.3 implies that $\mathcal{M}_{D_n}(\mathsf{v}, \delta_1)$ is nonempty if and only if it is a single point. Such dimension vectors are described by the following lemma.

Lemma 4.8. *The quiver variety $\mathcal{M}_{D_n}(\mathsf{v}, \delta_1)$ is nonempty if and only if v is one of the following vectors:*

- $\mathsf{v} = (1^k, 0^{n-k})$ *for* $k \in [0, n]$,
- $\mathsf{v} = (1^k, 2^{n-k-2}, 1, 1)$ *for* $k \in [0, n-3]$,
- $\mathsf{v} = (1^{n-2}, 0, 1)$.

There are exactly $2n$ *such* v.

Proof. From Proposition 4.2, one can see that the dimension vectors in the statement of the lemma give rise to a zero dimensional quiver variety. There are a total of $(n+1) + (n-2) + 1 = 2n$ such vectors. Since this is the dimension of the first fundamental representation of type D_n, the results of [Nak94] imply that we have exhibited all such v. □

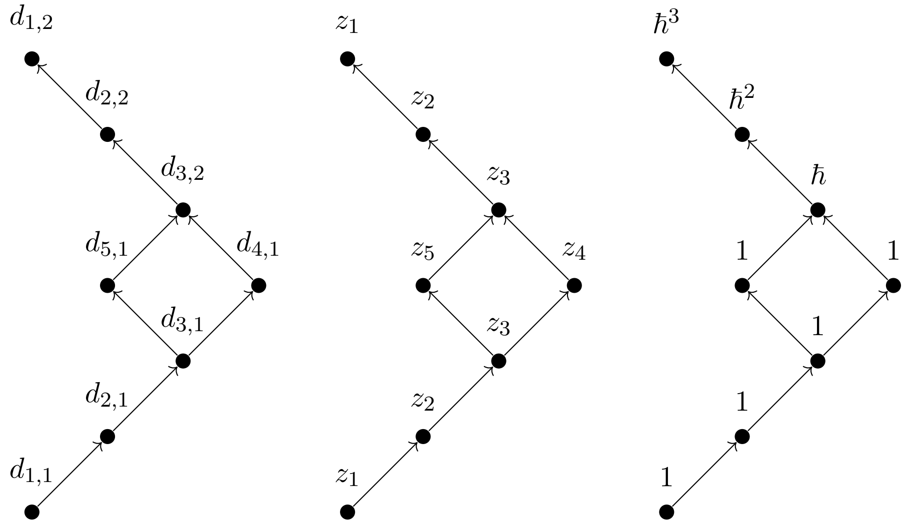

Figure 3: A D_4 poset of type ω_1, with $\mathsf{v} = (2, 2, 1, 1)$.

As above, we next exhibit quiver representations which are representatives of the unique point in $\mathcal{M}_{D_n}(\mathsf{v}, \delta_1)$ for the v described above.

Definition 4.9. The type D_n fundamental representation poset $\mathcal{P} = (P, \leq)$ is defined by

$$P = \{d_{i,j}\}_{(i,j)\in I},$$

where $I = \{(i, j) : i \in [1, n-2], j \in [1, 2]\} \cup \{(n-1, 1), (n, 1)\}$. The partial order $\leq$ is defined by the following relations:

$$d_{i,1} \leq d_{i+1,1} \quad \text{for all } i \in [1, n-2],$$

$$d_{i,2} \geq d_{i+1,2} \quad \text{for all } i \in [1, n-3],$$

$$d_{n-2,1} \leq d_{n,1}, \quad d_{n-2,2} \geq d_{n,1}, \quad d_{n-2,2} \geq d_{n-1,1}.$$

Moreover, the coloring function satisfies $\mathrm{c}(d_{i,j}) = i$ for all $i \in [1, n]$ and for all j, that is, each element $d_{i,j}$ is colored by vertex i. The weight function satisfies $\mathrm{wt}(d_{i,1}) = 1$ for all $i \in [1, n]$ and $\mathrm{wt}(d_{i,2}) = \hbar^{n-i-1}$ for all $i \in [1, n-2]$.

Definition 4.10. An order ideal of the type D_n fundamental poset, along with the induced coloring and weight function, is called a D_n poset of type ω_1.

Given a D_n poset $\mathcal{P} = (P, \leq)$ of type ω_1, we construct the corresponding quiver representation in the same way as before. For each vertex $i \in Q_0$, assign a vector space V_i over $\mathbb{C}$ whose dimension is equal to the number of elements in P colored by i, i.e., $\dim V_i = \left|\mathrm{c}^{-1}(i)\right|$. For each element $d_{i,j} \in P$, assign a basis vector $e_{i,j} \in V_i$. We define linear maps $X_e : V_{o(e)} \to V_{i(e)}$ and

$Y_e : V_{i(e)} \to V_{o(e)}$ for each edge $e \in Q_1$ in terms of the vectors $e_{i,j}$ by asserting that

- X_e maps $e_{o(e),j}$ to $e_{i(e),k}$ whenever there is a covering relation $d_{o(e),j} \leq d_{i(e),k}$,
- Y_e maps $e_{i(e),j}$ to $e_{o(e),k}$ whenever there is a covering relation $d_{i(e),j} \leq d_{o(e),k}$,
- the remaining matrix coefficients are zero.

We also define $A_1 : \mathbb{C} \to V_1$ by $A_1(1) = d_{1,1}$ and $A_i = 0$ for $i > 1$. We set $B_i = 0$ for all i. Overall, we obtain a collection of linear maps (X, Y, A, B) which is the corresponding quiver representation of the D_n poset $\mathcal{P}$ of type ω_1.

Proposition 4.11. *The quiver representation associated to a D_n poset of type ω_1 is stable and satisfies the moment map equations.*

Proof. The proof is analogous to the proof of Proposition 4.6. □

Putting together the results of this section, we obtain the following.

Theorem 4.12. *The nonempty (and hence zero-dimensional) quiver varieties $\mathcal{M}_{D_n}(\mathsf{v}, \delta_1)$ are in canonical bijection with D_n posets of type ω_1.*

5 Vertex functions

In this section, we will review the definition of vertex functions, our main objects of interest. Vertex functions for Nakajima quiver varieties were defined by A. Okounkov in [Oko17] using quasimap machinery developed in [CFKM14]. Later we will restrict to the case of type D quiver varieties and spell out the definitions more explicitly in this context.

5.1 General definitions

Let $\mathcal{M} := \mathcal{M}_Q(\mathsf{v}, \mathsf{w})$ be a quiver variety. Recall that $\mathcal{M}$ is defined as $\mu^{-1}(0) /\!\!/_\theta G_\mathsf{v}$. Then there is an open embedding

$$\mathcal{M} \hookrightarrow [\mu^{-1}(0)/G_\mathsf{v}] =: \mathfrak{M}.$$

A quasimap from $\mathbb{P}^1$ to $\mathcal{M}$ is by definition a map f from $\mathbb{P}^1 \to \mathfrak{M}$. A quasimap is said to be stable if $f(p) \in \mathcal{M}$ for all but finitely many $p \in \mathbb{P}^1$. Points p such that $f(p) \in \mathfrak{M} \setminus \mathcal{M}$ are called singularities of f. Let QM be the stack parameterizing stable quasimaps from $\mathbb{P}^1$ to $\mathcal{M}$.

Unpacking the definition, one sees that the data of a quasimap f consists vectors bundles $\mathcal{V}_i$ and topologically trivial bundles $\mathcal{W}_i$ for $i \in Q_0$ over $\mathbb{P}^1$

along with a section of a certain bundle built from $\mathcal{V}_i$ and $\mathcal{W}_i$; see Section 4 of [Oko17]. The degree of the quasimap f is defined to be

$$\deg f := (\deg \mathcal{V}_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}.$$

For $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, let $\mathsf{QM}^{\boldsymbol{d}}$ be the stack parameterizing degree $\boldsymbol{d}$ stable quasimaps. Thus

$$\mathsf{QM} = \bigsqcup_{\boldsymbol{d}\in\mathbb{Z}^{Q_0}} \mathsf{QM}^{\boldsymbol{d}}.$$

Let $\mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty} \subset \mathsf{QM}^{\boldsymbol{d}}$ be the space of quasimaps which are nonsingular at $\infty \in \mathbb{P}^1$. There is a diagram

$$\mathfrak{M} \xleftarrow{\;\mathrm{ev}_0\;} \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty} \xrightarrow{\;\mathrm{ev}_\infty\;} \mathcal{M}$$

where the maps are given by evaluating quasimaps at 0 and ∞. The action of the maximal torus $\mathsf{T} \subset \mathrm{Aut}(\mathcal{M})$ induces an action on $\mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty}$. In addition, there is an action of $\mathbb{C}^\times$ on $\mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty}$ induced by the usual action on $\mathbb{P}^1$. We denote this torus by $\mathbb{C}^\times_q$. The restriction of ev_∞ to $\left(\mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty}\right)^{\mathbb{C}^\times_q}$ is known to be proper; see Section 7.3 of [Oko17]. Hence the pushforward $\mathrm{ev}_{\infty,*}$ can be defined in localized equivariant K-theory.

It is known that $\mathsf{QM}^{\boldsymbol{d}}$ is equipped with a canonical perfect obstruction theory, which gives rise to a virtual structure sheaf $\mathcal{O}^{\boldsymbol{d}}_{\mathrm{vir}}$. For technical reasons, it is better to study the symmetrized virtual structure sheaf $\hat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}}$, which differs from the virtual structure sheaf by a twist by certain line bundle; see Section 6 of [Oko17]. This twist depends on a choice of polarization of $\mathcal{M}$, and we make the choice (4).

Definition 5.1 ([Oko17])**.** Let $\tau \in K_{\mathsf{T}\times\mathbb{C}^\times_q}(\mathfrak{M})$. The vertex function with descendant τ is the formal power series

$$V^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{\infty,*}\left(\hat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \mathrm{ev}_0^*(\tau)\right) \boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}^\times_q}(\mathcal{M})_{loc}[[z]],$$

where $\boldsymbol{z}^{\boldsymbol{d}} := \prod_{i\in Q_0} z_i^{d_i}$.

The formal parameters z_i are called Kähler parameters. To write explicit formulas for vertex functions, we will need some additional notation. Let $\varphi(x) := \prod_{i\geq 0}\left(1 - xq^i\right)$. The q-Pochhammer symbol is defined by

$$(x)_d := \frac{\varphi(x)}{\varphi(xq^d)} = \begin{cases} 1 & \text{if } d = 0, \\ \prod_{i=0}^{d-1}(1 - xq^i) & \text{if } d > 0, \\ \prod_{i=1}^{-d}(1 - xq^{-i})^{-1} & \text{if } d < 0, \end{cases}$$

and the q-binomial series is defined by

$$F(z) := \sum_{d=0}^{\infty} \frac{(\hbar)_d}{(q)_d} z^d.$$

Proposition 5.2 ([GR04])**.** *The q-binomial series $F(z)$ satisfies the following identity:*

$$F(z) = \prod_{n=1}^{\infty} \frac{1 - z\hbar q^n}{1 - zq^n}.$$

5.2 Type D vertex functions

Let Q be the type D_n quiver and let $\mathsf{v}, \mathsf{w} \in \mathbb{N}^{Q_0}$. We are interested in explicit formulas for vertex functions, so we will restrict our attention to cases where $\mathcal{M}^{\mathsf{T}}$ consists of isolated points. By Proposition 2.3, it is sufficient to assume that $\mathsf{w}_i = 0$ unless i is a minuscule vertex. We make this assumption now. In other words, we only allow nontrivial framings at vertices 1, $n-1$, and n.

We have $\mathsf{T} = (\mathbb{C}^\times)^{|\mathsf{w}|} \times \mathbb{C}^\times_\hbar$. The set $\mathcal{M}^{\mathsf{T}}$ is described completely and explicitly by combining (3) and the results of Section 4.

Let $p \in \mathcal{M}^{\mathsf{T}}$. In this description, the fixed point p is given by a $|\mathsf{w}|$-tuple

$$\left(\mathcal{P}_p^{i,j}\right)_{\substack{i \in Q_0 \\ 1 \le j \le \mathsf{w}_i}}$$

of colored posets where $\mathcal{P}_p^{i,j}$ is of type ω_i for all $j \in \{1, \ldots, \mathsf{w}_i\}$ such that the total number of elements of color k is equal to v_k for all $k \in Q_0$. We denote

$$\mathcal{P}_p = \bigsqcup_{\substack{i \in Q_0 \\ 1 \le j \le \mathsf{w}_i}} \mathcal{P}_p^{i,j}.$$

There is a coloring function $c_p : \mathcal{P}_p \to Q_0$ which restricts to the coloring functions of Sections 4.1 and 4.2 on each $\mathcal{P}_p^{i,j}$. As in Sections 4.1 and 4.2, each $\mathcal{P}_p^{i,j}$ has a weight function $\mathrm{wt}_{i,j} : \mathcal{P}_p^{i,j} \to \mathbb{Z}[\hbar^{\pm 1}]$. We define a weight function

$$\mathrm{wt} : \mathcal{P}_p \to \mathrm{Rep}(\mathsf{T}) = \mathbb{Z}\left[\left\{a_{i,j}^{\pm 1}\right\}_{\substack{i \in Q_0 \\ 1 \le j \le \mathsf{w}_i}}, \hbar^{\pm 1}\right]$$

on $\mathcal{P}_p$ by $\mathrm{wt}(x) = \mathrm{wt}_{i,j}(x) a_{i,j}$ for $x \in \mathcal{P}_p^{i,j}$.

The results of Section 4 imply that the T-weights of the restrictions $\mathscr{V}_i|_p$ are given by

$$\mathscr{V}_i|_p = \sum_{x \in c_p^{-1}(i)} \mathrm{wt}(x). \tag{11}$$

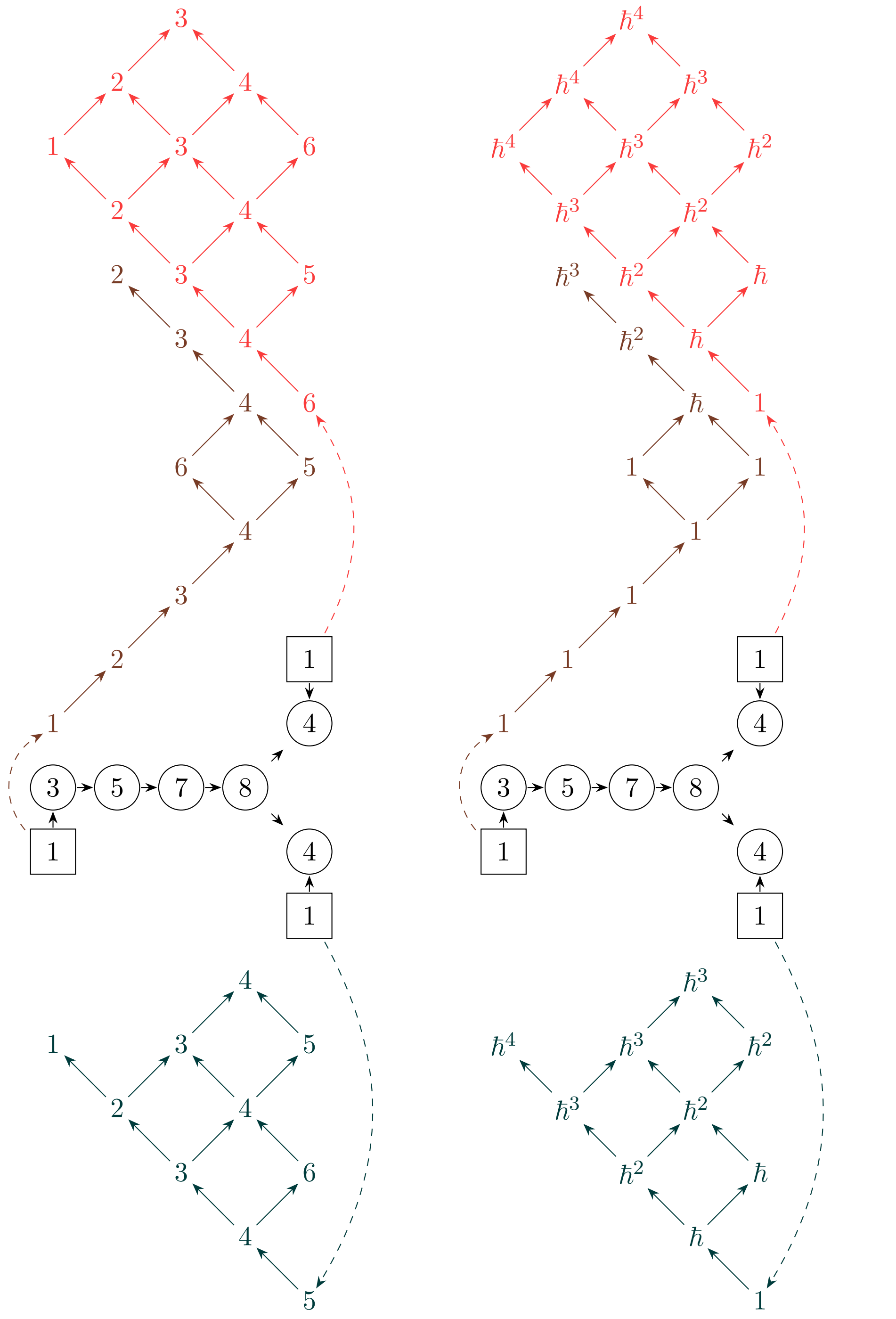

Figure 4: The poset of a D_6 quiver variety with $\mathsf{v} = (3, 5, 7, 8, 4, 4)$ and $\mathsf{w} = (1, 0, 0, 0, 1, 1)$, elements colored with colors i for $i \in [1, 6]$ on the left, and $\mathrm{wt}_{i,1}(x)$ filled in for each $x \in \mathcal{P}_p^{i,1}$ for each $i \in \{1, n-1, n\}$ on the right.

Example 5.3. Consider the D_6 quiver variety $\mathcal{M}$ with dimension vector $\mathsf{v} = (3,5,7,8,4,4)$ and framing $\mathsf{w} = (1,0,0,0,1,1)$. From Proposition 4.2, we see that $\dim_{\mathbb{C}} \mathcal{M} = 0$, and $\mathcal{M}^{\mathsf{T}}$ consists of isolated points. Let $p \in \mathcal{M}^{\mathsf{T}}$. From the description of the torus fixed points and the poset $\mathcal{P}_p$ above, we notice that $\mathcal{P}_p$ splits into a disjoint union of the three posets $\mathcal{P}_p^{1,1}$, $\mathcal{P}_p^{n-1,1}$, and $\mathcal{P}_p^{n,1}$. Then the dimension vector v also respectively splits into the sum of following three vectors: $\mathsf{v}^{1,1} = (1,2,2,2,1,1)$, $\mathsf{v}^{n-1,1} = (1,2,3,3,2,1)$, and $\mathsf{v}^{n,1} = (1,1,2,3,1,2)$, whose posets are illustrated in Figure 4. One can compute the tautological bundle weights from Equation 11 explicitly as follows:

- $\mathscr{V}_1|_p = a_{1,1} + a_{n-1,1}\hbar^4 + a_{n,1}\hbar^4$,
- $\mathscr{V}_2|_p = a_{1,1}(1+\hbar^4) + a_{n-1,1}(\hbar^3+\hbar^4) + a_{n,1}\hbar^3$,
- $\mathscr{V}_3|_p = a_{1,1}(1+\hbar^3) + a_{n-1,1}(\hbar^2+\hbar^3+\hbar^4) + a_{n,1}(\hbar^2+\hbar^3)$,
- $\mathscr{V}_4|_p = a_{1,1}(1+\hbar^2) + a_{n-1,1}(\hbar+\hbar^2+\hbar^3) + a_{n,1}(\hbar+\hbar^2+\hbar^3)$,
- $\mathscr{V}_5|_p = a_{1,1} + a_{n-1,1}\hbar + a_{n,1}(1+\hbar^2)$,
- $\mathscr{V}_6|_p = a_{1,1} + a_{n-1,1}(1+\hbar^2) + a_{n,1}\hbar$.

Definition 5.4. A reverse plane partition over a poset $\mathcal{P}$ is a function $\pi : \mathcal{P} \to \mathbb{N}$ such that $x \le y \implies f(y) \le f(x)$. If $\mathcal{P}$ is colored by a coloring function $c : \mathcal{P} \to I$, then we define the degree of a reverse plane partition π to be

$$\deg(\pi) := \left(\sum_{x \in c^{-1}(i)} \pi(x) \right)_{i \in I} \in \mathbb{N}^I.$$

Let $\operatorname{rpp}(\mathcal{P})$ be the set of reverse plane partitions over $\mathcal{P}$ and, if a coloring of $\mathcal{P}$ is provided, let $\operatorname{rpp}^{\boldsymbol{d}}(\mathcal{P})$ be the set of degree $\boldsymbol{d}$ reverse plane partitions over $\mathcal{P}$.

Proposition 5.5. *Let $p \in \mathcal{M}^{\mathsf{T}}$. Then*

$$\left(\mathsf{QM}^{\boldsymbol{d}}_{ns,\infty}\right)^{\mathsf{T}\times\mathbb{C}_q^\times} = \bigoplus_{p \in \mathcal{M}^{\mathsf{T}}} \left(\mathsf{QM}^{\boldsymbol{d}}_{ns,\infty,p}\right)^{\mathsf{T}\times\mathbb{C}_q^\times}$$

where

$$\mathsf{QM}^{\boldsymbol{d}}_{ns,\infty,p} = \{f \in \mathsf{QM}^{\boldsymbol{d}}_{ns,\infty} \mid f(\infty) = p\}.$$

Furthermore, $\left(\mathsf{QM}^{\boldsymbol{d}}_{ns,\infty,p}\right)^{\mathsf{T}\times\mathbb{C}_q^\times}$ *is in canonical bijection with* $\operatorname{rpp}^{\boldsymbol{d}}(\mathcal{P}_p)$.

Proof. The first claim follows from the fact that ∞ is a nonsingular point fixed by $\mathbb{C}_q^\times$. The second claim is standard, and the argument is identical to that in Section 5.1.3 of [Din22a]. □

The class τ can be viewed as a function of variables $x_{i,j}$ for $i \in Q_0$ and $1 \leq j \leq \mathsf{w}_i$ which is symmetric in $x_{i,1}, \ldots, x_{i,\mathsf{w}_i}$ for each i separately. Choose some bijection between $\{x_{i,j}\}_{1 \leq j \leq \mathsf{w}_i}$ and $\mathrm{c}_p^{-1}(i)$ for each i so that we can view τ as a function of the elements of $\mathcal{P}_p$. For $\pi \in \mathrm{rpp}(\mathcal{P}_p)$, let $\tau(\pi) := \tau|_{x=\mathrm{wt}(x)q^{\pi(x)}}$.

This is sufficient information to compute the vertex function $V^{(\tau)}|_p$ by localization. The proof is analogous to the computations of [DS20b] and [PSZ20] since the only type D specific ingredients are given by the poset arising in Proposition 5.5. See also Proposition 1 of [AFO18].

Theorem 5.6. *The restriction of the vertex function of $\mathcal{M}$ to a point $p \in \mathcal{M}^{\mathsf{T}}$ is given by the following formula:*

$$V^{(\tau)}(\boldsymbol{z})|_p = \sum_{\pi \in \mathrm{rpp}(\mathcal{P}_p)} \left(-\frac{q}{\hbar^{1/2}}\right)^{N(\pi)} \tau(\pi) \boldsymbol{z}^{\deg(\pi)}$$
$$\cdot \left(\prod_{i \in Q_0} \prod_{\substack{x \in \mathrm{c}_p^{-1}(i) \\ 1 \leq j \leq \mathsf{w}_i}} \frac{\left(\hbar \frac{\mathrm{wt}(x)}{a_{i,j}}\right)_{\pi(x)}}{\left(q \frac{\mathrm{wt}(x)}{a_{i,j}}\right)_{\pi(x)}} \right) \left(\prod_{e \in Q_1} \prod_{\substack{x \in \mathrm{c}_p^{-1}(o(e)) \\ y \in \mathrm{c}_p^{-1}(i(e))}} \frac{\left(\hbar \frac{\mathrm{wt}(y)}{\mathrm{wt}(x)}\right)_{\pi(y)-\pi(x)}}{\left(q \frac{\mathrm{wt}(y)}{\mathrm{wt}(x)}\right)_{\pi(y)-\pi(x)}} \right)$$
$$\cdot \left(\prod_{i \in Q_0} \prod_{x,y \in \mathrm{c}_p^{-1}(i)} \frac{\left(q \frac{\mathrm{wt}(y)}{\mathrm{wt}(x)}\right)_{\pi(y)-\pi(x)}}{\left(\hbar \frac{\mathrm{wt}(y)}{\mathrm{wt}(x)}\right)_{\pi(y)-\pi(x)}} \right),$$

where $N(\pi) = \sum_{e \in Q_1} \left(\mathsf{v}_{o(e)} \deg(\pi)_{i(e)} - \mathsf{v}_{i(e)} \deg(\pi)_{o(e)}\right) + \sum_{i \in Q_0} \mathsf{w}_i \deg(\pi)_i$.

Remark. The quantity $N(\pi)$ can also be written as

$$N(\pi) = \sum_{j \in Q_0} \left(\sum_{\substack{e \in Q_1 \\ i(e)=j}} \mathsf{v}_{o(e)} - \sum_{\substack{e \in Q_1 \\ o(e)=j}} \mathsf{v}_{i(e)} \right) \deg(\pi)_j.$$

So we can absorb the $(-q/\hbar^{1/2})^{N(\pi)}$ term into a shift of a Kähler parameters. We make this assumption in all that follows and will henceforth write $V^{(\tau)}(\boldsymbol{z})$ for this shifted vertex function.

Example 5.7. For concreteness, consider the simplest example of $Q = D_1$, $\mathsf{v} = (1)$, $\mathsf{w} = (1)$. For this example, we have

$$V^{(1)}(\boldsymbol{z}) = \sum_{k \geq 0} \frac{(\hbar)_k}{(q)_k} z_1^k = F(z_1),$$

which is a product of q-binomial series.

While Theorem 5.6 is entirely algorithmic, it is combinatorially complicated. Nevertheless, we will show that there is a beautiful summation formula for it, providing a vast generalization of the q-binomial theorem, when $|\mathsf{w}| = 1$.

5.3 Quasimaps to type D points

Now assume further that $\mathsf{w} = \delta_k$ where $k \in \{1, n-1, n\}$. Let v be such that the quiver variety $\mathcal{M}$ is nonempty, which by Proposition 2.3 means that it must be a single point. Furthermore, the torus acting on $\mathcal{M}$ is $\mathbb{C}^\times \times \mathbb{C}^\times_\hbar$. Since the diagonal subtorus of the framing torus always acts trivially, we can ignore the first factor and define $\mathsf{T} = \mathbb{C}^\times_\hbar$.

So $K_\mathsf{T}(\mathcal{M}) = K_\mathsf{T}(pt) = \mathbb{Z}[\hbar^{\pm 1}]$, and hence the vertex function is an element of $K_{\mathsf{T}\times\mathbb{C}^\times_q}(\mathcal{M})_{loc}[[z]] = \mathbb{Q}(q,\hbar)[[z_i]]_{i\in Q_0}$.

The moduli space $\mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty}$ was defined for arbitrary Nakajima quiver varieties in Section 5.1. Intuitively, $\mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty}$ is supposed to be a moduli space of maps from $\mathbb{P}^1$ to the quiver variety $\mathcal{M}$. Nevertheless, even in the case when $\mathcal{M}$ is a single point, we can spell out the definition of quasimaps to see that $\mathsf{QM}^{\boldsymbol{d}}_{\mathrm{ns},\infty}$ is something nontrivial.

Proposition 5.8. *Suppose that $\mathsf{w} = \delta_k$ where $k \in \{1, n-1, n\}$. The moduli space $\mathsf{QM}^{\boldsymbol{d}}_{ns,\infty}$ is the stack classifying the following data:*

- *rank v_i vector bundles $\mathcal{V}_i$ over $\mathbb{P}^1$ of degrees $\deg(\mathcal{V}_i) = d_i$ for $i \in Q_0$,*
- *a stable section s of the vector bundle $\mathscr{P} \oplus \mathscr{P}^*$ with*

$$\mathscr{P} = \mathcal{V}_k \oplus \left(\bigoplus_{i \le n-2} \operatorname{Hom}(\mathcal{V}_i, \mathcal{V}_{i+1}) \right) \oplus \operatorname{Hom}(\mathcal{V}_{n-2}, \mathcal{V}_n),$$

which is non-singular at $\infty \in \mathbb{P}^1$ and satisfies the moment map equations.

In the previous proposition, stability of the section s means the following. For each $p \in \mathbb{P}^1$, one can view $s(p)$ as an element of $T^* \operatorname{Rep}_Q(\mathsf{v}, \mathsf{w})$. Then the section s is stable if and only if $s(p)$ is stable in the sense of Proposition 2.2 for all but finitely many p.

Thus in this case, the vertex function $V^{(\tau)}(\boldsymbol{z})$ is the generating function for the equivariant Euler characteristics of certain K-theory classes on an interesting moduli space.

6 Vertex functions for zero-dimensional varieties

Our main result is a product formula for the vertex function for zero-dimensional type D_n Nakajima quiver varieties with framing at exactly one minuscule vertex. To write our formula, we need some notation.

Fix the D_n quiver as before. The associated root system can be constructed as follows. Let ϵ_i for $1 \leq i \leq n$ be an orthonormal basis of a Euclidean space. The simple roots of the D_n root system are constructed as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. The set of all roots is $\Phi = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}$. We use the rule $z_i = e^{\alpha_i}$, extended in the obvious way, to associate a monomial in the Kähler parameters to each root. Let ω_i for $1 \leq i \leq n$ be the fundamental weights.

As is standard when working with quiver varieties, see [Nak94], we view w and v as encoding two weights, λ and μ, by the rules $\lambda = \sum_i \mathsf{w}_i \omega_i$ and $\mu = \lambda - \sum_i \mathsf{v}_i \alpha_i$.

Theorem 6.1. *Fix* w *such that* $\mathsf{w}_i = 1$ *for a single minuscule vertex* i*, and* $\mathsf{w}_i = 0$ *otherwise. Let* v *be such that* $\mathcal{M}(\mathsf{v}, \mathsf{w})$ *is nonempty. Let* $\mu = \sum_i \mathsf{w}_i \omega_i - \sum_i \mathsf{v}_i \alpha_i$*. Let* $\Phi' = \{\alpha \in \Phi \mid (\mu, \alpha) < 0\}$*. Then the vertex function* $V(\boldsymbol{z})$ *of* $\mathcal{M}(\mathsf{v}, \mathsf{w})$ *satisfies the following identity:*

$$V(\boldsymbol{z}) = \prod_{\alpha \in \Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha),$$

where $z_\alpha := \prod_i z_i^{n_i}$ *and* $\mathsf{a}_\alpha := \sum_i n_i \mathsf{a}_i$ *for* $\alpha = \sum_i n_i \alpha_i$ *expressed as a sum of simple roots* α_i*, with* $\mathsf{a} \in \mathbb{Z}^{Q_0}$ *and* $n_i \in \mathbb{Z}$ *for each* $i \in Q_0$*.*

Remark. The integers a_α are determined as follows. There exist some $\mathsf{a}_i \in \mathbb{Z}$ depending on v and w such that if $\alpha = \sum_i c_i \alpha_i$, then $\mathsf{a}_\alpha = \sum_i c_i \mathsf{a}_i$. In other words, the $(q/\hbar)^{\mathsf{a}_\alpha}$ can be removed by a shift of the Kähler parameters. We will write explicit formulas for the a_i in Theorems 6.2 and 6.5 below.

We will prove the theorem by considering the cases $\mathsf{w} = \delta_i$ for $i \in \{1, n-1, n\}$ separately. By symmetry, the case of $\mathsf{w} = \delta_{n-1}$ is identical to the case of $\mathsf{w} = \delta_n$, and so we will only consider the latter below. In Theorem 6.1, we state the precise formulation for the $\mathsf{w} = \delta_n$ case, and prove it. In Theorem 6.5, we do the same for the $\mathsf{w} = \delta_1$ case.

Note that Remark 5.15 of [KP21] establishes that set of roots Φ' precisely correspond to weights of $T_0\mathcal{M}^!$, the tangent space at 0 of the variety 3d mirror dual to $\mathcal{M}(\mathsf{v}, \mathsf{w})$. Thus the factorization in Theorem 6.1 can also be viewed as a proof of the 3*d* mirror symmetry for vertex functions of these type D Nakajima varieties. In fact, limits in the equivariant parameters of vertex functions of arbitrary quiver varieties split into the product of vertex functions for quiver varieties with one framing. Thus Theorem 6.1 also implies a Conjecture 1 from [DS20a] for type D quiver varieties with framings only at minuscule vertices. See [DKK] for more details.

Remark. Theorem 6.1, along with our method of proof, allows one to write explicit formulas for $V^{(\tau)}(\boldsymbol{z})$, the vertex with descendant τ, whenever τ is in the subalgebra generated by exterior powers of tautological vector bundles and their duals. In fact, Lemmas 8.2 and 8.4 below express the vertex functions in terms of Macdonald polynomials. One immediately sees that applying various Macdonald difference operators to the expressions in these lemmas gives vertex functions with these descendants. Applying the same difference operator to the right hand side of Theorem 6.1 and using the identity $F(xq) = \left(\frac{1-x}{1-\hbar x}\right) F(x)$ then gives an explicit rational formula for $V^{(\tau)}(\boldsymbol{z})/V^{(1)}(\boldsymbol{z})$. Up to global scalar, this ratio is exactly the capped vertex function with descendant. See [DS22] where this was carried out in the type A setting.

6.1 Case of $\mathsf{w} = (0^{n-1}, 1)$

When the framing is at a spin node, Theorem 6.1 reads as follows.

Theorem 6.2. *Fix* v *and* $\mathsf{w} = (0^{n-1}, 1)$ *such that* $\mathcal{M}(\mathsf{v}, \mathsf{w})$ *is nonempty. Let* $\Phi' = \{\alpha \in \Phi \mid (\mu, \alpha) < 0\}$. *Then, the vertex function* $V(\boldsymbol{z})$ *of* $\mathcal{M}(\mathsf{v}, \mathsf{w})$ *satisfies the following identity:*

$$V(\boldsymbol{z}) = \prod_{\alpha \in \Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha),$$

where

$$\mathsf{a}_i = \begin{cases} \mathsf{v}_i - \mathsf{v}_{i-1} & \text{if } i \in [1, n-2], \\ \left\lfloor \frac{\mathsf{v}_{n-1} + \mathsf{v}_n}{2} \right\rfloor - \mathsf{v}_{n-2} & \text{if } i = n-1, \\ \left\lfloor \frac{\mathsf{v}_{n-1} + \mathsf{v}_n + 1}{2} \right\rfloor - \mathsf{v}_{n-1} - \mathsf{v}_n - \mathbb{1}_{2 \nmid n} \mathbb{1}_{\mathsf{v}_{n-2} = \mathsf{v}_{n-1} + \mathsf{v}_n} & \text{if } i = n, \end{cases}$$

where $\mathbb{1}_\phi$ *is the indicator function, equal to* 1 *if* ϕ *is true and* 0 *otherwise.*

Example 6.3. Let $\mathsf{v} = (1, 2, 3, 4, 5, 3, 3)$ and $\mathsf{w} = (0^6, 1)$. The corresponding poset is drawn in Figure 5. One can check that the set Φ' is in bijection with boxes in the diagram. Here, $\mathsf{a} = (1, 1, 1, 1, 1, -2, -3)$, and the corresponding values of a_α are written inside.

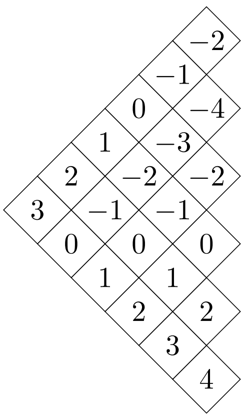

Figure 5: The poset description of $Q = D_7$ with $\mathsf{v} = (1, 2, 3, 4, 5, 3, 3)$ and $\mathsf{w} = (0^6, 1)$, along with the values of a_α.

Example 6.4. The smallest nontrivial example of Theorem 6.2 (for which the identity cannot be proven by iterated applications of the q-binomial theorem) is $Q = D_4$ with $\mathsf{v} = (1, 2, 1, 1)$ and $\mathsf{w} = (0, 0, 0, 1)$. Then Theorem 6.2 reads

$$V(\boldsymbol{z}) = F((q/\hbar)z_2) \cdot F(z_2 z_3) \cdot F((q/\hbar)^2 z_1 z_2) \cdot F((q/\hbar)z_1 z_2 z_3) \cdot F(z_1 z_2^2 z_3 z_4). \quad (12)$$

Even in this example, this identity is nontrivial; in particular, there is no bijection matching reverse plane partitions over the corresponding poset with the terms in the summation on the right hand side. To match with the root-theoretic description, we calculate

$$\mu = \omega_4 - (\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) = \omega_4 - \omega_2.$$

Using the fact that $(\omega_i, \alpha_j) = \delta_{ij}$, one can compute $(\mu, \alpha) = (\omega_4 - \omega_2, \alpha)$ for each $\alpha \in \Phi$. We find that

$$\begin{aligned} (\mu, \alpha_2) &= -1, \\ (\mu, \alpha_2 + \alpha_3) &= -1, \\ (\mu, \alpha_1 + \alpha_2) &= -1, \\ (\mu, \alpha_1 + \alpha_2 + \alpha_3) &= -1, \\ (\mu, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) &= -1, \end{aligned}$$

and $(\mu, \alpha) \geq 0$ otherwise. Up to a shift by powers of $(q/\hbar)$, this matches the z-monomials in the arguments of the q-binomial series. For the shifts, we compute $\mathsf{a} = (1, 1, -1, -2)$. So the a_α, in the order written in (12), are 1, 0, 2, 1, and 0.

6.2 Case of $\mathsf{w} = (1, 0^{n-1})$

When the framing is at the fundamental node, Theorem 6.1 reads as follows.

Theorem 6.5. *Fix* v *and* $\mathsf{w} = (1, 0^{n-1})$ *such that* $\mathcal{M}(\mathsf{v}, \mathsf{w})$ *is nonempty. Let* $\Phi' = \{\alpha \in \Phi \mid (\mu, \alpha) < 0\}$*. Then, the vertex function* $V(\boldsymbol{z})$ *of* $\mathcal{M}(\mathsf{v}, \mathsf{w})$ *satisfies the following identity:*

$$V(\boldsymbol{z}) = \prod_{\alpha \in \Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha),$$

where a_i *are given by the following:*

$$\mathsf{a}_i = \begin{cases} \mathsf{v}_1 - 1 & \text{if } i = 1, \\ \mathsf{v}_i - \mathsf{v}_{i-1} & \text{if } i \in [2, n-2], \\ -1 & \text{if } i = n-1 \text{ or } i = n. \end{cases}$$

Example 6.6. Let us write a nontrivial example of Theorem 6.5. For example, take D_5 poset of type ω_1 with $\mathsf{v} = (2,2,2,1,1)$. Then Theorem 6.5 reads

$$V(\boldsymbol{z}) = F((q/\hbar)z_1)F((q/\hbar)z_1z_2)F((q/\hbar)z_1z_2z_3)F(z_1z_2z_3z_4)F(z_1z_2z_3z_5) \\ F((\hbar/q)z_1z_2z_3z_4z_5)F((\hbar/q)z_1z_2z_3^2z_4z_5)F((\hbar/q)z_1z_2^2z_3^2z_4z_5), \tag{13}$$

which is quite nontrivial; there is no bijection matching reverse plane partitions over the corresponding poset with the terms in the summation on the right hand side. To match with the root-theoretic description, we calculate

$$\mu = \omega_1 - (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) = -\omega_1.$$

Using the fact that $(\omega_i, \alpha_j) = \delta_{ij}$, we can compute $(\mu, \alpha) = (-\omega_1, \alpha)$ for each $\alpha \in \Phi$. We find that

$$\begin{aligned}
(\mu, \alpha_1) &= -1, \\
(\mu, \alpha_1 + \alpha_2) &= -1, \\
(\mu, \alpha_1 + \alpha_2 + \alpha_3) &= -1, \\
(\mu, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) &= -1, \\
(\mu, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5) &= -1, \\
(\mu, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) &= -1, \\
(\mu, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) &= -1, \\
(\mu, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5) &= -1,
\end{aligned}$$

and that $(\mu, \alpha) \geq 0$ otherwise. Up to a shift by powers of $q/\hbar$, this matches the z-monomials in the arguments of the q-binomial series. For the shifts, we compute $\mathsf{a} = (1,0,0,-1,-1)$. So the a_α, in the order written in (13), are $1,1,1,0,0,-1,-1,-1$.

7 Proof of Theorem 6.2

7.1 Part 1

First, from Theorem 5.6 and Definition 4.4, we can explicitly write down $V(\boldsymbol{z})$ in terms of $\hbar$'s and q's:

$$V(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \boldsymbol{z}^{\boldsymbol{d}} \left(\prod_{j=1}^{\mathsf{v}_n} \frac{(\hbar^{2j-1})_{d_{n,j}}}{(q\hbar^{2j-2})_{d_{n,j}}} \right) \left(\prod_{i=1}^{n-3} \prod_{j=1}^{\mathsf{v}_i} \prod_{k=1}^{\mathsf{v}_{i+1}} \frac{\left(\hbar^{k-j}\right)_{d_{i+1,k}-d_{i,j}}}{\left(q\hbar^{k-j-1}\right)_{d_{i+1,k}-d_{i,j}}} \right)$$
$$\left(\prod_{j=1}^{\mathsf{v}_{n-2}} \prod_{k=1}^{\mathsf{v}_{n-1}} \frac{\left(\hbar^{2k-j}\right)_{d_{n-1,k}-d_{n-2,j}}}{\left(q\hbar^{2k-1-j}\right)_{d_{n-1,k}-d_{n-2,j}}} \right) \left(\prod_{j=1}^{\mathsf{v}_{n-2}} \prod_{k=1}^{\mathsf{v}_{n}} \frac{\left(\hbar^{2k-1-j}\right)_{d_{n,k}-d_{n-2,j}}}{\left(q\hbar^{2k-j-2}\right)_{d_{n,k}-d_{n-2,j}}} \right)$$
$$\left(\prod_{i=1}^{n-2} \prod_{j=1}^{\mathsf{v}_i} \prod_{k=1}^{\mathsf{v}_i} \frac{\left(q\hbar^{k-j}\right)_{d_{i,k}-d_{i,j}}}{\left(\hbar^{k-j+1}\right)_{d_{i,k}-d_{i,j}}} \right) \left(\prod_{j=1}^{\mathsf{v}_{n-1}} \prod_{k=1}^{\mathsf{v}_{n-1}} \frac{\left(q\hbar^{2(k-j)}\right)_{d_{n-1,k}-d_{n-1,j}}}{\left(\hbar^{2(k-j)+1}\right)_{d_{n-1,k}-d_{n-1,j}}} \right)$$
$$\left(\prod_{j=1}^{\mathsf{v}_n} \prod_{k=1}^{\mathsf{v}_n} \frac{\left(q\hbar^{2(k-j)}\right)_{d_{n,k}-d_{n,j}}}{\left(\hbar^{2(k-j)+1}\right)_{d_{n,k}-d_{n,j}}} \right) \tag{14}$$

where the multi-index $\boldsymbol{d}$ gives a reverse plane partition over the corresponding poset. We will first rewrite this as a sum over certain tuples of interlacing partitions. To express the interlacing relations, we assign a sign to each of the uppermost edges in the poset diagram in the following way.

Definition 7.1. The sign $\tau(i) \in \{+,-\}$ of each edge $e \in \omega$ is defined by

$$\tau(i) := \begin{cases} + & \text{if } i \le n-2 \text{ and } \mathsf{v}_i - \mathsf{v}_{i-1} = 1, \\ - & \text{if } i \le n-2 \text{ and } \mathsf{v}_i - \mathsf{v}_{i-1} = 0, \\ + & \text{if } i = n-1 \text{ and } \mathsf{v}_{n-1} + \mathsf{v}_n - \mathsf{v}_{n-2} = 1, \\ - & \text{if } i = n-1 \text{ and } \mathsf{v}_{n-1} + \mathsf{v}_n - \mathsf{v}_{n-2} = 0, \\ - & \text{if } i = n. \end{cases}$$

We say that an $(n-1)$-tuple of partitions $\boldsymbol{\lambda} := \left(\lambda^1, \ldots, \lambda^{n-1}\right)$ interlaces according to v if the following holds:

- $\tau(i) = + \implies \lambda^i \succ \lambda^{i+1}$ for each $i \in [1, n-2]$,
- $\tau(i) = - \implies \lambda^i \prec \lambda^{i+1}$ for each $i \in [1, n-2]$,
- $l(\lambda^i) \le \mathsf{v}_i$ for each $i \in [1, n-2]$,
- $l(\lambda^{n-1}) \le \mathsf{v}_{n-1} + \mathsf{v}_n$.

We write S_{v} for the set of all $(n-1)$-tuples of partitions that interlace according to v. We will always assume for $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^{n-1})$ that each λ^i is augmented with zeros such that its total length, with zeros included is v_i for $i < n-2$ and is $\mathsf{v}_{n-1} + \mathsf{v}_n$ for $i = n-1$. In what follows, $l(\lambda^i)$ denotes this notion of length.

Proposition 7.2. *We can rewrite the vertex function in (14) as*

$$V(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda} \in S_{\mathsf{v}}} \alpha_{\lambda^{n-1}} \left(\prod_{i=1}^{n-2} \beta_{\lambda^i, \lambda^{i+1}} \right) \left(\prod_{i=1}^{n-1} \gamma_{\lambda^i} \right) \boldsymbol{z}^{\boldsymbol{\lambda}}, \tag{15}$$

where the coefficients are given by

$$\alpha_{\lambda^{n-1}} = \prod_{2 \nmid j}^{l(\lambda^{n-1})} \frac{(\hbar^{l(\lambda^{n-1})-j+1})_{\lambda_j^{n-1}}}{(q\hbar^{l(\lambda^{n-1})-j})_{\lambda_j^{n-1}}},$$

$$\beta_{\lambda^i, \lambda^{i+1}} = \prod_{j=1}^{l(\lambda^i)} \prod_{k=1}^{l(\lambda^{i+1})} \frac{(\hbar^{l(\lambda^{i+1})-l(\lambda^i)-k+j})_{\lambda_k^{i+1}-\lambda_j^i}}{(q\hbar^{l(\lambda^{i+1})-l(\lambda^i)-k+j-1})_{\lambda_k^{i+1}-\lambda_j^i}} \text{ for each } i \in [1, n-2],$$

$$\gamma_{\lambda^i} = \begin{cases} \prod_{j,k=1}^{l(\lambda^i)} \frac{(q\hbar^{j-k})_{\lambda_k^i - \lambda_j^i}}{(\hbar^{j-k+1})_{\lambda_k^i - \lambda_j^i}} & \text{if } i \in [1, n-2], \\ \prod_{2 \mid j-k}^{l(\lambda^{n-1})} \frac{(q\hbar^{j-k})_{\lambda_k^i - \lambda_j^i}}{(\hbar^{j-k+1})_{\lambda_k^i - \lambda_j^i}} & \text{if } i = n-1, \end{cases}$$

and $\boldsymbol{z}^{\boldsymbol{\lambda}}$ is defined by

$$\boldsymbol{z}^{\boldsymbol{\lambda}} := \left(\prod_{i=1}^{n-2} z_i^{|\lambda^i|} \right) z_{n-1}^{\sum_{2 \mid j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1}} z_n^{\sum_{2 \nmid j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1}}.$$

Proof. Given $\boldsymbol{d}$, we define $\boldsymbol{\lambda}$ by $\lambda^i = (d_{i,\mathsf{v}_i}, d_{i,\mathsf{v}_i - 1}, \dots, d_{i,1})$ for each $i = 1, 2, \dots, n-2$ and $\lambda^{n-1} = (\dots, d_{n-1,2}, d_{n,2}, d_{n-1,1}, d_{n,1})$. Now straightforward algebra gives the result. $\square$

We next interpret the right-hand side of Equation (15) in terms of Macdonald polynomials. Recall the formulas in Section 3. Note the following formula for $\mathcal{E}_\lambda$ (see Proposition 2.1 of [BBCW18]):

$$\mathcal{E}_\lambda(x) = \Phi(x)^{-1} \sum_{\mu' \text{ even}} b_\mu^{\text{el}} P_{\lambda/\mu}(x).$$

Note that $\mathcal{E}_\lambda$ specializes to a single variable with

$$\mathcal{E}_\lambda(x)|_{x=(x_1)} = b_{e(\lambda)}^{\text{el}} \psi_{e(\lambda)/\lambda} x_1^{|e(\lambda)|-|\lambda|}$$

where $e(\lambda)$ is the unique partition such that $e(\lambda)'$ is even and $\lambda \prec e(\lambda)$.

Let ω be the path of the maximal edges in the half-Young diagram arising from v, including the vertical part, which is counted as a single edge. We write $E^{\nearrow}(\omega), E^{\searrow}(\omega), E^{\downarrow}(\omega)$ for the set of edges in ω which are $\nearrow$, $\searrow$, and $\downarrow$, respectively. As in [BBC20], define the weight of each edge $e = \kappa \to \mu \in \omega$ as

$$\mathcal{W}(e) = \begin{cases} \mathcal{E}_\kappa(x_e) & \text{if } e \in E^{\downarrow}(\omega), \\ Q_{\kappa/\mu}(x_e) & \text{if } e \in E^{\searrow}(\omega), \\ P_{\mu/\kappa}(x_e) & \text{if } e \in E^{\nearrow}(\omega), \end{cases}$$

where x_e stands for a collection of variables for each edge e. Moreover, define the weight of interlacing partitions $\boldsymbol{\lambda}$ as

$$\mathcal{W}(\boldsymbol{\lambda}) = \prod_{e \in E(\omega)} \mathcal{W}(e).$$

The functions $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ specialize to a single variable with

$$P_{\lambda/\mu}(x)|_{x=(x_1)} = \begin{cases} \psi_{\lambda/\mu} x_1^{|\lambda|-|\mu|} & \text{if } \lambda \succ \mu, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_{\lambda/\mu}(x)|_{x=(x_1)} \begin{cases} \varphi_{\lambda/\mu} x_1^{|\lambda|-|\mu|} & \text{if } \lambda \succ \mu, \\ 0 & \text{otherwise.} \end{cases}$$

We identify the coefficients of the vertex function in Proposition 7.2 with the weight of $\boldsymbol{\lambda}$:

Lemma 7.3. *Let $\boldsymbol{\lambda} \in S_{\mathsf{v}}$. Then, taking single-variable specialization for all variables x_i gives*

$$\mathcal{W}(\boldsymbol{\lambda})|_{x_i=(x_i)} = \alpha_{\lambda^{n-1}} \left(\prod_{i=1}^{n-2} \beta_{\lambda^i, \lambda^{i+1}} \right) \left(\prod_{i=1}^{n-1} \gamma_{\lambda^i} \right) \hat{\boldsymbol{z}}^{\boldsymbol{\lambda}},$$

where $\hat{z}_i = (q/\hbar)^{\mathsf{a}_i} z_i$ and

$$z_i = \begin{cases} x_i^{\tau(i)} x_{i+1}^{-\tau(i+1)} & \textit{if } i \in [1, n-2], \\ x_{n-1}^{\tau(n-1)} x_n^{-1} & \textit{if } i = n-1, \\ x_{n-1}^{\tau(n-1)} x_n & \textit{if } i = n. \end{cases}$$

Proof. We expand $\mathcal{W}(\boldsymbol{\lambda})|_{x_i=(x_i)}$ as follows:

$$\mathcal{W}(\boldsymbol{\lambda})|_{x_i=(x_i)} = \left(\prod_{\kappa \to \mu \in E^{\nearrow}(\omega)} \psi_{\mu/\kappa} x_e^{|\mu|-|\kappa|} \right) \left(\prod_{\kappa \to \mu \in E^{\searrow}(\omega)} \varphi_{\kappa/\mu} x_e^{|\kappa|-|\mu|} \right)$$

$$
\left(\prod_{\kappa \to \mu \in E^{\downarrow}(\omega)} \sum_{\substack{\kappa \prec \lambda \\ \lambda' \text{ even}}} b^{\mathrm{el}}_{\lambda} \psi_{\lambda/\kappa} x_e^{|\lambda|-|\kappa|} \right)
$$

$$
= \left(\prod_{i=1}^{n-2} f_{\lambda^i,\lambda^{i+1}} \right) b^{\mathrm{el}}_{e(\lambda^{n-1})} \psi_{e(\lambda^{n-1})/\lambda^{n-1}}
\left(\prod_{i=1}^{n-2} \left(x_i^{\tau_\lambda(i)} x_{i+1}^{-\tau_\lambda(i+1)} \right)^{|\lambda^i|} \right) (x_{n-1}^{\tau_\lambda(n-1)} x_n^{-1})^{|\lambda^{n-1}|} x_n^{|e(\lambda^{n-1})|},
$$

where we have

$$
f_{\lambda^i,\lambda^{i+1}} = \begin{cases} \psi_{\lambda^{i+1}/\lambda^i} & \text{if } i \to i+1 \in E^{\nearrow}(\omega), \\ \varphi_{\lambda^i/\lambda^{i+1}} & \text{if } i \to i+1 \in E^{\searrow}(\omega). \end{cases}
$$

Concretely,

$$
f_{\lambda^i,\lambda^{i+1}} = \begin{cases} \prod_{s \in R_{\lambda^{i+1}/\lambda^i} - C_{\lambda^{i+1}/\lambda^i}} \frac{b_{\lambda^{i+1}}(s)}{b_{\lambda^i}(s)} & \text{if } i \to i+1 \in E^{\nearrow}(\omega), \\ \prod_{s \in C_{\lambda^{i+1}/\lambda^i}} \frac{b_{\lambda^{i+1}}(s)}{b_{\lambda^i}(s)} & \text{if } i \to i+1 \in E^{\searrow}(\omega). \end{cases}
$$

Changing variables, $\mathcal{W}(\boldsymbol{\lambda})|_{x_i=(x_i)}$ is equal to

$$
\sum_{\boldsymbol{\lambda}} \left(\prod_{i=1}^{n-2} f_{\lambda^i,\lambda^{i+1}} \right) b^{\mathrm{el}}_{e(\lambda^{n-1})} \psi_{e(\lambda^{n-1})/\lambda^{n-1}}
\left(\prod_{i=1}^{n-2} \hat{z}_i^{|\lambda^i|} \right) \hat{z}_{n-1}^{\sum_{2|j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1}} \hat{z}_n^{\sum_{2\nmid j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1}}
$$

$$
= \sum_{\boldsymbol{\lambda}} \left(\prod_{i=1}^{n-2} f_{\lambda^i,\lambda^{i+1}} \right) b^{\mathrm{el}}_{e(\lambda^{n-1})} \psi_{e(\lambda^{n-1})/\lambda^{n-1}} \hat{\boldsymbol{z}}^{\boldsymbol{\lambda}}
$$

due to the identity

$$
\sum_{2\nmid j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1} - \sum_{2|j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1} = |e(\lambda^{n-1})| - |\lambda^{n-1}|.
$$

Thus, it suffices to show

$$
\left(\prod_{i=1}^{n-2} f_{\lambda^i,\lambda^{i+1}} \right) b^{\mathrm{el}}_{e(\lambda^{n-1})} \psi_{e(\lambda^{n-1})/\lambda^{n-1}} \left(\frac{q}{\hbar} \right)^t
= \alpha_{\lambda^{n-1}} \left(\prod_{i=1}^{n-2} \beta_{\lambda^i,\lambda^{i+1}} \right) \left(\prod_{i=1}^{n-1} \gamma_{\lambda^i} \right),
$$

where $t \in \mathbb{Z}$ satisfies

$$t = \left(\sum_{i=1}^{n-2}(l(\lambda^i) - l(\lambda^{i-1}))|\lambda^i|\right)$$
$$+ \left(\left\lfloor \frac{l(\lambda^{n-1})}{2} \right\rfloor - l(\lambda^{n-2})\right) \left(\sum_{2|j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1}\right)$$
$$+ \left(\left\lfloor \frac{l(\lambda^{n-1})+1}{2} \right\rfloor - l(\lambda^{n-1}) - \mathbb{1}_{2\nmid n}\mathbb{1}_{l(\lambda^{n-2})=l(\lambda^{n-1})}\right) \left(\sum_{2\nmid j} \lambda^{n-1}_{l(\lambda^{n-1})-j+1}\right).$$

This identity follows by a straightforward, though tedious, computation analogous to Section 3.7 of [DS20b]. □

Remark. The previous proposition shows that these vertex functions coincide with the partition functions of the half-space Macdonald processes from [BBC20].

In the next subsection, we convert the expression on the right-hand side to a product form.

7.2 Part 2

Proposition 2.2 of [BBC20] gives a formula for $\sum_{\boldsymbol{\lambda} \in S_{\mathsf{v}}} \mathcal{W}(\boldsymbol{\lambda})$. When the variable sets are assumed to consist of a single variable, it reads

$$\begin{aligned}
&\sum_{\boldsymbol{\lambda}} \mathcal{W}(\boldsymbol{\lambda})|_{x_i=(x_i)} \\
&= \left(\prod_{\substack{e<e' \\ e,e' \in E^{\nearrow}(\omega)}} F(x_e x'_e)\right) \left(\prod_{\substack{e<e': \\ e\in E^{\nearrow}(\omega), e' \in E^{\searrow}(\omega)\cup E^{\downarrow}(\omega)}} F(x_e x'_e)\right) \\
&= \prod_{\substack{e<e' \\ e\in E^{\nearrow}(\omega), e'\in\omega}} F(x_e x'_e). \qquad (16)
\end{aligned}$$

This identity (and its generalization for arbitrary variable sets) is proven in [BBC20] by repeated applications of Cauchy identities. We now prove that terms in the product match certain roots.

Lemma 7.4. *Fix* v*. Let* w *and* Φ' *be as in Theorem 6.2. Then*

$$\{x_e x'_e : e < e', e \in E^{\nearrow}(\omega), e' \in \omega\} = \{(q/\hbar)^{\mathsf{a}_\alpha} z_\alpha : \alpha \in \Phi'\}$$

for each v*. Thus*

$$\prod_{\substack{e<e' \\ e\in E^{\nearrow}(\omega), e'\in\omega}} F(x_e x'_e) = \prod_{\alpha\in\Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha).$$

Proof. Let C be the Cartan matrix for Φ, defined by

- $C_{i,i} = 2$ for $i \in [1, n]$,
- $C_{i,i+1} = C_{i+1,i} = -1$ for all $i \in [1, n-2]$,
- $C_{n-2,n} = C_{n,n-2} = -1$.

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$. Note that $C\boldsymbol{\omega} = \boldsymbol{\alpha}$ by the natural identification. We also have $\boldsymbol{\epsilon} = D\boldsymbol{\alpha}$, where D is the matrix satisfying

- $D_{i,j} = 1$ for all $i \in [1, n]$ and $j \in [i, n-2]$,
- $D_{i,n-1} = D_{i,n} = \frac{1}{2}$ for $i \in [1, n-1]$,
- $D_{n,n-1} = -\frac{1}{2}$, and $D_{n,n} = \frac{1}{2}$.

Every $\alpha \in \Phi$ falls into one of three distinct possibilities:

- $\alpha = \epsilon_i + \epsilon_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ for all $i < j \in [1, n-1]$,
- $\alpha = \epsilon_i + \epsilon_n = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n$ for all $i \in [1, n-1]$,
- $\alpha = \epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for all $i \neq j \in [1, n]$.

We explicitly evaluate $(\mu, \alpha) = (\omega_n - \sum_i \mathsf{v}_i \alpha_i, \alpha)$ and find all such α that give $(\mu, \alpha) < 0$. Let $\boldsymbol{c} = (c_1, \ldots, c_n)$ be the coefficients for $\alpha = \sum_{i=1}^n c_i \alpha_i$. In this notation, the condition $(\mu, \alpha) < 0$ is equivalent to $\mathsf{v}^T \boldsymbol{C}\boldsymbol{c} > c_n$. The roots can be expanded in terms of simple roots in the following way:

- $\epsilon_i - \epsilon_j$ corresponds to $\boldsymbol{c} = (0^{i-1}, 1^{j-i}, 0^{n-j+1})$,
- $\epsilon_i + \epsilon_j$ corresponds to $\boldsymbol{c} = (0^{i-1}, 1^{j-i}, 2^{n-j-1}, 1^2)$,
- $\epsilon_i + \epsilon_n$ corresponds to $\boldsymbol{c} = (0^{i-1}, 1^{n-i-1}, 0, 1)$,

where the superscripts stand for the multiplicity of the number.

We are now ready to prove the lemma by induction on $|\mathsf{v}|$. For the base case, when $\mathsf{v} = 0$, we see that both sets in the statement of the lemma are empty.

Suppose that the lemma holds for v. Denote the set in the left (resp. right) side of the lemma by L_v (resp. R_v). Suppose that v' is such that $\mathsf{v}'_m = \mathsf{v}_m + 1$ for some m and $\mathsf{v}'_i = \mathsf{v}_i$ for $i \neq m$. It suffices to show that $L_{\mathsf{v}'} \setminus L_\mathsf{v} = R_{\mathsf{v}'} \setminus R_\mathsf{v}$, and that $L_\mathsf{v} \setminus L_{\mathsf{v}'} = R_\mathsf{v} \setminus R_{\mathsf{v}'}$. Let $\alpha = \sum_i c_i \alpha_i$ and $\Delta(\alpha) = \mathsf{v}'^T \boldsymbol{C}\boldsymbol{c} - \mathsf{v}^T \boldsymbol{C}\boldsymbol{c}$. Then $\Delta(\alpha)$ takes the following form depending on the value of m:

- If $m \leq n-3$, then $\Delta = -c_{m-1} + 2c_m - c_{m+1}$.

- If $m = n-2$, then $\Delta = -c_{n-3} + 2c_{n-2} - c_{n-1} - c_n$.
- If $m = n-1$, then $\Delta = -c_{n-2}$.
- If $m = n$, then $\Delta = -c_{n-2}$.

Now $c_i \in \{0, 1, 2\}$ for all i, so we can explicitly consider all the cases for which $\Delta(\alpha) \neq 0$. This gives us the following data, which again depends on the value of m and now also on the type of root:

- $m \leq n-3$. We write $c_{m-1}c_mc_{m+1}$ on the left (before $\to$), and $\Delta(\alpha)$ on the right.
  - $\alpha = \epsilon_i - \epsilon_j$: 001, 011, 110, 100 $\to$ -1, 1, 1, -1.
  - $\alpha = \epsilon_i + \epsilon_j$ $(i, j \leq n-1)$: 001, 011, 122, 112, 221, 211 $\to$ -1, 1, 1, -1, 1, -1.
  - $\alpha = \epsilon_i + \epsilon_n$: 001, 011, 110, 100, 010, 101 $\to$ -1, 1, 1, -1, 2, -2.
- $m = n-2$. We write $c_{n-3}c_{n-2}c_{n-1}c_n$ on the left, and $\Delta(\alpha)$ on the right.
  - $\epsilon_i - \epsilon_j$: 0001, 0010, 0011, 0100, 0110, 1000, 1100, 1111 $\to$ -1, -1, -2, 2, 1, -1, 1, -1.
  - $\epsilon_i + \epsilon_j$ $(i, j \leq n-1)$: 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1111 $\to$ -1, -1, -2, 2, 1, 1, -1, -2, -2, -3, 1, -1.
  - $\epsilon_i + \epsilon_n$: 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1111 $\to$ -1, -1, -2, 2, 1, 1, -1, -2, -2, -3, 1, -1.
- $m = n-1$. We write c_{n-2} on the left, and $\Delta(\alpha)$ on the right.
  - $\epsilon_i - \epsilon_j$: 1 $\to$ -1.
  - $\epsilon_i + \epsilon_j$ $(i, j \leq n-1)$: 1, 2 $\to$ -1, -2.
  - $\epsilon_i + \epsilon_n$: 1, 2 $\to$ -1, -2.
- $m = n$. We write c_{n-2} on the left, and $\Delta(\alpha)$ on the right.
  - $\epsilon_i - \epsilon_j$: 1 $\to$ -1.
  - $\epsilon_i + \epsilon_j$ $(i, j \leq n-1)$: 1, 2 $\to$ -1, -2.
  - $\epsilon_i + \epsilon_n$: 1, 2 $\to$ -1, -2.

We check manually for each m that the new contributions are exactly the desired roots, concluding the induction. □

Combining Lemma 7.3, Equation (16), and Lemma 7.4 concludes the proof of Theorem 6.2.

8 Proof of Theorem 6.5

8.1 Reduction to two cases

Recall Lemma 4.8, which describes the v such that $\mathcal{M}(\mathsf{v},\mathsf{w})$ is nonempty. If $\mathsf{v}_i < 2$ for all i, then Theorem 6.5 follows by repeated applications of the q-binomial theorem. So we can assume that v is of the form $\mathsf{v} = (1^k, 2^{n-k-2}, 1, 1)$ for some $k \in [0, n-2]$. We can further reduce the number of possibilities by the following lemma.

Lemma 8.1. *Let* $\mathsf{v} = (1^k, 2^{n-k-2}, 1, 1)$ *where* $k \geq 2$*. Assuming Lemma 8.5, which we prove later, we have*

$$V(\boldsymbol{z}) = \prod_{\alpha\in\Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha).$$

Proof. The formula in Theorem 5.6 implies that

$$\begin{aligned}
V(\boldsymbol{z}) = \sum_{\boldsymbol{d}} &\left(\prod_{i=1}^{k-1} \frac{(\hbar)_{d_{i+1,1}-d_{i,1}}}{(q)_{d_{i+1,1}-d_{i,1}}}\right) \frac{(\hbar)_{d_{k+1,1}-d_{k,1}}}{(q)_{d_{k+1,1}-d_{k,1}}} \frac{(\hbar^{n-k-1})_{d_{k+1,2}-d_{k,1}}}{(q\hbar^{n-k-2})_{d_{k+1,2}-d_{k,1}}} \\
&\prod_{i=k+1}^{n-3} \frac{(\hbar)_{d_{i+1,1}-d_{i,1}}}{(q)_{d_{i+1,1}-d_{i,1}}} \frac{(\hbar^{n-i-1})_{d_{i+1,2}-d_{i,1}}}{(q\hbar^{n-i-2})_{d_{i+1,2}-d_{i,1}}} \frac{(\hbar^{2+i-n})_{d_{i+1,1}-d_{i,2}}}{(q\hbar^{1+i-n})_{d_{i+1,1}-d_{i,2}}} \frac{(1)_{d_{i+1,2}-d_{i,2}}}{(q\hbar^{-1})_{d_{i+1,2}-d_{i,2}}} \\
&\cdot \frac{(\hbar)_{d_{n-1,1}-d_{n-2,1}}}{(q)_{d_{n-1,1}-d_{n-2,1}}} \frac{(1)_{d_{n-1,1}-d_{n-2,2}}}{(q\hbar^{-1})_{d_{n-1,1}-d_{n-2,2}}} \frac{(\hbar)_{d_{n,1}-d_{n-2,1}}}{(q)_{d_{n,1}-d_{n-2,1}}} \frac{(1)_{d_{n,1}-d_{n-2,2}}}{(q\hbar^{-1})_{d_{n,1}-d_{n-2,2}}} \\
&\cdot \frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}} \left(\prod_{i=k+1}^{n-2} \frac{(q\hbar^{n-i-1})_{d_{i,2}-d_{i,1}}}{(\hbar^{n-i})_{d_{i,2}-d_{i,1}}} \frac{(q\hbar^{i+1-n})_{d_{i,1}-d_{i,2}}}{(\hbar^{i+2-n})_{d_{i,1}-d_{i,2}}}\right) \prod_{i=1}^{n}\prod_{j=1}^{\mathsf{v}_i} z_i^{d_{i,j}}.
\end{aligned}$$

Rewrite $d_{i,j} = d'_{i,j} + d_{1,1}$ for all $i \in [1,n]$ and $j \in [1, \mathsf{v}_i]$. Then we have

$$\begin{aligned}
V(\boldsymbol{z}) = \sum_{\boldsymbol{d}'} &\frac{(\hbar)_{d'_{2,1}}}{(q)_{d'_{2,1}}} \left(\prod_{i=2}^{k-1} \frac{(\hbar)_{d'_{i+1,1}-d'_{i,1}}}{(q)_{d'_{i+1,1}-d'_{i,1}}}\right) \frac{(\hbar)_{d'_{k+1,1}-d'_{k,1}}}{(q)_{d'_{k+1,1}-d'_{k,1}}} \frac{(\hbar^{n-k-1})_{d'_{k+1,2}-d'_{k,1}}}{(q\hbar^{n-k-2})_{d'_{k+1,2}-d'_{k,1}}} \\
&\prod_{i=k+1}^{n-3} \frac{(\hbar)_{d'_{i+1,1}-d'_{i,1}}}{(q)_{d'_{i+1,1}-d'_{i,1}}} \frac{(\hbar^{n-i-1})_{d'_{i+1,2}-d'_{i,1}}}{(q\hbar^{n-i-2})_{d'_{i+1,2}-d'_{i,1}}} \frac{(\hbar^{2+i-n})_{d'_{i+1,1}-d'_{i,2}}}{(q\hbar^{1+i-n})_{d'_{i+1,1}-d'_{i,2}}} \frac{(1)_{d'_{i+1,2}-d'_{i,2}}}{(q\hbar^{-1})_{d'_{i+1,2}-d'_{i,2}}} \\
&\cdot \frac{(\hbar)_{d'_{n-1,1}-d'_{n-2,1}}}{(q)_{d'_{n-1,1}-d'_{n-2,1}}} \frac{(1)_{d'_{n-1,1}-d'_{n-2,2}}}{(q\hbar^{-1})_{d'_{n-1,1}-d'_{n-2,2}}} \frac{(\hbar)_{d'_{n,1}-d'_{n-2,1}}}{(q)_{d'_{n,1}-d'_{n-2,1}}} \frac{(1)_{d'_{n,1}-d'_{n-2,2}}}{(q\hbar^{-1})_{d'_{n,1}-d'_{n-2,2}}} \\
&\cdot \frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}} \left(\prod_{i=k+1}^{n-2} \frac{(q\hbar^{n-i-1})_{d'_{i,2}-d'_{i,1}}}{(\hbar^{n-i})_{d'_{i,2}-d'_{i,1}}} \frac{(q\hbar^{i+1-n})_{d'_{i,1}-d'_{i,2}}}{(\hbar^{i+2-n})_{d'_{i,1}-d'_{i,2}}}\right) \\
&\cdot \left(\prod_{i=1}^{k} z_i^{d_{1,1}+d'_{i,1}}\right) \left(\prod_{i=k+1}^{n-2} z_i^{2d_{1,1}+d'_{i,1}+d'_{i,2}}\right) z_{n-1}^{d_{1,1}+d'_{n-1,1}} z_n^{d_{1,1}+d'_{n,1}}.
\end{aligned}$$

We can further simplify $V(\boldsymbol{z})$ to the following expression:

$$\sum_{d_{1,1},\boldsymbol{d'}} \frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}} \frac{(\hbar)_{d'_{2,1}}}{(q)_{d'_{2,1}}} \left(\left(\prod_{i=1}^{k} z_i \right) \left(\prod_{i=k+1}^{n-2} z_i^2 \right) z_{n-1} z_n \right)^{d_{1,1}}$$
$$\cdot \left(\prod_{i=2}^{k-1} \frac{(\hbar)_{d'_{i+1,1}-d'_{i,1}}}{(q)_{d'_{i+1,1}-d'_{i,1}}} \right) \frac{(\hbar)_{d'_{k+1,1}-d'_{k,1}}}{(q)_{d'_{k+1,1}-d'_{k,1}}} \frac{(\hbar^{n-k-1})_{d'_{k+1,2}-d'_{k,1}}}{(q\hbar^{n-k-2})_{d'_{k+1,2}-d'_{k,1}}}$$
$$\prod_{i=k+1}^{n-3} \frac{(\hbar)_{d'_{i+1,1}-d'_{i,1}}}{(q)_{d'_{i+1,1}-d'_{i,1}}} \frac{(\hbar^{n-i-1})_{d'_{i+1,2}-d'_{i,1}}}{(q\hbar^{n-i-2})_{d'_{i+1,2}-d'_{i,1}}} \frac{(\hbar^{2+i-n})_{d'_{i+1,1}-d'_{i,2}}}{(q\hbar^{1+i-n})_{d'_{i+1,1}-d'_{i,2}}} \frac{(1)_{d'_{i+1,2}-d'_{i,2}}}{(q\hbar^{-1})_{d'_{i+1,2}-d'_{i,2}}}$$
$$\cdot \frac{(\hbar)_{d'_{n-1,1}-d'_{n-2,1}}}{(q)_{d'_{n-1,1}-d'_{n-2,1}}} \frac{(1)_{d'_{n-1,1}-d'_{n-2,2}}}{(q\hbar^{-1})_{d'_{n-1,1}-d'_{n-2,2}}} \frac{(\hbar)_{d'_{n,1}-d'_{n-2,1}}}{(q)_{d'_{n,1}-d'_{n-2,1}}} \frac{(1)_{d'_{n,1}-d'_{n-2,2}}}{(q\hbar^{-1})_{d'_{n,1}-d'_{n-2,2}}}$$
$$\cdot \left(\prod_{i=k+1}^{n-2} \frac{(q\hbar^{n-i-1})_{d'_{i,2}-d'_{i,1}}}{(\hbar^{n-i})_{d'_{i,2}-d'_{i,1}}} \frac{(q\hbar^{i+1-n})_{d'_{i,1}-d'_{i,2}}}{(\hbar^{i+2-n})_{d'_{i,1}-d'_{i,2}}} \right)$$
$$\cdot \left(\prod_{i=2}^{k} z_i^{d'_{i,1}} \right) \left(\prod_{i=k+1}^{n-2} z_i^{d'_{i,1}+d'_{i,2}} \right) z_{n-1}^{d'_{n-1,1}} z_n^{d'_{n,1}},$$

which can be factored as

$$\sum_{d_{1,1}} \frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}} \left(\left(\prod_{i=1}^{k} z_i \right) \left(\prod_{i=k+1}^{n-2} z_i^2 \right) z_{n-1} z_n \right)^{d_{1,1}}$$
$$\cdot \sum_{\boldsymbol{d'}} \frac{(\hbar)_{d'_{2,1}}}{(q)_{d'_{2,1}}} \left(\prod_{i=2}^{k-1} \frac{(\hbar)_{d'_{i+1,1}-d'_{i,1}}}{(q)_{d'_{i+1,1}-d'_{i,1}}} \right) \frac{(\hbar)_{d'_{k+1,1}-d'_{k,1}}}{(q)_{d'_{k+1,1}-d'_{k,1}}} \frac{(\hbar^{n-k-1})_{d'_{k+1,2}-d'_{k,1}}}{(q\hbar^{n-k-2})_{d'_{k+1,2}-d'_{k,1}}}$$
$$\prod_{i=k+1}^{n-3} \frac{(\hbar)_{d'_{i+1,1}-d'_{i,1}}}{(q)_{d'_{i+1,1}-d'_{i,1}}} \frac{(\hbar^{n-i-1})_{d'_{i+1,2}-d'_{i,1}}}{(q\hbar^{n-i-2})_{d'_{i+1,2}-d'_{i,1}}} \frac{(\hbar^{2+i-n})_{d'_{i+1,1}-d'_{i,2}}}{(q\hbar^{1+i-n})_{d'_{i+1,1}-d'_{i,2}}} \frac{(1)_{d'_{i+1,2}-d'_{i,2}}}{(q\hbar^{-1})_{d'_{i+1,2}-d'_{i,2}}}$$
$$\cdot \frac{(\hbar)_{d'_{n-1,1}-d'_{n-2,1}}}{(q)_{d'_{n-1,1}-d'_{n-2,1}}} \frac{(1)_{d'_{n-1,1}-d'_{n-2,2}}}{(q\hbar^{-1})_{d'_{n-1,1}-d'_{n-2,2}}} \frac{(\hbar)_{d'_{n,1}-d'_{n-2,1}}}{(q)_{d'_{n,1}-d'_{n-2,1}}} \frac{(1)_{d'_{n,1}-d'_{n-2,2}}}{(q\hbar^{-1})_{d'_{n,1}-d'_{n-2,2}}}$$
$$\cdot \left(\prod_{i=k+1}^{n-2} \frac{(q\hbar^{n-i-1})_{d'_{i,2}-d'_{i,1}}}{(\hbar^{n-i})_{d'_{i,2}-d'_{i,1}}} \frac{(q\hbar^{i+1-n})_{d'_{i,1}-d'_{i,2}}}{(\hbar^{i+2-n})_{d'_{i,1}-d'_{i,2}}} \right)$$
$$\cdot \left(\prod_{i=2}^{k} z_i^{d'_{i,1}} \right) \left(\prod_{i=k+1}^{n-2} z_i^{d'_{i,1}+d'_{i,2}} \right) z_{n-1}^{d'_{n-1,1}} z_n^{d'_{n,1}}$$
$$= F\left(\left(\prod_{i=1}^{k} z_i \right) \left(\prod_{i=k+1}^{n-2} z_i^2 \right) z_{n-1} z_n \right) V(z_2, z_3, \ldots, z_n),$$

which implies

$$V(\boldsymbol{z}) = F(x_1 x_{k+1}) V(z_2, z_3, \ldots, z_n)$$

$$= F(x_1x_{k+1})F(x_2x_{k+1})\dots F(x_{k-1}x_{k+1})V(z_k,\dots,z_n)$$
$$= \left(\prod_{i<k} F(x_ix_{k+1})\right) V(z_k,\dots,z_n).$$

We note that Lemma 8.5 implies that

$$V(z_k,\dots,z_n) = \left(\prod_{i=2}^{n-k} F(x_{k+1}x_{k+i})\right) F(x_{k+1}x_k) \left(\prod_{i>1} F(x_{k+1}x_{k+i}^{-1})\right)$$
$$= F(x_{k+1}x_k) \prod_{j>k+1} F(x_{k+1}x_j)F(x_{k+1}x_j^{-1})$$

after the variable shift $(x_1,\dots,x_{n-k+1}) \mapsto (x_k,\dots,x_n)$. This implies that

$$V(\boldsymbol{z}) = \left(\prod_{i<k} F(x_ix_{k+1})\right) F(x_{k+1}x_k) \left(\prod_{j>k+1} F(x_{k+1}x_j)F(x_{k+1}x_j^{-1})\right)$$
$$= \left(\prod_{i<k+1} F(x_ix_{k+1})\right) \left(\prod_{j>k+1} F(x_{k+1}x_j)F(x_{k+1}x_j^{-1})\right).$$

Note that $(\omega_1,\epsilon_1) = 1$, $(\omega_1,\epsilon_j) = 0$ for all $j \neq 1$. Now $\mu = \omega_1 - (\epsilon_1 + \epsilon_{k+1})$, if $\alpha = \epsilon_i + \epsilon_j$ then $(\mu,\alpha) = (\omega_1,\epsilon_i) + (\omega_1,\epsilon_j) - \delta_{1,i} - \delta_{1,j} - \delta_{k+1,i} - \delta_{k+1,j}$. If $i = 1$, then $-\delta_{k+1,j}$, so $\alpha = \epsilon_1 + \epsilon_{k+1}$. If $i > 1$, then $\alpha = -\delta_{k+1,i} - \delta_{k+1,j}$ so if $i = k+1$ or $j = k+1$ we have $(\mu,\alpha) < 0$, that is, $\alpha = \epsilon_{k+1} + \epsilon_j$ where $j > k+1$ or $\alpha = \epsilon_i + \epsilon_{k+1}$ for $1 < i < k+1$. In total, $\alpha = \epsilon_i + \epsilon_{k+1}$ for $i < k+1$ or $\alpha = \epsilon_{k+1} + \epsilon_j$ for $j > k+1$. If $\alpha = \epsilon_i - \epsilon_j$ then $(\mu,\alpha) = (\omega_1 - \epsilon_1 - \epsilon_{k+1}, \epsilon_i - \epsilon_j) = (\omega_1,\epsilon_i) - (\omega_1,\epsilon_j) - (\epsilon_1,\epsilon_i) + (\epsilon_1,\epsilon_j) - (\epsilon_{k+1},\epsilon_i) + (\epsilon_{k+1},\epsilon_j)$. Now if $i = 1$ then $(\mu,\alpha) = (\epsilon_{k+1},\epsilon_j) \geq 0$. So $i > 1$, and thus $j > i > 1$, then $(\mu,\alpha) = -(\epsilon_{k+1},\epsilon_i) + (\epsilon_{k+1},\epsilon_j)$. If $j = k+1$, then $(\mu,\alpha) \geq 0$. So $j \neq k+1$, and $i = k+1$ in order for $(\mu,\alpha) < 0$. That is, $(\mu,\alpha) < 0$ if and only if $\alpha = \epsilon_{k+1} - \epsilon_j$ where $j > k+1$. In sum, $(\mu,\alpha) < 0$ if and only if $\alpha = \epsilon_i + \epsilon_{k+1}$, $\alpha = \epsilon_{k+1} + \epsilon_j$, or $\alpha = \epsilon_{k+1} - \epsilon_j$. Note that $\mathsf{a} = (0^k, 1, 0^{n-k-3}, -1, -1)$. For $\alpha = \epsilon_i + \epsilon_{k+1}$, we have $\mathsf{a}_\alpha = 0 + 2 - 2 = 0$. For $\alpha = \epsilon_{k+1} + \epsilon_j$, we again have $\mathsf{a}_\alpha = 0 + 2 - 2 = 0$. For $\alpha = \epsilon_{k+1} - \epsilon_j$, we have $\mathsf{a}_\alpha = 1$. Recall the substitutions made in Section 8.3, that is, $z_{k+1} = (\hbar/q)x_{k+1}/x_{k+2}$, $z_i = x_i/x_{i+1}$ for $i \in [k+2, n-2]$, $z_{n-1} = (q/\hbar)x_{n-1}/x_n$, $z_n = (q/\hbar)x_{n-1}x_n$ after the variable shift. Hence, we recover

$$V(\boldsymbol{z}) = \prod_{\alpha\in\Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha)$$

as desired. □

We see that Lemma 8.1 implies Theorem 6.5, so it remains to show the two cases $\mathsf{w} = (2^{n-2}, 1, 1)$ and $\mathsf{w} = (1, 2^{n-3}, 1, 1)$.

8.2 Case of $\mathsf{v} = (2^{n-2}, 1^2)$

Our starting point is the explicit formula for the vertex function given by combining Theorem 5.6 and Definition 4.9, where the vertex function $V(\boldsymbol{z})$ is equal to the following expression:

$$\sum_{\boldsymbol{d}} \prod_{i=1}^{n-3} \frac{(\hbar)_{d_{i+1,1}-d_{i,1}}}{(q)_{d_{i+1,1}-d_{i,1}}} \frac{(\hbar^{n-i-1})_{d_{i+1,2}-d_{i,1}}}{(q\hbar^{n-i-2})_{d_{i+1,2}-d_{i,1}}} \frac{(\hbar^{2+i-n})_{d_{i+1,1}-d_{i,2}}}{(q\hbar^{1+i-n})_{d_{i+1,1}-d_{i,2}}} \frac{(1)_{d_{i+1,2}-d_{i,2}}}{(q\hbar^{-1})_{d_{i+1,2}-d_{i,2}}}$$
$$\cdot \frac{(\hbar)_{d_{n-1,1}-d_{n-2,1}}}{(q)_{d_{n-1,1}-d_{n-2,1}}} \frac{(1)_{d_{n-1,1}-d_{n-2,2}}}{(q\hbar^{-1})_{d_{n-1,1}-d_{n-2,2}}} \frac{(\hbar)_{d_{n,1}-d_{n-2,1}}}{(q)_{d_{n,1}-d_{n-2,1}}} \frac{(1)_{d_{n,1}-d_{n-2,2}}}{(q\hbar^{-1})_{d_{n,1}-d_{n-2,2}}}$$
$$\cdot \frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}} \frac{(\hbar^{n-1})_{d_{1,2}}}{(q\hbar^{n-2})_{d_{1,2}}} \left(\prod_{i=1}^{n-2} \frac{(q\hbar^{n-i-1})_{d_{i,2}-d_{i,1}}}{(\hbar^{n-i})_{d_{i,2}-d_{i,1}}} \frac{(q\hbar^{i+1-n})_{d_{i,1}-d_{i,2}}}{(\hbar^{i+2-n})_{d_{i,1}-d_{i,2}}} \right) \boldsymbol{z}^{\boldsymbol{d}}$$

where the multi-index $\boldsymbol{d}$ gives a reverse plane partition over the corresponding poset. Applying (10) and changing variables via $z_1 = (\hbar/q)x_1/x_2$, $z_i = x_i/x_{i+1}$ for $2 \le i \le n-2$, $z_{n-1} = (q/\hbar)x_{n-1}/x_n$, $z_n = (q/\hbar)x_{n-1}x_n$, we rewrite the above expression as

$$\sum_{\boldsymbol{d}} \left(\prod_{i=1}^{n-3} \frac{(\hbar)_{d_{i+1,1}-d_{i,1}}}{(q)_{d_{i+1,1}-d_{i,1}}} \frac{(\hbar^{n-i-1})_{d_{i+1,2}-d_{i,1}}}{(q\hbar^{n-i-2})_{d_{i+1,2}-d_{i,1}}} \frac{(\hbar^{n-i-1})_{d_{i,2}-d_{i+1,1}}}{(q\hbar^{n-i-2})_{d_{i,2}-d_{i+1,1}}} \frac{(\hbar)_{d_{i,2}-d_{i+1,2}}}{(q)_{d_{i,2}-d_{i+1,2}}} \right)$$
$$\cdot \frac{(\hbar)_{d_{n-1,1}-d_{n-2,1}}}{(q)_{d_{n-1,1}-d_{n-2,1}}} \frac{(\hbar)_{d_{n-2,2}-d_{n-1,1}}}{(q)_{d_{n-2,2}-d_{n-1,1}}} \frac{(\hbar)_{d_{n,1}-d_{n-2,1}}}{(q)_{d_{n,1}-d_{n-2,1}}} \frac{(\hbar)_{d_{n-2,2}-d_{n,1}}}{(q)_{d_{n-2,2}-d_{n,1}}}$$
$$\cdot \frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}} \frac{(\hbar^{n-1})_{d_{1,2}}}{(q\hbar^{n-2})_{d_{1,2}}} \left(\prod_{i=1}^{n-2} \frac{(q\hbar^{n-i-1})_{d_{i,2}-d_{i,1}}}{(\hbar^{n-i})_{d_{i,2}-d_{i,1}}} \frac{(q\hbar^{n-i-2})_{d_{i,2}-d_{i,1}}}{(\hbar^{n-i-1})_{d_{i,2}-d_{i,1}}} \right)$$
$$\cdot \left(\prod_{\substack{1 \le i \le n \\ i \ne n-1}} x_i^{d_i - d_{i-1}} \right) x_{n-1}^{d_{n-1}+d_n-d_{n-2}} \tag{17}$$

where $d_i := \sum_{j=1}^{\mathsf{v}_i} d_{i,j}$ for $1 \le i \le n$ and $d_0 := 0$. As in the proof of Theorem 6.2, we write this in terms of Macdonald polynomials.

Lemma 8.2. *The vertex function is equal to*

$$V(\boldsymbol{z}) = \sum \varphi_{\nu^{(0)}/(a^{n-2})} \left(\prod_{i=2}^{n} x_i \right)^{-a}$$
$$Q_{(a)}(x_1) P_{|\nu^{(0)}|-(n-2)a}(x_1) \prod_{i=0}^{n-2} P_{\nu^{(0)}/\nu^{(i+1)}}(x_{i+2}), \tag{18}$$

where the sum is taken over all n-tuples of partitions $(\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(n-1)})$ and all $a \in \mathbb{N}$ such that

$$\emptyset = \nu^{(n-1)} \prec \nu^{(n-2)} \prec \nu^{(n-1)} \prec \dots \prec \nu^{(0)} \succ (a^{n-2}).$$

Proof. Let $\nu^{(i)}$ for $0 \le i \le n-1$ and $a \in \mathbb{N}$ be as in the statement of the lemma. We write them explicitly as

- $\nu^{(i)} = (\nu_1^{(i)}, a^{n-3-i}, \nu_2^{(i)})$ for $i \in [0, n-3]$,
- $\nu^{(n-2)} = (\nu_1^{n-2})$.

The assignment

- $d_{1,1} = \nu_2^{(0)}$, $d_{1,2} = \nu_1^{(0)}$,
- $d_{i,1} = \nu_1^{(0)} + \nu_2^{(0)} - \nu_1^{(i-1)}$, $d_{i,2} = \nu_1^{(0)} + \nu_2^{(0)} - \nu_2^{(i-1)}$ for $i \in [2, n-2]$,
- $d_{n-1,1} = \nu_1^{(0)} + \nu_2^{(0)} - \nu_1^{(n-2)}$,
- $d_{n,1} = \nu_1^{(0)} + \nu_2^{(0)} - a$

defines a bijection between the set of $\big((\nu^{(i)})_{i=0,\dots,n-1}, a\big)$ satisfying the interlacing conditions and reverse plane partitions $\boldsymbol{d}$ over the poset corresponding to v. It is now straightforward using Equations (6) and (5) to check that the corresponding terms of Equations (17) and (18) coincide. □

We can now sum the right hand side of Equation (18).

Lemma 8.3. *The vertex function is equal to*

$$V(\boldsymbol{z}) = \prod_{i=2}^{n} F(x_1 x_i) F(x_1 x_i^{-1}).$$

Proof. We first apply the branching rule (9) to get

$$\sum \varphi_{\nu^{(0)}/(a^{n-2})} \left(\prod_{i=2}^{n} x_i\right)^{-a} Q_{(a)}(x_1) P_{|\nu^{(0)}|-(n-2)a}(x_1) \prod_{i=0}^{n-2} P_{\nu^{(0)}/\nu^{(i+1)}}(x_{i+2})$$
$$= \sum \varphi_{\nu^{(0)}/(a^{n-2})} \left(\prod_{i=2}^{n} x_i\right)^{-a} Q_{(a)}(x_1) P_{|\nu^{(0)}|-(n-2)a}(x_1) P_{\nu^{(0)}}(x_2, \dots, x_n)$$

where the first sum is over the set described in Lemma 8.2 and second sum is over the set of $a \ge 0$ and $\nu^{(0)}$ such that $\nu^{(0)} \succ (a^{n-1})$. Applying the Pieri rule (7) gives

$$\sum_{\substack{a \ge 0 \\ \nu^{(0)} \succ (a^{n-2})}} \varphi_{\nu^{(0)}/(a^{n-2})} \left(\prod_{i=2}^{n} x_i\right)^{-a} Q_{(a)}(x_1) P_{|\nu^{(0)}|-(n-2)a}(x_1) P_{\nu^{(0)}}(x_2, \dots, x_n)$$
$$= \sum_{a,b \ge 0} \left(\prod_{i=2}^{n} x_i\right)^{-a} Q_{(a)}(x_1) P_{(b)}(x_1) P_{(a^{n-2})}(x_2, \dots, x_n) Q_{(b)}(x_2, \dots, x_n).$$

Applying the inversion identity (8) gives

$$\sum_{a,b\geq 0}\left(\prod_{i=2}^{n}x_i\right)^{-a}Q_{(a)}(x_1)P_{(a^{n-2})}(x_2,\dots,x_n)P_{(b)}(x_1)Q_{(b)}(x_2,\dots,x_n)$$
$$=\sum_{a,b\geq 0}Q_{(a)}(x_1)P_{(a)}(x_2^{-1},\dots,x_n^{-1})P_{(b)}(x_1)Q_{(b)}(x_2,\dots,x_n).$$

Now the Cauchy identity finishes the proof. □

To finish the proof of Theorem 6.5 in this case, it remains to identify terms in the product with certain roots. Recall that the roots are $\epsilon_i \pm \epsilon_j$ for $i \neq j$. These are expressed in terms of positive roots as follows:

- $\alpha = \epsilon_i + \epsilon_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ for all $i < j \in [1, n-1]$,
- $\alpha = \epsilon_i + \epsilon_n = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n$ for all $i \in [1, n-1]$, or
- $\alpha = \epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for all $i \neq j \in [1, n]$,

Since $(\omega_i, \alpha_j) = \delta_{ij}$, by inspecting the three different types of roots, we see that $(-\omega_1, \alpha) < 0$ for roots of the form $\epsilon_1 \pm \epsilon_i$, and $(-\omega_1, \alpha) = 0$ otherwise. Thus

$$V(\boldsymbol{z}) = \prod_{i=2}^{n} F(x_1 x_i)F(x_1 x_i^{-1}) = \prod_{\alpha\in\Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha),$$

as desired.

8.3 Case of $\mathsf{v} = (1, 2^{n-3}, 1^2)$

Our starting point is the explicit formula for the vertex function given by combining Theorem 5.6 and Definition 4.9:

$$\begin{aligned}
V(\boldsymbol{z}) = \sum_{\boldsymbol{d}} &\left(\prod_{i=1}^{n-3}\frac{(\hbar)_{d_{i+1,1}-d_{i,1}}}{(q)_{d_{i+1,1}-d_{i,1}}}\frac{(\hbar^{n-i-1})_{d_{i+1,2}-d_{i,1}}}{(q\hbar^{n-i-2})_{d_{i+1,2}-d_{i,1}}}\right)\\
&\cdot\left(\prod_{i=2}^{n-3}\frac{(\hbar^{2+i-n})_{d_{i+1,1}-d_{i,2}}}{(q\hbar^{1+i-n})_{d_{i+1,1}-d_{i,2}}}\frac{(1)_{d_{i+1,2}-d_{i,2}}}{(q\hbar^{-1})_{d_{i+1,2}-d_{i,2}}}\right)\frac{(\hbar)_{d_{n-1,1}-d_{n-2,1}}}{(q)_{d_{n-1,1}-d_{n-2,1}}}\\
&\cdot\frac{(1)_{d_{n-1,1}-d_{n-2,2}}}{(q\hbar^{-1})_{d_{n-1,1}-d_{n-2,2}}}\frac{(\hbar)_{d_{n,1}-d_{n-2,1}}}{(q)_{d_{n,1}-d_{n-2,1}}}\frac{(1)_{d_{n,1}-d_{n-2,2}}}{(q\hbar^{-1})_{d_{n,1}-d_{n-2,2}}}\frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}}\\
&\cdot\left(\prod_{i=2}^{n-2}\frac{(q\hbar^{n-i-1})_{d_{i,2}-d_{i,1}}}{(\hbar^{n-i})_{d_{i,2}-d_{i,1}}}\frac{(q\hbar^{i+1-n})_{d_{i,1}-d_{i,2}}}{(\hbar^{i+2-n})_{d_{i,1}-d_{i,2}}}\right)\boldsymbol{z}^{\boldsymbol{d}}
\end{aligned}$$

where the multi-index $\boldsymbol{d}$ gives a reverse plane partition over the corresponding poset. Applying (10) and changing variables via $z_2 = (\hbar/q)x_2/x_3$, $z_i = x_i/x_{i+1}$ for $1 \le i \le n-2$ and $i \neq 2$, $z_{n-1} = (q/\hbar)x_{n-1}/x_n$, and $z_n = (q/\hbar)x_{n-1}x_n$ we can rewrite this as

$$\begin{aligned} V(\boldsymbol{z}) = \sum_{\boldsymbol{d}} & \left(\prod_{i=1}^{n-3} \frac{(\hbar)_{d_{i+1,1}-d_{i,1}}}{(q)_{d_{i+1,1}-d_{i,1}}} \frac{(\hbar^{n-i-1})_{d_{i+1,2}-d_{i,1}}}{(q\hbar^{n-i-2})_{d_{i+1,2}-d_{i,1}}} \right) \\ & \cdot \left(\prod_{i=2}^{n-3} \frac{(\hbar^{n-i-1})_{d_{i,2}-d_{i+1,1}}}{(q\hbar^{n-i-2})_{d_{i,2}-d_{i+1,1}}} \frac{(\hbar)_{d_{i,2}-d_{i+1,2}}}{(q)_{d_{i,2}-d_{i+1,2}}} \right) \frac{(\hbar)_{d_{n-1,1}-d_{n-2,1}}}{(q)_{d_{n-1,1}-d_{n-2,1}}} \\ & \cdot \frac{(\hbar)_{d_{n-2,2}-d_{n-1,1}}}{(q)_{d_{n-2,2}-d_{n-1,1}}} \frac{(\hbar)_{d_{n,1}-d_{n-2,1}}}{(q)_{d_{n,1}-d_{n-2,1}}} \frac{(\hbar)_{d_{n-2,2}-d_{n,1}}}{(q)_{d_{n-2,2}-d_{n,1}}} \frac{(\hbar)_{d_{1,1}}}{(q)_{d_{1,1}}} \\ & \cdot \left(\prod_{i=2}^{n-2} \frac{(q\hbar^{n-i-1})_{d_{i,2}-d_{i,1}}}{(\hbar^{n-i})_{d_{i,2}-d_{i,1}}} \frac{(q\hbar^{n-i-2})_{d_{i,2}-d_{i,1}}}{(\hbar^{n-i-1})_{d_{i,2}-d_{i,1}}} \right) \\ & \cdot \left(\prod_{\substack{1\le i\le n \\ i\neq n-1}} x_i^{d_i-d_{i-1}} \right) x_{n-1}^{d_{n-1}+d_n-d_{n-2}} \end{aligned} \tag{19}$$

where $d_i = \sum_{j=1}^{\mathsf{v}_i} d_{i,j}$ and $d_0 = 0$. As before, we write this in terms of Macdonald polynomials.

Lemma 8.4. *The vertex function is equal to*

$$V(\boldsymbol{z}) = \sum \varphi_{\nu^{(0)}/(a^{n-3})} \left(\prod_{i=3}^{n} x_i \right)^{-a} Q_{(a)}(x_2) P_b(x_2) Q_{(b)/(|\nu^{(0)}|-(n-3)a)}(x_1) \prod_{i=0}^{n-3} P_{\nu^{(0)}/\nu^{(i+1)}}(x_{i+3}), \tag{20}$$

where the sum is taken over the set of all $(n-1)$-tuples of partitions $(\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(n-2)})$ and all $a, b \in \mathbb{N}$ such that

$$\emptyset = \nu^{(n-2)} \prec \nu^{(n-1)} \prec \ldots \prec \nu^{(0)} \succ (a^{n-3}).$$

Proof. Let a, b, and $\nu^{(i)}$ be as in the statement of the lemma. We write them explicitly as

- $\nu^{(0)} = (\nu_1^{(0)}, a^{n-4}, \nu_2^{(0)})$,
- $\nu^{(i)} = (\nu_1^{(i)}, a^{n-4-i}, \nu_2^{(i)})$ for $i \in [1, n-4]$,
- $\nu^{(n-3)} = (\nu_1^{(n-3)})$.

The assignment

- $d_{1,1} = a + b - \nu_1^{(0)} - \nu_2^{(0)}$,
- $d_{2,1} = a + b - \nu_1^{(0)}$, $d_{2,2} = a + b - \nu_2^{(0)}$,
- $d_{i,1} = a + b - \nu_1^{(i-2)}$, $d_{i,2} = a + b - \nu_2^{(i-2)}$ for $i \in [3, n-2]$,
- $d_{n-1,1} = a + b - \nu_1^{(n-3)}$,
- $d_{n,1} = a$

defines a bijection between the set of $\big((\nu^{(i)})_{i=0,\ldots,n-2}, a, b\big)$ satisfying the interlacing conditions and reverse plane partitions $\boldsymbol{d}$ over the poset corresponding to v. It is now straightforward using Equations (6) and (5) to check that the corresponding terms of Equations (19) and (20) coincide. □

Lemma 8.5. *The vertex function is equal to*

$$V(\boldsymbol{z}) = \left(\prod_{i=3}^{n} F(x_2 x_i)\right)\left(\prod_{i\neq 2} F(x_2 x_i^{-1})\right).$$

Proof. We first apply the branching rule (9) to see that the right-hand side of Equation (8.4) is equal to

$$\sum_{\substack{a,b\geq 0\\ \nu^{(0)}\succ(a^{n-3})}} \varphi_{\nu^{(0)}/(a^{n-3})}\left(\prod_{i=3}^{n} x_i\right)^{-a} Q_{(a)}(x_2)P_b(x_2)$$
$$Q_{(b)/(|\nu^{(0)}|-(n-3)a)}(x_1)\cdot P_{\nu^{(0)}}(x_3,\ldots,x_n).$$

By the Pieri rule (7), this is equal to

$$\sum_{a,b,c\geq 0}\left(\prod_{i=3}^{n} x_i\right)^{-a} Q_{(a)}(x_2)P_{(a^{n-3})}(x_3,\ldots,x_n)$$
$$P_{(b)}(x_2)Q_{(b)/(c)}(x_1)Q_{(c)}(x_3,\ldots,x_n).$$

Applying the branching rule, the previous line becomes

$$\sum_{a,b\geq 0}\left(\prod_{i=3}^{n} x_i\right)^{-a} Q_{(a)}(x_2)P_{(a^{n-3})}(x_3,\ldots,x_n)P_{(b)}(x_2)Q_{(b)}(x_1,x_3,\ldots,x_n).$$

Applying the inversion identity (8), this is equal to

$$\sum_{a,b\geq 0} Q_{(a)}(x_2)P_{(a)}(x_3^{-1},\ldots,x_n^{-1})P_{(b)}(x_2)Q_{(b)}(x_1,x_3,\ldots,x_n).$$

Now the Cauchy identity finishes the proof. □

To finish the proof of Theorem 6.5 in this case, it remains to identify terms in the product with certain roots. Recall that the roots are $\epsilon_i \pm \epsilon_j$ for $i \neq j$. These are expressed in terms of positive roots as follows:

- $\alpha = \epsilon_i + \epsilon_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ for all $i < j \in [1, n-1]$,
- $\alpha = \epsilon_i + \epsilon_n = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n$ for all $i \in [1, n-1]$, or
- $\alpha = \epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for all $i \neq j \in [1, n]$,

Since $(\omega_i, \alpha_j) = \delta_{ij}$, by inspecting the three different types of roots, we see that $(\omega_1 - \omega_2, \alpha) < 0$ for roots of the form $\epsilon_2 + \epsilon_i$ for $i > 2$, and $\epsilon_2 - \epsilon_i$ for $i \neq 2$; otherwise $(\omega_1 - \omega_2, \alpha) = 0$. Thus

$$V(\boldsymbol{z}) = \left(\prod_{i=3}^{n} F(x_2 x_i)\right)\left(\prod_{i \neq 2} F(x_2 x_i^{-1})\right) = \prod_{\alpha \in \Phi'} F((q/\hbar)^{\mathsf{a}_\alpha} z_\alpha),$$

as desired.

References

[AFO18] Mina Aganagic, Edward Frenkel, and Andrei Okounkov, *Quantum q-Langlands correspondence*, Trans. Moscow Math. Soc. **79** (2018), 1–83.

[AO21] Mina Aganagic and Andrei Okounkov, *Elliptic stable envelopes*, J. Amer. Math. Soc. **34** (2021), no. 1, 79–133.

[BBC20] Guillaume Barraquand, Alexei Borodin, and Ivan Corwin, *Half-space Macdonald processes*, Forum Math. Pi **8** (2020), e11.

[BBCW18] Guillaume Barraquand, Alexei Borodin, Ivan Corwin, and Michael Wheeler, *Stochastic six-vertex model in a half-quadrant and half-line open asymmetric simple exclusion process*, Duke Math. J. **167** (2018), no. 13, 2457–2529.

[BC14] Alexei Borodin and Ivan Corwin, *Macdonald processes*, Probab. Theory Related Fields **158** (2014), no. 1, 225–400.

[BLPW14] Tom Braden, Anthony M. Licata, N. Proudfoot, and Ben Webster, *Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality*, Astérisque (2014), 75–179.

[CFKM14] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik, *Stable quasimaps to GIT quotients*, J. Geom. Phys. **75** (2014), 17–47.

[Cor12] Ivan Corwin, *The Kardar–Parisi–Zhang equation and universality class*, Random Matrices Theory Appl. **1** (2012), no. 1, 1130001.

[DEKMF24] Anne Dranowski, Balázs Elek, Joel Kamnitzer, and Calder Morton-Ferguson, *Heaps, crystals, and preprojective algebra modules*, Selecta Math. **30** (2024), 94.

[Din20] Hunter Dinkins, *3d mirror symmetry of the cotangent bundle of the full flag variety*, Lett. Math. Phys. **112** (2020), 100.

[Din22a] ———, *Exotic quantum difference equations and integral solutions*, Ph.D. thesis, University of North Carolina Chapel Hill, 2022.

[Din22b] ———, *Symplectic duality for* $T^*Gr(k,n)$, Math. Res. Lett. **29** (2022), no. 3, 663–690.

[DKK] Hunter Dinkins, Ivan Karpov, and Vasily Krylov, *The quantum Hikita conjecture via quasimaps*, In preparation.

[DS20a] Hunter Dinkins and Andrey Smirnov, *Characters of tangent spaces at torus fixed points and 3d-mirror symmetry*, Lett. Math. Phys. **110** (2020), no. 9, 2337–2352.

[DS20b] ———, *Quasimaps to zero-dimensional* A_∞*-quiver varieties*, Int. Math. Res. Not. **2022** (2020), no. 2, 1123–1153.

[DS22] ———, *Capped vertex with descendants for zero dimensional* A_∞ *quiver varieties*, Adv. Math. **401** (2022), 108324.

[Gas95] George Gasper, *Lecture notes for an introductory minicourse on q-series*, 1995.

[Gin09] Victor Ginzburg, *Lectures on Nakajima's quiver varieties*, 2009.

[GR04] George Gasper and Mizan Rahman, *Basic hypergeometric series*, 2 ed., Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2004.

[IS96] K. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories*, Phys. Lett. B **387** (1996), no. 3, 513–519.

[Kam22] Joel Kamnitzer, *Symplectic resolutions, symplectic duality, and Coulomb branches*, Bull. London Math. Soc. **54** (2022), 1515–1551.

[KP21] Vasily Krylov and Ivan Perunov, *Almost dominant generalized slices and convolution diagrams over them*, Adv. Math. **392** (2021), 108034.

[KPZ86] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986), no. 9, 889–892.

[KS22] Yakov Kononov and Andrey Smirnov, *Pursuing quantum difference equations II: 3D mirror symmetry*, Int. Math. Res. Not. **2023** (2022), no. 15, 13290–13331.

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2 ed., Oxford University Press, 1995.

[Nak94] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[Nak98] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

[Nak01] ———, *Quiver varieties and tensor products*, Invent. Math. **146** (2001), no. 2, 399–449.

[Nak04] ———, *Quiver varieties and t-analogs of q-characters of quantum affine algebras*, Ann. of Math. **160** (2004), no. 3, 1057–1097.

[NS21] Masatoshi Noumi and Ayako Sano, *An infinite family of higher-order difference operators that commute with Ruijsenaars operators of type A*, Lett. Math. Phys. **111** (2021), 91.

[Oko17] Andrei Okounkov, *Lectures on K-theoretic computations in enumerative geometry*, Geometry of Moduli Spaces and Representation Theory, IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., 2017, pp. 251–380.

[OSZ13] Neil O'Connell, Timo Seppäläinen, and Nikos Zygouras, *Geometric RSK correspondence, Whittaker functions and symmetrized random polymers*, Invent. Math. **197** (2013), no. 2, 361–416.

[Pro86] Robert A. Proctor, *A Dynkin diagram classification theorem arising from a combinatorial problem*, Adv. Math. **62** (1986), no. 2, 103–117.

[Pro99] ———, *Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets*, J. Algebraic Combin. **9** (1999), 61–94.

[PSZ20] Petr P. Pushkar, Andrey V. Smirnov, and Anton M. Zeitlin, *Baxter Q-operator from quantum K-theory*, Adv. Math. **360** (2020), 106919.

[RSVZ21] Richárd Rimányi, Andrey Smirnov, Alexander Varchenko, and Zijun Zhou, *Three-dimensional mirror symmetry and elliptic stable envelopes*, Int. Math. Res. Not. **2022** (2021), no. 13, 10016–10094.

[Ste94] John R. Stembridge, *On minuscule representations, plane partitions and involutions in complex Lie groups*, Duke Math. J. **73** (1994), no. 2, 469–490.

[SZ22] Andrey Smirnov and Zijun Zhou, *3d mirror symmetry and quantum K-theory of hypertoric varieties*, Adv. Math. **395** (2022), 108081.