Matching of frames of open Jacobi diagrams and chord diagrams

2023 Yulia's Dream Program Final Paper
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Abstract

This paper is dedicated to the study of similarities between two objects that arise from the theory of Vassiliev invariants: open Jacobi diagrams and chord diagrams, which are uni-trivalent graphs with "orientation" structure and trivalent graphs with the structure of the chosen Hamilton cycle, respectively. We define a space of open Jacobi diagrams as the span of Jacobi diagrams modulo IHX and AS relations and a space of chord diagrams as a span of chord diagrams modulo 4T relations. We define two "frame" maps as operations from sets of chord and Jacobi diagrams to trivalent graphs. These operations act by forgetting about the structures in Jacobi and chord diagrams, making them trivalent graphs. Using these operations, we find the correspondence between the frames of elements of bases of spaces of Jacobi diagrams and chord diagrams, and formulate the "Frame matching" conjecture. As an intermediate step for proving the "Frame matching" conjecture, we prove a useful lemma, which states that any open Jacobi diagrams can be presented as a linear combination of special open Jacobi diagrams, which we called chord type Jacobi diagram. Finally, we wrote Python code to check the "Frame matching" conjecture; we provide an algorithm of how to use this code for checking the "Frame matching" conjecture.

1 Introduction

The two objects that we work with in this paper, Jacobi diagrams and chord diagrams, arise from the study of Vassiliev invariants. The Vassiliev invariants, also called the finite type invariants, was defined by V. Vassiliev around 1989 in [4] when V. Vassiliev worked on complements of discriminants in spaces of maps [3, p.71]. It was discovered, that chord diagrams, which are trivalent graphs with the structure of the chosen Hamilton cycle, can be used to code certain information about singular knots. Specifically, the chord diagram $\sigma(K) \in A_n$ of a singular knot with n double points is obtained by marking on the parametrizing circle n pairs of points whose images are the n double points of the knot [3, p.80].

The study of Vassiliev knot invariants, at least complex-valued, is largely reduced to the study of the algebra of chord Diagrams [3, p. 127]. So, Bar-Natan introduced a new type of diagrams representing elements of this algebra called chinese character diagrams [2, p.8], another name for which is Jacobi diagrams, which first appeared in [3]. It is derived from Jacobi identity because of the fact that chinese character diagrams are naturally labeled by a Lie algebra equipped with a Lie algebra representation. This terminology, which we will use in this paper, allows for a new definition: open Jacobi diagrams, which are uni-trivalent graphs with an "orientation" structure. In this paper, we concentrate on the space of open Jacobi diagrams, which is a span of Jacobi diagrams modulo IHX and AS relations [3, p.142] and a space of chord diagrams as a span of chord diagrams modulo 4T relation.

In order to study the similarities between chord diagrams and open Jacobi diagrams, we defined two frame maps, which are maps from sets of chord and Jacobi diagrams to trivalent graphs. Chord diagrams and open Jacobi diagrams have different structures, so to compare them, we define frame maps as maps that work by forgetting selected structures in open Jacobi diagrams and chord diagrams: uni-trivalent vertices with an "orientation" structure for the former and the structure of the chosen Hamilton cycle for the later. We also provide a detailed explanation of why a single frame map that acts from the set of open Jacobi diagrams to chord diagrams would be poorly defined by considering an open Jacobi diagram containing two different Hamiltonian paths (Fig. 13).

Our colleague, Vasily Dolgushev, a mathematician from Temple University, suggested that there exists a correspondence between the frames of elements of bases of spaces of Jacobi diagrams and chord diagrams and formulated the “Frame matching” conjecture.

Conjecture 1. *$\forall\, m, n\, \exists\, B =< c_1, c_2, c_3, ... >$, a basis of $A = \bigoplus_i, A_i, c_i \in CD$ such that*

$$\forall\, j \in JD_{m,n},\ j = \sum_k t_k j_k\ : j_k \in JD_{m,n}\ ,\ F_{Jac}(j_k) \in\ \{F_{CD}(c_1), F_{CD}(c_2), F_{CD}(c_3), ...\}$$

To check this conjecture, all open Jacobi diagrams in space $Jac_(j,k)$ should be rewritten as a linear combination of open Jacobi diagrams that have at least one Hamiltonian cycle through all of its trivalent vertices, as demonstrated on Fig. 20. For convenience, we call such open Jacobi diagrams, chord type Jacobi diagrams.

The idea of rewriting a given Jacobi diagram as a linear combination of chord type Jacobi diagrams, motivated us to prove the “open Jacobi diagram frame decomposition” lemma, whose formulation was also proposed by Vasily Dolgushev. This lemma is proven by induction on the number of trivalent vertices that are not in the chosen cycle and properties of IHX relation, which allow us to rewrite a Jacobi diagram with n trivalent vertices, not in a chosen cycle as a sum of two Jacobi diagrams with $n-1$ trivalent vertices not in a chosen cycle.

As a part of our project we have also developed a code on Python. Details on the algorithms are presented in the Section 4.

2 Definitions

2.1 Chord diagrams

Definition 1 (Chord diagram)**.** A chord diagram is a graph with an external circle and chords that lie on that circle. This graph can be represented as a cyclic set that consists of m elements with 2 copies each, where m is the number of chords. To write the set, we label all chords with the numbers $1, 2, \ldots, m$, and then set a point on the external circle, and travel along the circle in any direction setting labels of chords into the cyclic set. Obviously, if we relabel chords, the corresponding cyclic representation is equivalent to the other cyclic representations with different chord labels.

Example 1. There is a chord diagram with 4 chords (Fig. 1). Moving along the counterclockwise direction, the diagram is written as a cyclic set with elements $[1, 2, 3, 2, 4, 1, 3, 4]$.

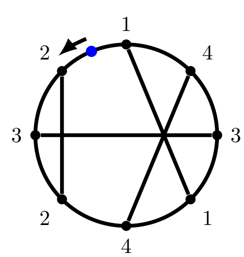

Figure 1: Example of a chord diagram.

Remark 1. A knot can be represented as a chord diagram. It can be done by labeling every intersection of a knot with the numbers $1, 2, \ldots, m$; and moving along the knot in an arbitrary direction, we append every intersection label to the cyclic set until we meet the starting point. However, there may be 2 non-isotopic knots that give us the same chord diagram.

Definition 2 (Set of all chord diagrams)**.** We write set of all chord diagrams with a fixed number of chords m as CD_m.

Definition 3 (Vector spaces of chord diagrams)**.** The space of chord diagrams is a vector space with a fixed number of chords spanned by CD_m and quotient by all 4T relations generated on CD_m:

$$A_m = span_{\mathbb{R}}(CD_m) \Big/ _{(all\ 4T\ relations)} \tag{1}$$

4T relation is given by the alteration of two arbitrary chords in a diagram (Fig. 2).

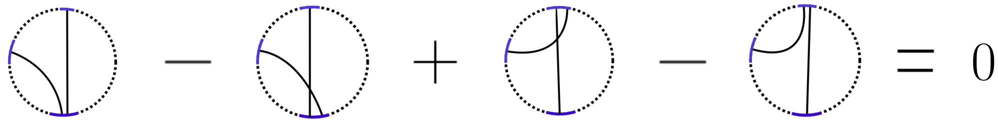

Figure 2: 4T relation.

The full space of chord diagrams:

$$A = \bigoplus_{k=1}^{\infty} A_k. \tag{2}$$

Example 2. 4T relationship gives us a linearly dependent set of vectors. A linear dependence starts from diagrams with 3 chords[1] (Fig. 3).

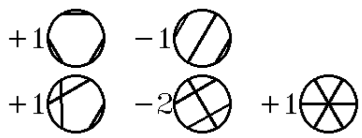

Figure 3: All distinct 4T relations that are generated on diagrams with 3 chords.

The more complicated cases of the linear dependence start to emerge in diagrams with 4 chords and more: take the relations from the following picture (Fig. 4). It appears that we have 3 equations, but due to the dependence between them, only 2 of them are linearly independent.

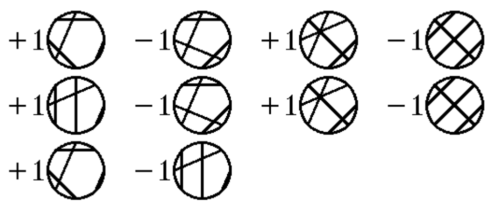

Figure 4: Non-trivial example of 4T relation's linear dependence (all 4T relations are not represented here).

Example 3. Examples of CD_m (Fig. 5) and basis of A_m (Fig. 6) for values of m in range from 2 to 4.

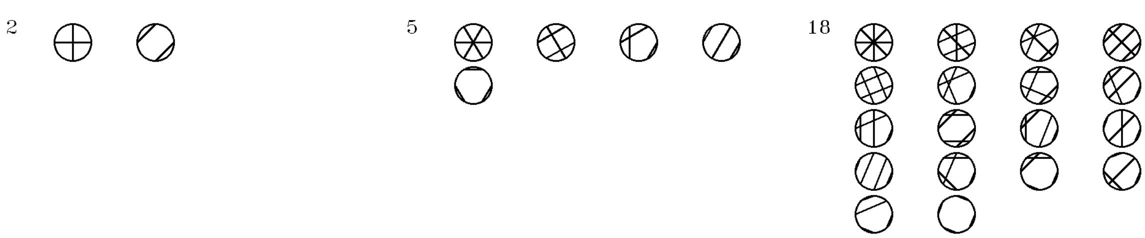

Figure 5: CD_2, CD_3, and CD_4.

[1]Code on chord diagram generation and its visualization: https://github.com/LoolzMe/ChordJacobiDiagrams

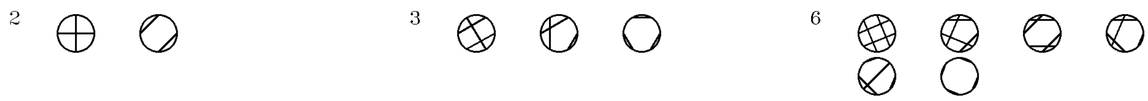

Figure 6: A_2, A_3, and A_4.

2.2 Jacobi diagrams

Definition 4. A closed Jacobi diagram is a connected trivalent graph with a distinguished embedded oriented cycle, called Wilson loop, and a fixed cyclic order of half-edges at each vertex not on the Wilson loop.

Definition 5. C is space of closed Jacobi diagrams factorised over STU relation where STU relation is:

Definition 6. An open Jacobi diagram is a graph with 1 and 3-valent vertices, cyclic order of half-edges at every 3-valent vertex and with at least one 1-valent vertex in every connected component.

Definition 7. JD is a set of open Jacobi diagrams.

Definition 8. $JD_{k,h}$ is a set of open Jacobi diagrams that have k trivalent and h univalent vertices.

Definition 9 (Vector space of open Jacobi diagrams with k trivalent and h univalent vertices)**.**

$$Jac_{k,h} = span_{\mathbb{R}}(JD_{k,h}) \Big/ (AS \text{ and } IHX \text{ relations}) \tag{3}$$

Where AS relation is the relation shown on Fig. 7, which describes the change of cyclic order on a chosen trivalent vertex.

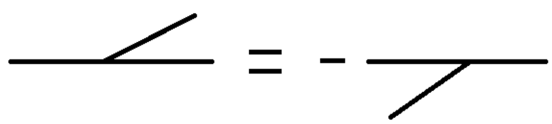

Figure 7: AS relation

IHX relation, shown on Fig. 8, describes how we can move one trivalent vertex through another trivalent vertex.

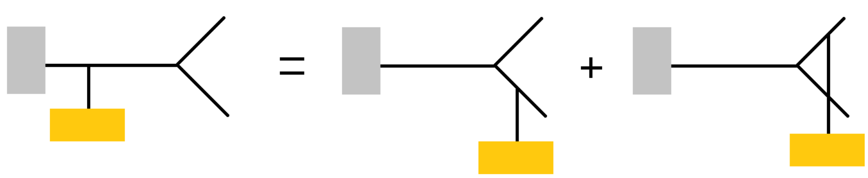

Figure 8: IHX relation

Definition 10 (Vector space of open Jacobi diagrams)**.**

$$Jac = \bigoplus_{k,h=0}^{\infty} Jac_{k,h}. \tag{4}$$

Theorem 1.

$$A \cong C \tag{5}$$

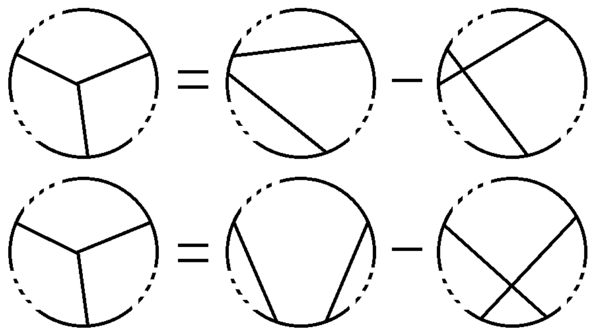

Figure 9: Different ways of reducing closed Jacobi diagrams

Proof. By STU relation we can reduce Jacobi diagram with k vertices with degree 3 to diagram with $k-1$ vertices with degree 3.We can do this operation until we have one vertice with degree 3. We can reduce it in 3 ways but they are equivalent because of 4T relation, as can be seen on Fig. 9.
Fig. 9 also shows that from STU we can get $4T$ in chord diagrams, so we prove this theorem. □

Theorem 2. *$Jac \cong C$, [3]*

2.3 Matching frames of CD and JD

Definition 11 (Frame of a chord diagram)**.** Frame of a Chord diagram $F_{CD} : CD \longrightarrow$ "trivalent graphs" is an operation that forgets a cycle on a Chord diagram and transforms it to a regular trivalent graph.

Example 4. On Fig. 10, frame of a chord diagram on the left is a corresponding simple trivalent graph on the right.

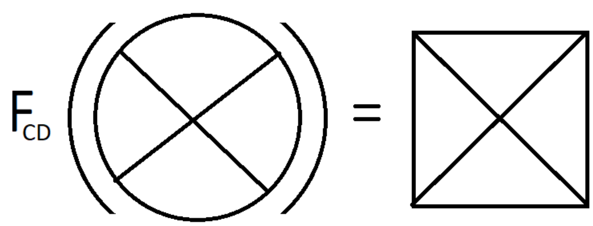

Figure 10

Definition 12 (Frame of a Jacobi diagram)**.** Frame of a Jacobi diagram $F_{Jac} : JD \longrightarrow$ "trivalent graphs" is an operation of deleting all univalent vertices, edges coming from them, and vertices connected to those univalent vertices from a given Jacobi diagram

Example 5. As can be seen on Fig. 11, F_{Jac} can be applied separately to each individual univalent vertex.

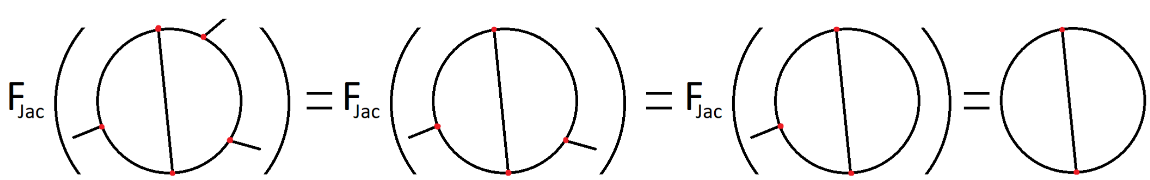

Figure 11

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Remark 2 (F_{Jac} of a wheel). There is a case we need to consider separately: F_{Jac} of any wheel is a circle, as can be seen on Fig. 12.

To be consistent with the definition $F_{Jac} : JD \longrightarrow$ "trivalent graphs" , F_{Jac} of a wheel, which is a circle, is considered to be a trivalent graph.

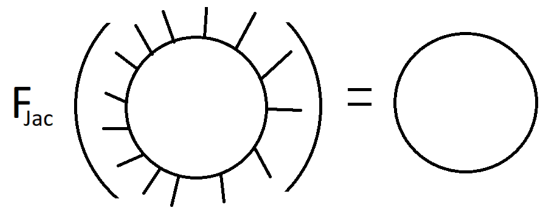

Figure 12: F_{Jac} of a wheel

Remark 3 (Important observation about frames). Important observation: We couldn't define frame operation as $F : JD \longrightarrow CD$ because if a Jacobi diagram has 2 different Hamiltonian cycles, as, for instance, the one on Fig. 13 then its frame can be interpreted as 2 different Chord diagrams.

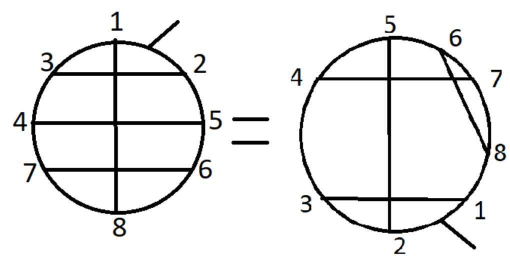

Figure 13: Jacobi diagrams with 2 different Hamiltonian paths

Two Jacobi diagrams above are equal by AS relation in the space of open Jacobi diagrams. However, if they were Chord diagrams, they wouldn't be equal in the space of chord diagrams by 4T relation, which is evident by the following expression on Fig. 14, which was obtained with our code.

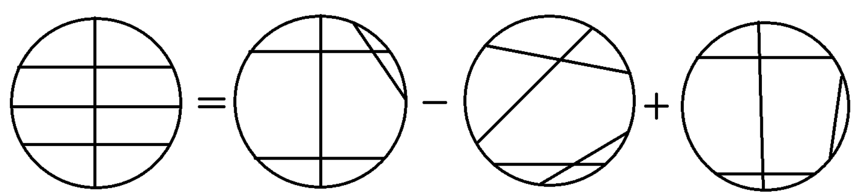

Figure 14: Last 2 CDs are the elements of the same basis of A_4 so their difference $\neq 0$

But two Jacobi diagrams on fig. 13, which we started with, are equal as trivalent graphs. That's why we define two separate frame operations: $F_{Jac} : Jac \longrightarrow$ "trivalent graphs" and $F_{CD} : CD \longrightarrow$ "trivalent graphs".

Definition 13 (Chord type Jacobi diagram)**.** We call a given Jacobi diagram a Chord type Jacobi diagram if it has at least 1 Hamiltonian cycle through all of its trivalent vertices. We denote a set of chord type Jacobi diagrams as $CTJD$.

Property 1. *$\forall\ a\ \in CTJD\ \exists\ \ b \in CD_j\ \ : F_{Jac}(a) = F_{CD}(b)$. Fig. 15 demonstrates this property.*

Lemma 1 (open Jacobi diagram frame decomposition)**.** Using IHX relation in the space of open Jacobi diagrams, every open Jacobi diagram J that is a connected graph and has a cycle can be rewritten as a finite sum:

$$J = \sum_k c_k J_k,\ \forall k\ J_k \in CTJD,\ c_k \in \mathbb{Z}$$

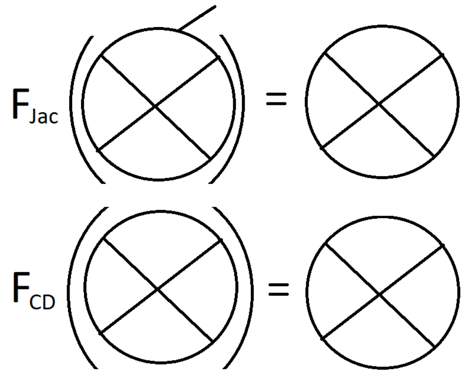

Figure 15: A simple illustration of property 1

Proof. If a given Jacobi diagram has a cycle (otherwise it's tree, which is 0 in the space of open Jacobi diagrams), then we can redraw this Jacobi diagram by redrawing this cycle and vertices that form it as a circle with the vertices on this circle. For example, Jacobi diagram on Fig. 16 can be redrawn in the following way.

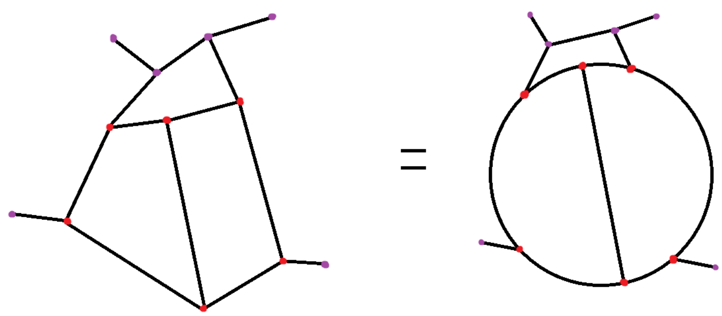

Figure 16: Example of drawing cycle as a circle. Notice that for each vertex we keep the cyclic order

If this cycle is a loop (1 vertex cycle), then it's well-known that because of AS relation, the Jacobi diagram containing a loop is equal to 0 in the space of open Jacobi diagrams. That's why from now on we will consider connected open Jacobi diagrams with a cycle of length 2 or more.

Since all vertices have either degrees 1 or 3, all vertices on the circle are 3-valent and have 1 edge that is different from the edges forming the circle.

Now, we would like to show that a given Jacobi diagram can be rewritten as a linear combination of Jacobi diagrams with 0 3-valent vertices not on the circle.

For this, we do an induction on the number of 3-valent vertices not on the circle.

Base: All open Jacobi diagrams with 0 3-valent vertices not on the circle.

Inductive step: Suppose that having n 3-valent vertices not on the circle you can rewrite a Jacobi diagram as a linear combination of Jacobi diagrams with $n-1$ 3-valent vertices on the circle.

Prove: show that we can rewrite Jacobi diagram with $n+1$ trivalent vertices not on the circle in the same way.

Notice that IHX relation rewrites a connected Jacobi diagram as a sum of two connected Jacobi diagrams and that trivalent vertices not on the circle can't be connected to the univalent vertices not on the circle that are connected to the circle. The obvious consequence of these facts is that if there is at least 1 trivalent vertex not on the circle, there is at least 1 trivalent vertex not on the circle that is connected to the circle.

This statement implies that we can always rewrite a Jacobi diagram with $n+1, n \in \mathbb{N}$ trivalent vertices not on the circle as the sum of two Jacobi diagrams with n trivalent vertices not on the circle by applying the IHX to the trivalent vertex not on the circle that is connected to some vertex on the circle, as demonstrated on the Fig. 17. Then, according to the inductive step, we can rewrite each of those two Jacobi diagrams as the linear combination of Jacobi diagrams with $n-1$ 3-valent vertices not on the circle. Obviously, we can continue this process and get the linear combination (with integer coefficients) of open Jacobi diagrams with 0 3-valent vertices not on the circle.

The example below shows an example of how the IHX is being implied in this proof.

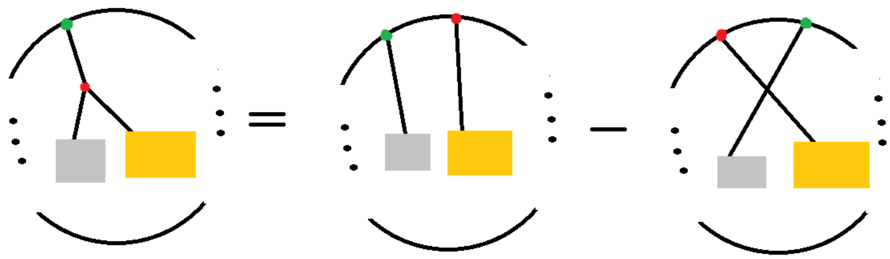

Figure 17: Here, grey and yellow boxes can have any trivalent graph, they can be connected together, and they can be connected to dotted parts of the circle.

So, we can rewrite a given Jacobi diagram as a linear combination of Jacobi diagrams with all of their trivalent vertices on the circle. In each of those Jacobi diagrams, a circle with all of its trivalent vertices is a Hamiltonian path through all of its trivalent vertices. Such Jacobi diagrams are chord type Jacobi diagrams by definition. □

3 Conjectures

3.1 "Frame matching" conjecture

Conjecture 2 (Frame matching)**.** *$\forall\, m, n\ \exists B =< c_1, c_2, c_3, ... >$, a basis of $A = \bigoplus_i A_i, c_i \in CD$:*

$$\forall\, j \in JD_{m,n}\, j = \sum_k t_k j_k\ : j_k \in JD_{m,n}\ ,\ F_{Jac}(j_k) \in\ \{F_{CD}(c_1), F_{CD}(c_2), F_{CD}(c_3), ...\}$$

Example 6 ("Frame matching" conjecture for 1 chord)**.** For 1 chord the statement is trivial because frames of all Jacobi diagrams with 1 chord are equal to the chord diagram with 1 chord, which is unique, so it obviously belongs to the basis of A.

Example 7 ("Frame matching" conjecture for 2 chords)**.** First, consider F_{JD} of all chord type Jacobi diagrams with 2 chords, which are on the left side on Fig. 18 and F_{CD} of all chord diagrams with 2 chords, which are on the right side on Fig. 18, and notice that these frames form equal pairs.

To find chord diagrams from basis A, we consider all 4T for chord diagrams with 2 chords, shown on Fig. 19.

This trivial 4T on Fig. 19 shows that both of the chord diagrams drawn above belong to the basis A. Now, we can see that frames of all chord type Jacobi diagrams with 2 chords are equal to the trivalent graphs that are identical to chord diagrams from basis A.

Remark 4 (Decompose JD that are not $CTJD$ first)*.* When working on the "Frame matching" conjecture, we need to use the Jacobi diagram decomposition lemma for Jacobi diagrams that are not CTJD, as demonstrated on Fig. 20.

3.2 Birman's Conjecture

Conjecture 3. *Every open Jacobi diagram with the odd number of univalent vertices is equal to zero in the factor space of open Jacobi diagrams.*

Using the Jacobi diagram decomposition theorem, we can simplify Birman's conjecture to

$$\forall\, k, n \in \mathbb{N}, \forall\, J\ \in JD_{k,2n-1} \cap CTJD,\ J = 0 \text{ in space } Jac$$

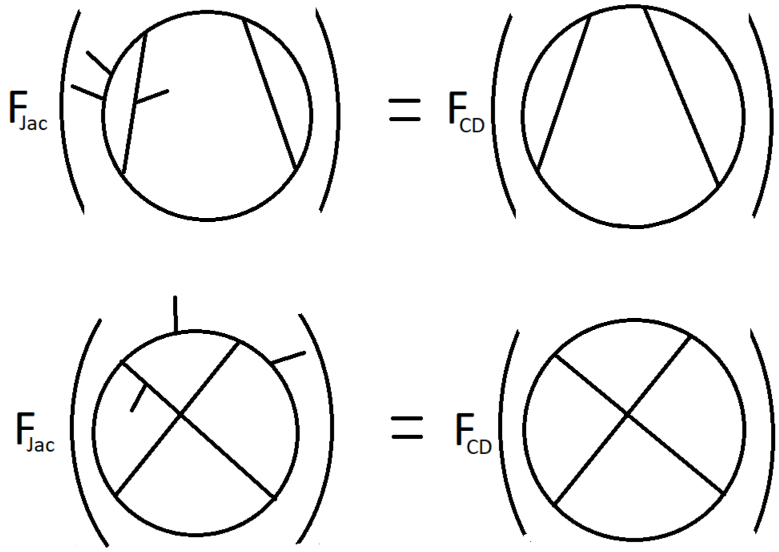

Figure 18

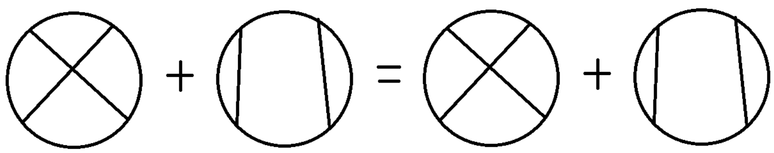

Figure 19

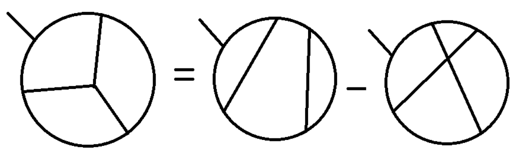

Figure 20

Obviously, proving this simplified conjecture is sufficient to prove Birman's conjecture because, according to the Jacobi diagram decomposition theorem, every Jacobi diagram is a linear combination of chord type Jacobi diagrams with the same number of univalent vertices.

Theorem 3 (Generalized IHX relation [3]). *The statement of this theorem can be found on Fig. 21*

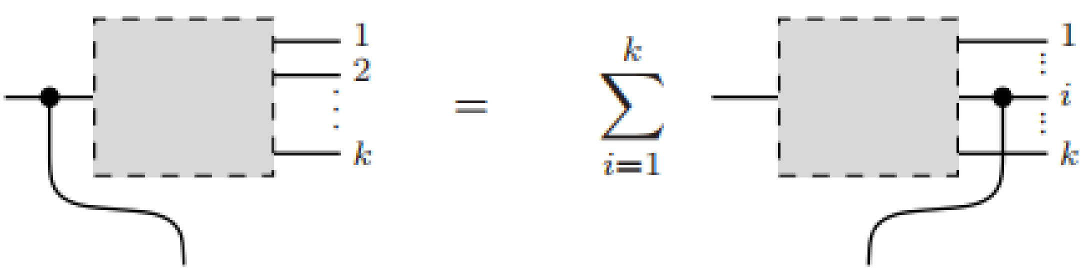

Figure 21: Where the gray area only contains 3-valent vertices

Theorem 4. *Every chord-type Jacobi diagram in which all univalent vertices are on one arch of the circle (with no trivalent vertices among them) is equal to 0 in the space of open Jacobi diagrams [3].*

Proof. Any chord-type Jacobi diagram can be put into the form on the left of the next picture. Then, by the generalized IHX relation, it is equal to the diagram on the right on Fig. 23 [3].

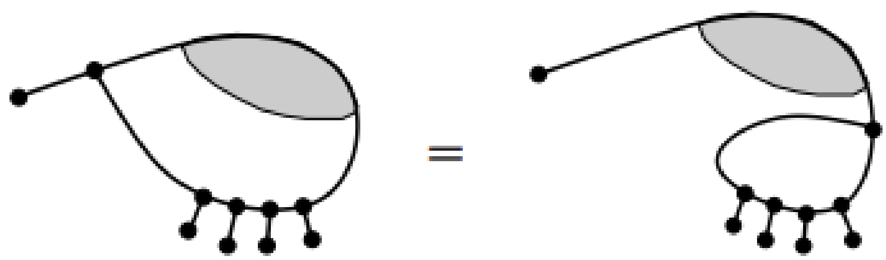

Figure 22

Now, notice that applying AS to one point and then to $2n$ points apart from the first one we can get the following chains of equalities on Fig. 23.

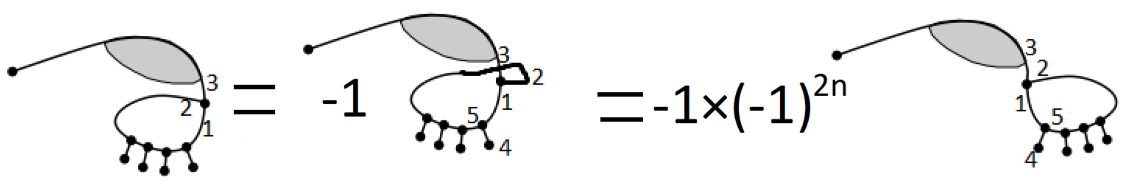

Figure 23

Now, we observe that the rightmost Jacobi diagram is equivalent to the leftmost one, so such chord-type Jacobi diagrams with odd number of univalent vertices are all equal to 0 in the space of open Jacobi diagrams. □

3.3 Comparing relations between elements of A_m and $CTJD$ in $Jac_{m,x}$

Remark 5 (4T in Jac). CTJD with no hairs have the same 4T relation as chord diagrams, which are made by applying IHX twice, as shown on the Fig. 24.

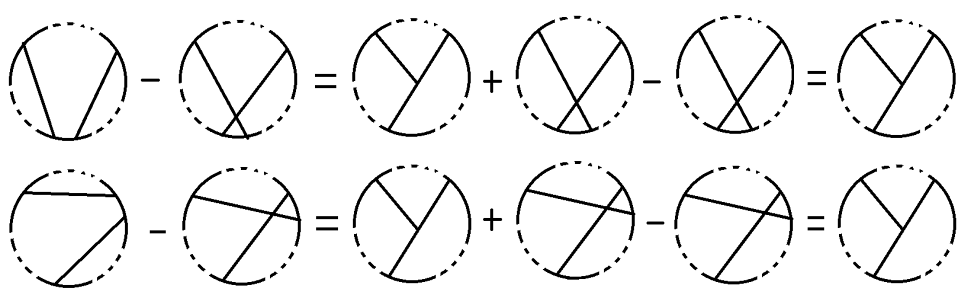

Figure 24: Leftmost difference in the 1st row is equal to the leftmost difference in the 3rd row, which is exactly 4T for Jacobi diagrams.

But, not all relations between CTJD have analogous relations in A, i.e. the following $4T'$ relation shown on 25.
QUESTION: Is the list of relations between CTJD exhaustive by $4T$, $4T'$, and internal symmetries?

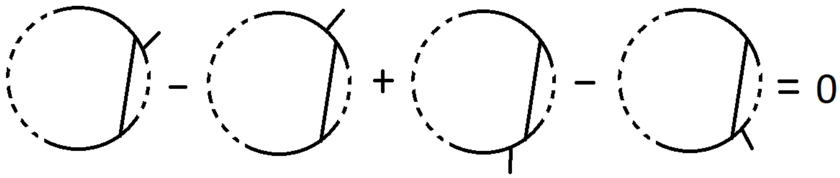

Figure 25: $4T'$ relation that is unique to chord type Jacobi diagrams

4 Computational tools

4.1 Frame map

When working with chord diagrams, we needed to know how to algorithmically identify Jacobi diagrams, whose frames are chord diagrams (from now on we'll call them chord-type Jacobi diagrams.)

Property 2. *If a Jacobi diagram has a Hamiltonian path through all of its trivalent vertices, then we can place all 3-valent vertices in a cycle, which, as discussed in the proof of theorem 2, after taking a frame gives us a chord diagram.*

Property 3. *Property 2 implies that if a Jacobi diagram has 2 different Hamiltonian paths through its 3-valent vertices, then this Jacobi diagram is equal to 2 different chord-type diagrams.*

4.2 Generation of factor space of open Jacobi diagrams

Definition 14 (Relation matrix)**.** Matrix $g \times j$, where g is the number of diagrams in the space and j is the number of relations. If we rearrange every relation in the way that one side of the equation is exactly zero, then every row in this matrix represents constants before every diagram on the other side of the equation.

We have developed a program that generates factor space of open Jacobi diagrams. The program was based on surface-dynamics package for SageMath.

The algorithm that generates the factor space $Jac_{k,h}$ with fixed number of trivalent nodes k and hairs h is:

1. Generate the whole set of open Jacobi diagrams with given parameters k, h: $JD_{k,h}$.
2. Firstly, we apply all possible AS relations to this space by adding corresponding rows to the relation matrix, therefore creating intermediate space $span_{\mathbb{R}}(JD_{k,h}) \,/\, AS$.
3. Secondly, we apply IHX to all diagrams in the intermediate space in the same way as with AS.
4. Output: $Jac_{k,h}$ with relation matrix, where we can get all the needed information (e.g., bases).

4.3 The algorithm for verifying Vasiliy's conjecture

The algorithm for verifying Vasiliy's conjecture for factor space $Jac_{k,h}$ (Fig. 26):

1. Input: $Jac_{k,h}$.
2. We find bases that consist only of CTJD, by checking whether the given diagram has Hamiltonian path, and creating sets with number of $dim(Jac_{k,h})$ chord type Jacobi diagrams.
3. We create separate sets of frames of the elements from those bases.
4. Check if there are bases such that frames of their elements can be found among sets from the third step.
5. Output: all sets with base elements with property described in the fourth step; those sets prove that Vasiliy's conjecture is true for this single space. For multiple spaces, we should implement a more sophisticated algorithm with efficient operation of "merging spaces" (finding basis elements from the spaces from which we take frame operation).

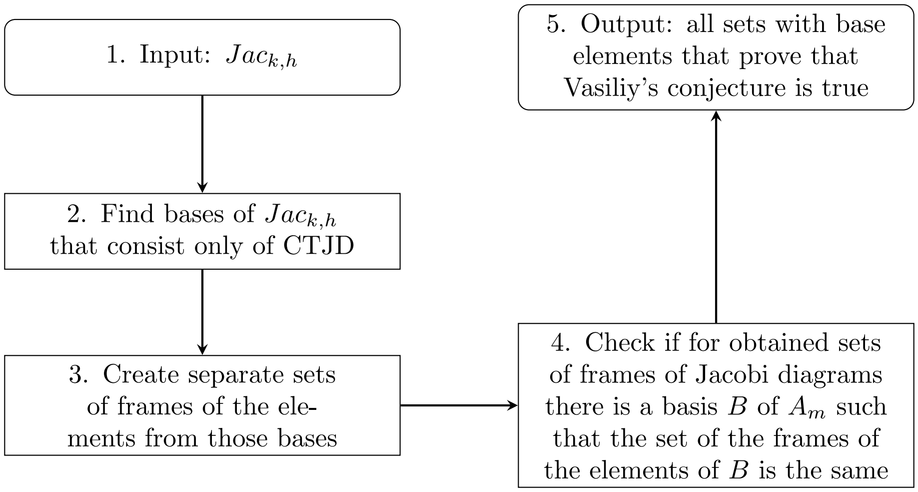

Figure 26: Summarization graph of the algorithm.

5 Conclusion

We defined two frame maps as maps from sets of chord and Jacobi diagrams to trivalent graphs that work by forgetting structures in open Jacobi diagrams and chord diagrams, making them trivalent graphs. Using these operations, the "Frame matching" conjecture was formulated. We provided proof of this conjecture for cases with 1 and 2 chords.

We defined the chord type Jacobi diagram and proved the "open Jacobi diagram frame decomposition" lemma, which allowed us to rewrite every open Jacobi diagram as a linear combination of chord type Jacobi diagrams. This way of rewriting open Jacobi diagrams allowed us to check the "frame matching" conjecture for more complicated cases, for example, as the one on Fig. 20.

With the knowledge that we could rewrite any open Jacobi diagram as a linear combination of chord type Jacobi diagrams, we further checked the "Frame matching" conjecture for more complicated cases, for example for diagrams with 3 chords. For this purpose, we wrote a Python code, described an algorithm of how to check the "Frame matching" conjecture with our code, and left a GitHub link [1] for it.

We also use the "open Jacobi diagram frame decomposition" lemma to simplify the conjecture that any open Jacobi diagram with an odd number of hairs is equal to 0 in the space of open Jacobi diagrams [2, p.50]. This important conjecture is equivalent to the conjecture that Vassiliev invariants do not distinguish the orientation of knots. Our Jacobi diagram frame decomposition lemma simplifies mentioned conjecture to a conjecture, saying that all chord-type Jacobi diagrams with an odd number of hairs are equal to 0 in the space of Jacobi diagrams.

Finally, we learned that 4T relation also existed in open Jacobi diagrams. Since our goal in the "Frame matching" conjecture was to compare bases of chord diagrams with bases of open Jacobi diagrams that consist only of chord type Jacobi diagrams, we arrived at a logical question: is the list of relations between CTJD exhaustive by $4T$, $4T'$, and internal symmetries?

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