EXTENDING BENSON'S CONJECTURE TO ARBITRARY PRIMES

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Abstract. Let G be a finite p-group and $\Bbbk$ be an algebraically closed field of characteristic p. Dave Benson has conjectured that when $p = 2$, if V is an odd-dimensional indecomposable representation for a finite 2-group G, then all non-trivial summands of the tensor product $V \otimes V^*$ have even dimension. It is known that the analogous result for general p is false. In this paper, we investigate the class of graded representations V which have dimension coprime to p and for which $V \otimes V^*$ has a non-trivial summand of dimension coprime to p, for a graded group scheme closely related to $\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^s\mathbb{Z}$, for two nonnegative integers r and s. We produce an infinite family of such representations.

1. Introduction

In modular representation theory of finite groups, that is, the study of representations of a finite group G whose order is divisible by a prime p over a field of characteristic p, many seemingly easy-to-state questions about the decompositions of tensor products remain unsolved. One such fundamental question that has generated much research interest is the following.

Question 1.1. *For what finite-dimensional indecomposable G-representations V does $V \otimes V^*$ break down into a direct sum of the trivial representation $\Bbbk$ and indecomposable representations of dimension divisible by p?*

By the Benson–Carlson Theorem [BC86, Theorem 2.1], the trivial representation $\Bbbk$ appears as a direct summand of $V \otimes W$, with V and W indecomposable, if and only if $W \cong V^*$ and the dimension of V is not divisible by p. In this case the multiplicity of $\Bbbk$ in the decomposition of $V \otimes V^*$ is 1. We adopt the terminology of [Ben20]: when V has dimension non-divisible by p, we say that V is a p' representation, and when $V \otimes V^*$ breaks down into a direct sum of $\Bbbk$ and representations of dimension divisible by p we call V p'-*invertible.* By the Benson–Carlson Theorem, Question 1.1 can now be restated as: which p'-representations are p'-invertible?

This question can be restated in yet another way in the language of semisimplifications of tensor categories. The semisimplification of a tensor category (see [EO22]) is a semisimple tensor category, where simple objects correspond to those indecomposable objects of the original category whose dimension (considered as an element of $\Bbbk$) is nonzero. Therefore, Question 1.1 can be restated as: what are the tensor-invertible representations of the semisimplification of G-representations? Since semisimplifications of categories of representations for finite groups have recently been an important tool for studying general symmetric tensor categories [CEO23, BEO23], this question is pressing.

Based on extensive evidence via computer algebra systems, Dave Benson has made a striking conjecture for the answer to Question 1.1 in characteristic 2, see [Ben20, Conjecture 1.1]:

Conjecture 1.2 (Benson's Conjecture)**.** *If G is a finite 2-group, $\Bbbk$ a field of characteristic 2, and V an odd-dimensional representation of G, then all indecomposable summands of $V \otimes V^*$ have dimension divisible by 2. That is, every $2'$-representation is 2'-invertible.*

In fact, this is a weak version of Benson's Conjecture; he also gives a strong version of the conjecture, which is that all summands of $V \otimes V^*$ have dimension divisible by 4.

One useful consequence of Conjecture 1.2 would be that all tensor powers of a $2'$-representation V would have a unique $2'$-summand [FH91]. In this case, Benson has further conjectures on the growth of the dimension of these $2'$-summands [CV, Conjecture 1.0.2], which has been verified in a few small examples [CV, Theorem 1.0.3]. Alternative growth functions related to either the number of indecomposable summands in a tensor power or the dimensions of non-projective summands in tensor powers have also received recent attention [BS20, Upa21, Upa22, CHU22, COT23, CEO].

For $p > 2$, however, there is not even a conjectural description of the p'-invertible representations. The naive extension of Benson's Conjecture– that all p'-representations are p'-invertible– fails in even very low-dimensional examples starting with $p = 3$. In this paper, we study Question 1.1 for cyclic graded representations of a finite group scheme $\alpha_p(r, s)$ which is closely related to $\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^s\mathbb{Z}$, which has generators x and y. Given a cyclic representation V, we draw a graded diagram for V with rows and columns as below in Section 2.

Theorem 1.3. *Let V_h be the cyclic module of dimension h generated in degree $(0, 0)$ for $\alpha_p(r, s)$ such that x acts by 0. Let $\mathrm{pow}(h)$ be the smallest number such that $h \leq p^{\mathrm{pow}(h)}$. If V_h is not p'-invertible, then neither is V, where V satisfies either*

(1) every column of V is equal to either 0 or h modulo $p^{\mathrm{pow}(h)}$;
(2) every row of V is equal to either 0 or h modulo $p^{\mathrm{pow}(h)}$.

Based on computational evidence collected using the computer algebra system Magma, the only cyclic p'-representations of $\alpha_p(r, s)$ appear to be those appearing in Theorem 1.3.

Question 1.4. *If V is a cyclic p'-representation for $\alpha_p(r, s)$ not satisfying the conditions of Theorem 1.3, is V p'-invertible?*

2. Background

Let $\Bbbk$ denote an algebraically closed field of characteristic p. We denote $G := \mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^s\mathbb{Z}$. The generating set of G is given by $\{g, h\}$, where $g^{p^r} = h^{p^s} = 1$ and $gh = hg$ (g and h commute). Let g' and h' denote the corresponding vectors of g and h in the group algebra $\Bbbk G$, and define $x := g' - 1$ and $y := h' - 1$. Note that x and y are both nilpotent because

$$x^{p^r} = (g - 1)^{p^r} = g^{p^r} - 1 = 1 - 1 = 0,$$

and the analogous relation holds for y. Thus, it is sufficient to define the values that x and y map to fully define a map from $\Bbbk G$, as g' and h' form a generating set for $\Bbbk G$.

We define $\alpha_p(r, s)$ to be the group algebra of $\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^s\mathbb{Z}$ as an algebra, with an alternative comultiplication given by

$$x \mapsto x \otimes 1 + 1 \otimes x, \quad y \mapsto y \otimes 1 + 1 \otimes y.$$

This alternative definition of comultiplication (for a description of the canonical structure of a cocommutative Hopf algebra on $\Bbbk G$, refer to [EGH+11, Section 4.4] or [Kas94, Section 3.3]) provides $\alpha_p(r, s)$ with the structure of a Hopf algebra, or the coordinate ring of a group scheme ([Kas94]). To achieve a $\mathbb{Z}^2$ grading, we define the degree of x to be $(1, 0)$ and the degree of y to be $(0, 1)$, respectively, with x and y as elements of $\alpha_p(r, s)$ as before. This grading gives $\alpha_p(r, s)$ the structure of a graded Hopf algebra, allowing us to form graded diagrams of these modules.

To formally define these graded diagrams, consistent with the notation used in [CV] for the more general notion of a monomial diagram, consider a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ for which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. The representation corresponding to such a partition that we consider has basis elements $v_{i,j}$ such that $0 \leq i \leq n - 1$ and $0 \leq j \leq \lambda_{i+1}$. In such a representation, the x action applied to the basis element $v_{i,j}$ yields $v_{i+1,j}$, and similarly the y action applied to $v_{i,j}$ yields $v_{i,j+1}$. If $v_{i+1,j}$ or $v_{i,j+1}$ do not exist in these respective cases, we say that the basis element $v_{i,j}$ is sent to 0.

It is important to note that for this to be a valid representation, there must be at most p^s values of j for each i such that $v_{i,j}$ exists, and similarly, at most p^r values of i for each j for which $v_{i,j}$ exists. We can further pictorially depict the grading of these representations with graded diagrams. For all i and j for which $v_{i,j}$ is a basis element, if we draw in the grid box, or cell, in degree (i, j), we can produce such a graded diagram.

Example 2.1. For $G := \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$, the graded diagram corresponding to the partition $[6, 3, 2, 2]$ is shown below. As can be seen from the diagram, there are basis vectors in $v_{i,j}$ for $(i, j) \in S$ for

$$S := \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0), (3, 1)\}.$$

1			
1			
1			
1	1		
1	1	1	1
1	1	1	1

Throughout this paper, we often refer to the *rows* and *columns* within the graded diagrams corresponding to the representations mentioned above. Row a refers to the group of cells corresponding to $v_{i,j}$ with $j = a$. Similarly, column b refers to the group of cells corresponding to $v_{i,j}$ for which $i = b$.

3. Classifying certain maps between tensor products

Let V be a cyclic representation of $\alpha_p(r,s)$ generated in degree (0,0). Let W be the cyclic representation of $\alpha_p(r,s)$ of dimension $n+m+1$ generated in degree $-n$, where n and m are positive integers, such that x acts by 0. In this section, we classify the G-representation homomorphisms $V \to V \otimes W$, which will be useful in the following section.

Since V is cyclic, any graded map f is determined by the collection $(a_0, ..., a_n) \in \Bbbk^{n+1}$, where

$$f(v_{00}) = a_0 v_{0,0} \otimes w_{0,0} + a_1 v_{0,1} \otimes w_{0,-1} + ... + a_n v_{0,n} \otimes w_{0,-n}.$$

Lemma 3.1. *Let f be as above. Then*

$$\begin{aligned} f(v_{ij}) =& a_0 \left(\sum_{k=0}^{j} \binom{j}{k} v_{i,k} \otimes w_{0,j-k} \right) \\ &+ a_1 \left(\sum_{k=0}^{j} \binom{j}{k} v_{i,k+1} \otimes w_{0,j-k-1} \right) \\ &+ ... \\ &+ a_n \left(\sum_{k=0}^{j} \binom{j}{k} v_{i,k+n} \otimes w_{0,j-k-n} \right). \end{aligned}$$

Proof. We have

$$\begin{aligned} f(v_{i,j}) &= f(x^i y^j v_{0,0}) \\ &= x^i y^j f(v_{0,0}) \\ &= x^i y^j (a_0 v_{0,0} \otimes w_{0,0} + a_1 v_{0,1} \otimes w_{0,-1} + ... + a_n v_{0,n} \otimes w_{0,-n}) \\ &= (x \otimes 1 + 1 \otimes x)^i (y \otimes 1 + 1 \otimes y)^j (a_0 v_{0,0} \otimes w_{0,0} + a_1 v_{0,1} \otimes w_{0,-1} + ... + a_n v_{0,n} \otimes w_{0,-n}) \\ &= (y \otimes 1 + 1 \otimes y)^j (a_0 v_{i,0} \otimes w_{0,0} + a_1 v_{i,1} \otimes w_{0,-1} + ... + a_n v_{i,n} \otimes w_{0,-n}) \\ &= \left(\sum_{k=0}^{j} \binom{j}{k} y^k \otimes y^{j-k} \right) (a_0 v_{i,0} \otimes w_{0,0} + a_1 v_{i,1} \otimes w_{0,-1} + ... + a_n v_{i,n} \otimes w_{0,-n}). \end{aligned}$$

The last line then yields the claimed formula. ∎

Corollary 3.2. *Maps $V \to V \otimes W$ are in bijection with solutions $(a_0, ..., a_n)$ in $\Bbbk^{n+1}$ to the system of equations*

$$\begin{aligned}
0 &= a_0\binom{j}{1} + a_1\binom{j}{2} + ... + a_n\binom{j}{n+1} \\
&= a_0\binom{j}{2} + a_1\binom{j}{3} + ... + a_n\binom{j}{n+2} \\
&... \\
&= a_0\binom{j}{\min\{j,m\}} + a_1\binom{j}{\min\{j,m\}+1} + ... + a_n\binom{j}{\min\{j,m\}+n}
\end{aligned}$$

over all pairs (i, j) where $v_{i,j} = 0$ and $v_{i,j-1} \neq 0$.

Proposition 3.3. *Let f and g be the maps $V \to V \otimes W$ corresponding to $(a_0, ..., a_n)$ and $(b_0, ..., b_n) \in \Bbbk^{n+1}$ as in Corollary 3.2, respectively. Denote by $\hat{f}$ the map $W \to V \otimes V^*$ and $\overline{g}$ the map $V \otimes V^* \to W$ the maps corresponding to g and f as in [EGNO15, Proposition 2.10.8]. Then $\overline{g} \circ \hat{f} \neq 0$ if and only if $b_n \neq 0$, and*

$$\sum_{i,j}\left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + ... + a_n\binom{j}{2n}\right) = \sum_j \operatorname{length}(j)\left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + ... + a_n\binom{j}{2n}\right) \neq 0,$$

where $\operatorname{length}(j)$ is the length of the row at height j.

Proof. We have the formulas

$$\begin{aligned}
\hat{f} &= (\mathrm{id}_{V^*\otimes V} \otimes \mathrm{ev}_W) \circ (\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_W) \circ (\mathrm{coev}_V \otimes \mathrm{id}_W), \\
\overline{g} &= (\mathrm{ev}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_{V^*} \otimes g),
\end{aligned}$$

where we use implicitly that $W \cong W^*$ and that the tensor product commutes up to natural isomorphism. The composition $\overline{g} \circ \hat{f}$ is nonzero if and only if it sends $w_{0,-n}$ to something nonzero. The proof then follows from a direct computation.

□

4. Proof of the main theorem

In this section, we state and prove our main theorem concerning general p-groups. We first state our theorem below.

Theorem 4.1. *For an arbitrary integer u, define $\operatorname{pow}(u)$ to be the unique integer value for which $p^{\operatorname{pow}(u)-1} < u \leq p^{\operatorname{pow}(u)}$. If V_{2n+1} is a direct summand of $V_h \otimes V_h^*$, then V_{2n+1} is a summand of $V \otimes V^*$, where V is a cyclic module, all columns are either 0 or h modulo p^g, where $g = \max(\operatorname{pow}(h), \operatorname{pow}(2n+1))$.*

Proof. If V_{2n+1} is a summand of $V_h \otimes V_h^*$, by Corollary 3.2 and Proposition 3.3, there must exist some solution $(a_0, a_1, \ldots, a_n)$ which satisfy

$$\begin{aligned}
0 &= a_0\binom{2n}{1} + a_1\binom{2n}{2} + ... + a_n\binom{2n}{n+1} \\
&= a_0\binom{2n}{2} + a_1\binom{2n}{3} + ... + a_n\binom{2n}{n+2} \\
&\vdots \\
&= a_0\binom{2n}{n} + a_1\binom{2n}{n+1} + ... + a_n\binom{2n}{2n},
\end{aligned}$$

and likewise for $(b_0, b_1, \ldots, b_n)$. For V_{2n+1} to be a direct summand of $V \otimes V^*$, the same set of equations must hold, with the only difference being in the values of j for which there exist i such that $v_{i,j} = 0$ and $v_{i,j-1} \neq 0$. For $V_h \otimes V_h^*$, the only such j was $j = 2n+1$. However, for $V \otimes V^*$, all j which are congruent to 0 or h modulo p^g must satisfy this system of equations.

If $j \equiv 0 \pmod{p^g}$, it can be seen that $\binom{j}{l}$, for $1 \leq l \leq 2n$, are all $\equiv 0 \pmod{p}$, as if we expand these binomial coefficients, there will always be more factors of p in the numerator than in the denominator.

More specifically, upon expansion, the numerator and denominator will both be products of l consecutive $s(s-1)\cdots(s-l+1)$, for $s=j$ and $s=l$, respectively. For such products of consecutive elements, having $s \equiv 0 \pmod{p^g}$ will achieve the maximum number of factors of p in the product, and the inverse holds as well: if s is not a multiple of p^g, this maximum number of factors of p will not be achieved. Since $\binom{j}{l}=0$ for all $1 \leq l \leq 2n$, the system does not yield any restrictions on the values of $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$.

There must exist at least one j for which $j \equiv 2n+1 \pmod{p^g}$, as otherwise the dimension of V would be $\equiv 0 \pmod{p^g}$. In the case when $j \equiv 2n+1 \pmod{p^g}$, it can be seen by Lucas's Theorem that all the binomial coefficients $\binom{j}{l}$ will be equivalent to $\binom{2n+1}{l}$ modulo p. Let the base p expansion of $2n+1$ be expressed as

$$\beta_{g-1}p^{g-1} + \beta_{g-2}p^{g-2} + \cdots + \beta_1 p + \beta_0.$$

Then, since $j \equiv 2n+1 \pmod{p^g}$, we can express j as

$$\beta_e p^e + \beta_{e-1}p^{e-1} + \cdots + \beta_g p^g + (\beta_{g-1}p^{g-1} + \beta_{g-2}p^{g-2} + \cdots + \beta_1 p + \beta_0).$$

In addition, denote the base p expansion of l, where $l < 2n+1$, by

$$\gamma_{g-1}p^{g-1} + \gamma_{g-2}p^{g-2} + \cdots + \gamma_1 p + \gamma_0.$$

By Lucas's Theorem (see [Me , Theorem 2.1]), it follows that $\binom{2n+1}{l} \equiv \prod_{i=0}^{g-1}\binom{\beta_i}{\gamma_i} \pmod{p}$. Analogously, we have that

$$\binom{j}{l} \equiv \prod_{i=0}^{e}\binom{\beta_i}{\gamma_i} = \prod_{i=0}^{g-1}\binom{\beta_i}{\gamma_i}\cdot\left(\prod_{i=g}^{e}\binom{\beta_i}{\gamma_i}\right) = \prod_{i=0}^{g-1}\binom{\beta_i}{\gamma_i}\cdot\left(\prod_{i=g}^{e}\binom{\beta_i}{0}\right) = \prod_{i=0}^{g-1}\binom{\beta_i}{\gamma_i}.$$

Hence, we have that $\binom{2n+1}{l} \equiv \binom{j}{l}$ for $1 \leq l \leq 2n$, and it follows that a tuple $(a_0, a_1, \ldots, a_n)$ in $\Bbbk^{n+1}$ is a solution to the system corresponding to a map from $V_{2n+1} \to V_{2n+1} \otimes W$ if and only if it is a solution to the system corresponding to a map from $V \to V \otimes W$.

We further show that a pair of solutions $(a_0, a_1, \ldots, a_n), (b_0, b_1, \ldots, b_n)$ will satisfy $b_n \neq 0$ and make the expression

$$\sum_{i,j}\left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \ldots + a_n\binom{j}{2n}\right) = \sum_{j}\text{length}(j)\left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \ldots + a_n\binom{j}{2n}\right)$$

nonzero in the case of V_h if and only if it does so for all V of the described form.

The $b_n \neq 0$ condition clearly satisfies the if and only if criteria, as the value of b_n is the same for both cases. In the case of V_h, the expression above that we desire to be nonzero simplifies to

$$\sum_{j=0}^{h-1}\left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \ldots + a_n\binom{j}{2n}\right) = a_0\binom{h}{n+1} + a_1\binom{h}{n+2} + \cdots + a_n\binom{h}{2n+1}$$

by the Hockey Stick Identity (see [Wes21, Theorem 1.2.3]. For the remainder of this proof, we use c to denote the sum above.

In the case of V, note that the length(j) value will be equal among many terms in this sum. Because all the column heights are equivalent to 0 or h modulo p^g, we can consider the summands in this expression ranging between two adjacent column heights. If we denote those two adjacent heights by j_1 and j_2, then we are considering the expression

$$\text{length}([j_1, j_2-1])\sum_{j=j_1}^{j_2-1}\left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \ldots + a_n\binom{j}{2n}\right),$$

where we have factored out the constant length(j) term from row j_1 to row j_2-1, inclusive, as these rows are all of the same length. If $j_1 \equiv j_2 \pmod{p^g}$, whether it be that they are both equivalent to 0 or both equivalent to $2n+1$ modulo p^g, then it can be seen that this expression evaluates to 0 because

$$
\begin{aligned}
&\sum_{j=j_1}^{j_2-1}\left(a_0\binom{j}{n}+a_1\binom{j}{n+1}+\ldots+a_n\binom{j}{2n}\right)\\
&=\frac{j_2-j_1}{p^g}\cdot\sum_{j=0}^{p^g-1}\left(a_0\binom{j}{n}+a_1\binom{j}{n+1}+\ldots+a_n\binom{j}{2n}\right)\\
&=\frac{j_2-j_1}{p^g}\cdot\left(\sum_{i=0}^{n}\sum_{j=0}^{p^g-1}a_i\binom{j}{n+i}\right)\\
&=\frac{j_2-j_1}{p^g}\cdot\left(\sum_{i=0}^{n}a_i\binom{p^g}{n+i}\right)\\
&=0.
\end{aligned}
$$

To see the final step, note that because $g=\max(\mathrm{pow}(h),\mathrm{pow}(2n+1))$, we must have that $g>\mathrm{pow}(2n+1)$. Note that pow is an increasing function, so this would also imply that $g>\mathrm{pow}(i)$ for $i=n+1,n+2,\ldots,2n+1$. Thus, using analogous reasoning to when we were showing that $\binom{j}{l}\equiv 0 \pmod{p}$ for $1\leq l\leq 2n$ when $j\equiv 0 \pmod{p^g}$ above, we have that $\binom{p^g}{n+i}\equiv 0$ for $i=0,1,\ldots,n$, and the final step follows.

Suppose $j_1\equiv 0 \pmod{p^g}$ and $j_2\equiv h \pmod{p^g}$. If we let $j_1=p^gc_1$ and $j_2=p^gc_2+h$, then the desired expression evaluates to

$$
\begin{aligned}
&\mathrm{length}([j_1,j_2-1])\sum_{j=p^gc_1}^{p^gc_2+h-1}\left(a_0\binom{j}{n}+a_1\binom{j}{n+1}+\ldots+a_n\binom{j}{2n}\right)\\
&=\mathrm{length}([j_1,j_2-1])\sum_{j=p^gc_1}^{p^gc_2-1}\left(a_0\binom{j}{n}+a_1\binom{j}{n+1}+\ldots+a_n\binom{j}{2n}\right)\\
&+\mathrm{length}([j_1,j_2-1])\sum_{j=p^gc_2}^{p^gc_2+h-1}\left(a_0\binom{j}{n}+a_1\binom{j}{n+1}+\ldots+a_n\binom{j}{2n}\right)\\
&=\mathrm{length}([j_1,j_2-1])\sum_{j=p^gc_2}^{p^gc_2+h-1}\left(a_0\binom{j}{n}+a_1\binom{j}{n+1}+\ldots+a_n\binom{j}{2n}\right)\\
&=\mathrm{length}([j_1,j_2-1])\sum_{j=0}^{h-1}\left(a_0\binom{j}{n}+a_1\binom{j}{n+1}+\ldots+a_n\binom{j}{2n}\right)\\
&=c\cdot\mathrm{length}([j_1,j_2-1]).
\end{aligned}
$$

In the last case if $j_1\equiv h \pmod{p^g}$ and $j_2\equiv 0 \pmod{p^g}$, let $j_1=p^gd_1+h$ and $j_2=p^gd_2$. In this case, the desired expression evaluates to

$$\begin{aligned}
&\text{length}([j_1, j_2-1]) \sum_{j=p^g d_1+h}^{p^g d_2-1} \left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \dots + a_n\binom{j}{2n}\right) \\
&= \text{length}([j_1, j_2-1]) \sum_{j=p^g d_1+h}^{p^g d_2+h-1} \left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \dots + a_n\binom{j}{2n}\right) \\
&- \text{length}([j_1, j_2-1]) \sum_{j=p^g d_2}^{p^g d_2+h-1} \left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \dots + a_n\binom{j}{2n}\right) \\
&= -\text{length}([j_1, j_2-1]) \sum_{j=p^g d_2}^{p^g d_2+h-1} \left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \dots + a_n\binom{j}{2n}\right) \\
&= -\text{length}([j_1, j_2-1]) \sum_{j=0}^{h-1} \left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \dots + a_n\binom{j}{2n}\right) \\
&= -c \cdot \text{length}([j_1, j_2-1]).
\end{aligned}$$

For row j, let l_j denote the unique column for which the height of column l is $j+1$ and the height of column $l+1$ is less than $j+1$. If column $l+1$ does not exist, assume its height is 0. Let rowlength(i), for column i, denote the length of a row j for which $l_j = i$. Note that such a row j does not necessarily always exist, but in this proof we will only use it in contexts where it does exist. Furthermore, for column i, (where we start counting from the left), let j_i denote the height of the ith column.

Then from our work above, we have that

$$\begin{aligned}
&\sum_j \text{length}(j) \left(a_0\binom{j}{n} + a_1\binom{j}{n+1} + \dots + a_n\binom{j}{2n}\right) \\
&= \left(\sum_{i \text{ with } j_i \equiv j_{i+1}} 0\right) + \left(\sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} c \cdot \text{rowlength}(i)\right) + \left(\sum_{i \text{ with } j_i \equiv h, j_{i+1} \equiv 0} -c \cdot \text{rowlength}(j)\right) \\
&= c \cdot \left(\sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} \text{rowlength}(i) - \sum_{j \text{ with } j_i \equiv h, j_{i+1} \equiv 0} \text{rowlength}(i)\right).
\end{aligned}$$

Now we consider the expression $\sum_j \text{length}(j)$ in which we sum all the row lengths. Using the same notation as above, we can observe that

$$\sum_j \text{length}(j) = \left(\sum_{j \text{ with } j_l \equiv j_{l+1}} \text{length}(j)\right) + \left(\sum_{j \text{ with } j_l \equiv 0, j_{l+1} \equiv h} \text{length}(j)\right) + \left(\sum_{j \text{ with } j_l \equiv h, j_{l+1} \equiv 0} \text{length}(j)\right).$$

Consider each of the three expressions in parentheses. With respect to the expression in the first parentheses, for every row j with $j_l \equiv j_{l+1}$, there exist $j_{l+1} - j_l - 1$ other adjacent rows which also appear in the sum all with the same l_j value and identical lengths as well. But this group of $j_{l+1} - j_l$ rows with equal lengths will evaluate to 0 in this sum because $p^g | j_{l+1} - j_l$. Thus, the expression in the first parentheses evaluates to 0 by arranging all the rows in these aforementioned groups.

For the expression in the second parentheses, similar to the expression in the first parentheses, for every row j with $j_l \equiv 0$ and $j_{l+1} \equiv h$, there exist $j_{l+1} - j_l - 1$ other adjacent rows in the sum with the same l_j value and identical lengths. If we let r denote this constant length and l the constant l_j, then the sum of all the lengths of these rows will be $(j_{l+1} - j_l) \cdot r \equiv h \cdot r$. Note that the constant length of each of these groups of rows corresponds uniquely to the value of rowlength(i) for $i = l$. This correspondence establishes that

$$\sum_{j \text{ with } j_l \equiv 0, j_{l+1} \equiv h} \text{length}(j) \equiv \sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} h \cdot \text{rowlength}(i),$$

where we iterate over columns instead of rows in the latter expression.

With respect to the expression in the third parentheses, similar to the expressions in the other two parentheses, for every row j with $j_l \equiv h$ and $j_{l+1} \equiv 0$, there exist $j_{l+1} - j_l - 1$ other adjacent rows in the sum with identical l_j value and length. Using the same notation as for the second parentheses, if we let r denote this constant length and l the constant l_j, then the sum of all the lengths of these rows will be $(j_{l+1} - j_l) \cdot r \equiv -h \cdot r$. Note that the constant length of each of these groups of rows corresponds uniquely to the value of rowlength(i) for $i = l$. This correspondence establishes that

$$\sum_{j \text{ with } j_l \equiv h, j_{l+1} \equiv 0} \text{length}(j) \equiv \sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} -h \cdot \text{rowlength}(i),$$

where we again iterate over columns instead of rows in the second expression.

In summary, we have demonstrated that

$$\begin{aligned}\sum_j \text{length}(j) &\equiv 0 + \sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} h \cdot \text{rowlength}(i) + \sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} -h \cdot \text{rowlength}(i) \\ &= h \cdot \left(\sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} \text{rowlength}(i) - \sum_{j \text{ with } j_i \equiv h, j_{i+1} \equiv 0} \text{rowlength}(i) \right).\end{aligned}$$

Recall that the expression that we wish to be nonzero is

$$\begin{aligned}&\sum_j \text{length}(j) \left(a_0 \binom{j}{n} + a_1 \binom{j}{n+1} + \ldots + a_n \binom{j}{2n} \right) \\ &= c \cdot \left(\sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} \text{rowlength}(i) - \sum_{j \text{ with } j_i \equiv h, j_{i+1} \equiv 0} \text{rowlength}(i) \right).\end{aligned}$$

If c is 0, then the expressions for both V_h and V evaluate to 0, and hence V_{2n+1} is not present in the summands of $V_h \otimes V_h^*$ or $V \otimes V^*$.

If $c \neq 0$, V_{2n+1} is summand in $V_h \otimes V_h^*$. Furthermore, since c is nonzero, it follows that the desired expression for V is nonzero if and only if

$$\sum_{i \text{ with } j_i \equiv 0, j_{i+1} \equiv h} \text{rowlength}(i) - \sum_{j \text{ with } j_i \equiv h, j_{i+1} \equiv 0} \text{rowlength}(i)$$

is nonzero. But since $h \neq 0$, this expression is nonzero if and only if $\sum_j \text{length}(j)$, or the number of cells in the grid, is nonzero. The latter is always true: thus V_{2n+1} is also a summand of $V \otimes V^*$ if c is nonzero.

Since we have checked for all possible values of c, we have shown that V_{2n+1} is a summand in $V_h \otimes V_h^*$ if it is a summand in $V \otimes V^*$, and we are done with the proof.

□

5. Examples in characteristic 3

Providing a complete characterization of all p'-invertible representations remains an extremely difficult question. In this section, we explore a few examples of p'-representations that are not p'-invertible for $p = 3$. Using Theorem 4.1, we can quickly produce many infinite classes of representations that satisfy this criterion.

Let M_5 denote the cyclic $\alpha(1, 0)$ representation corresponding to [5]. It can be verified that $M_5 \otimes M_5^* \cong k \oplus M_3 \oplus M_5 \oplus M_7 \oplus M_9$, where M_3, M_5, M_7, M_9 denote subrepresentations of dimension $3, 5, 7, 9$ which correspond to the monomial diagrams shown below.

$\Bbbk$	M_3	M_5	M_7	M_9
1	1	1	1	1
	1	1	1	1
	1	1	1	1
		1	1	1
		1	1	1
			1	1
			1	1
				1
				1

For instance, to show that M_5 is a summand of $M_5 \otimes M_5^*$, we can examine the system of equivalences produced by Corollary 3.2. We get that

$$2a_0 + a_1 + a_2 \equiv 0,$$
$$a_0 + 2a_1 + 2a_2 \equiv 0,$$

which implies that (a_0, a_1, a_2) must be of the form $(x, 0, x)$. (b_0, b_1, b_2) must be of this form as well, but the only value of our concern is that there exists such a tuple for which $b_2 = x \neq 0$, which there is.

All that is left to check is that there exists some solution $(a_0, a_1, a_2) = (x, 0, x)$ for which

$$\sum_{j=0}^{4} \text{length}(j) \left(a_0 \binom{j}{2} + a_1 \binom{j}{3} + a_2 \binom{j}{5} \right) \neq 0.$$

Substituting in the form of (a_0, a_1, a_2), we get that the expression simplifies to $x \cdot \sum_{j=0}^{4} \text{length}(j) \cdot \left(\binom{j}{2} + \binom{j}{4}\right)$. Since $\text{length}(j) = 1$ for $j = 0, 1, 2, 3, 4$, we get that this expression is equal to $x \cdot \left(\binom{5}{3} + \binom{5}{5}\right) = 2x$, and thus setting x equal to any nonzero value produces the desired nonzero composition of maps, showing that M_5 is a summand of $M_5 \otimes M_5^*$.

Corollary 5.1. *Let V be a cyclic representation of $\alpha_3(r, s)$ generated in degree $(0, 0)$ with dimension coprime to 3 corresponding to a graded diagram with heights of $a_1, a_2, \ldots, a_n$ such that $a_i \equiv 0, 5 \pmod 9$ for $i = 1, 2, \ldots, n$. For all such V, $M_5 \otimes M_5^*$ is in the decomposition of $V \otimes V^*$, and thus, specifically M_5 and M_7 are in the decomposition of $V \otimes V*$.*

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