The Action of the Cactus Group on Arc Diagrams

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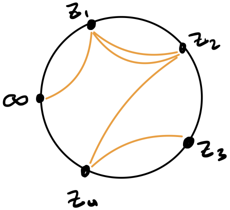
Arc Diagram

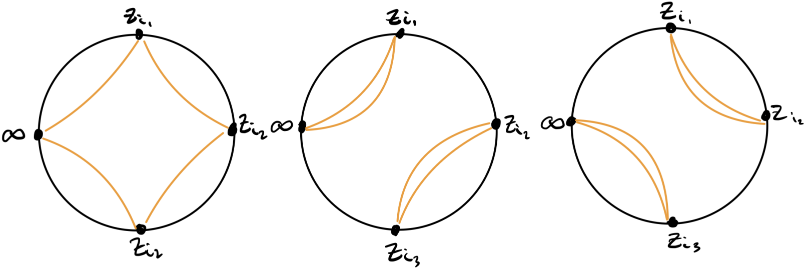
Definition (Arc Diagram)

- Place $n+1$ points on a circle and label them $z_1, z_2, \dots, z_n, z_\infty$
- $z_1, z_2, \dots, z_n$ can be in any order
- Connect points with non-intersecting arcs
- Valence of z_i is called ℓ_i



The Set of Arc Diagrams

- $X(\ell_1, \ell_2, \ldots, \ell_n, \ell_\infty)$ is the set of all arc diagrams with valences $\ell_1, \ell_2, \ldots, \ell_n, \ell_\infty$ for all orderings of the corresponding $z_1, z_2, \ldots, z_n$.
- Here is $X(2, 2, 2, 2)$ (for all choices of distinct $i_1, i_2, i_3 \in \{1, 2, 3\}$):



Group

Definition (Group)

A group is a set G with an operation $\times : G \times G \to G$ satisfying:

- Associativity: $a \times (b \times c) = (a \times b) \times c$
- Identity: $a \times e = e \times a = a$
- Inverses: $a \times a^{-1} = a^{-1} \times a = e$

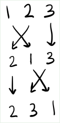
Note that $a \times b$ is often written as ab.

Example

Symmetric Group

S_3 (permutations of 3 elements) under composition ($\circ$) is a group:

- The set is $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$
- Operation is composition: apply permutations one after the other from right to left.
- $(1,2,3)$ is the identity.
- $(1,3,2) \circ (2,1,3) = (2,3,1)$



Group Action

Definition (Group Action)

Given a group G and a set X, a group action is a function $\alpha : G \times X \to X$. Notationally $\alpha(g, x) = g \cdot x$.

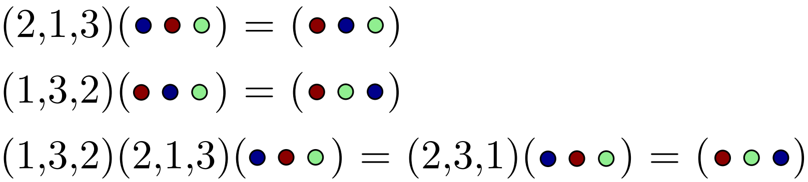
- Identity: $e \cdot x = x$
- Compatibility: $g \cdot (h \cdot x) = (gh) \cdot x$

Essentially each $g \in G$ is assigned some transformation of X such that it is compatible with the group structure.

Example

S_3 acts on a set of 3 ordered points

- Permute the points according to the element of S_3.



Generators and Relations

- Generators are a set of group elements which can be multiplied to make elements in the group

Free Group

- Example: $\langle a, b \rangle$ is the set of strings consisting of a, b, a^{-1} and b^{-1} ($\times$ is concatenation)
- Thus $aba \times a^{-1}ba = abaa^{-1}ba = abba$

Generators and Relations

- Relations are imposed on the generators

Relations

- Example: $\langle a, b \mid a^2 = b^2 = e \rangle$ is the set of strings consisting of a, b, a^{-1} and b^{-1} except we declare that $aa = bb = e$
- Thus $aba \times a^{-1}ba = abaa^{-1}bb = abba = aa = e$

- Groups are often defined in this way

The Cactus Group

Definition (Cactus Group J_n)

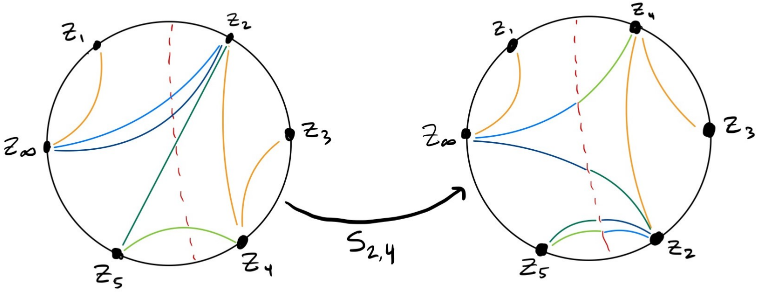
The cactus group is defined by the set of generators $\{s_{p,q} \mid 1 \leq p < q \leq n\}$ and relations:

- $s_{p,q}^2 = e$ where e is the identity for any $s_{p,q}$.
- $s_{p,q}s_{p',q'} = s_{p',q'}s_{p,q}$ if $q' < p$ or $q < p'$, that is, the intervals $[p, q]$ and $[p', q']$ are disjoint.
- $s_{p,q}s_{p',q'}s_{p,q} = s_{p+q-q',p+q-p'}$ if $p \leq p' < q' \leq q$, that is, the interval $[p', q']$ falls inside the interval $[p, q]$.

Action of the Cactus Group on Arc Diagrams

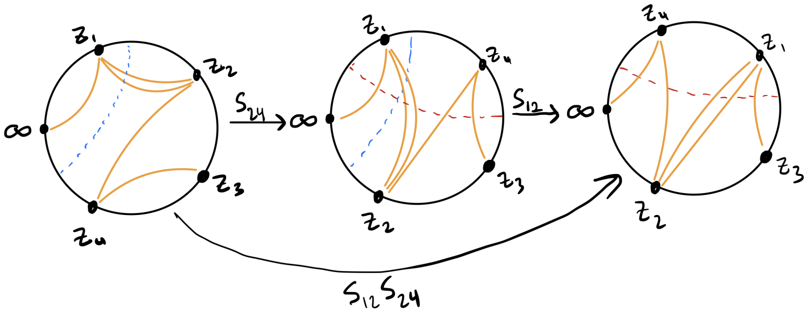
For the action of a generator $s_{p,q}$:

- Isolate the smallest section of the diagram containing points p through q.
- Reflect this section to reverse the order of the points.
- Broken connecting lines are reconnected in opposite order



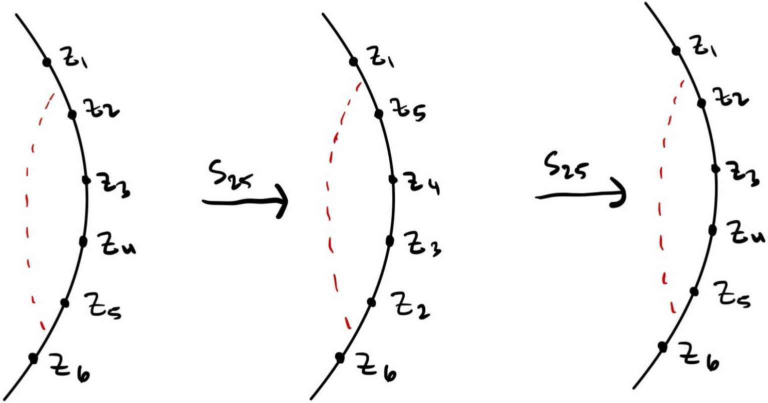
Example

Operation extended by composition:



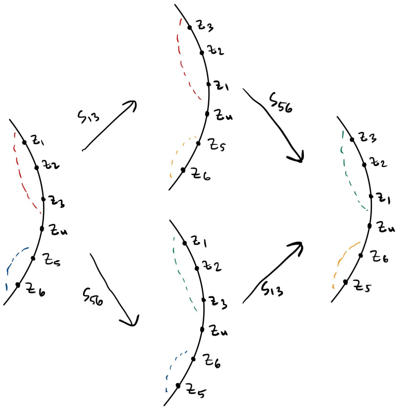
Proof that it's a Group Action

- $s_{p,q}^2 = e$ where e is the identity for any $s_{p,q}$.



Proof that it's a Group Action

- $s_{p,q} s_{p',q'} = s_{p',q'} s_{p,q}$ if $q' < p$ or $q < p'$, that is the intervals $[p, q]$ and $[p', q']$ are disjoint.



Proof that it's a Group Action

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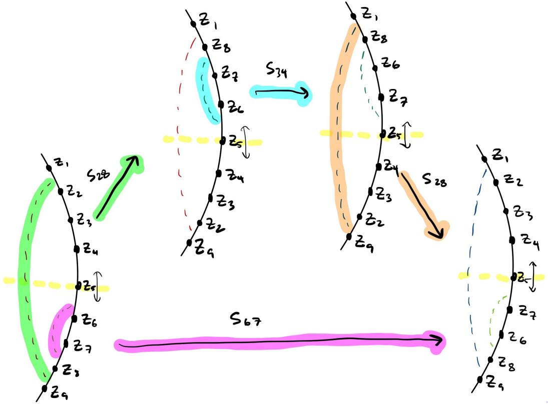
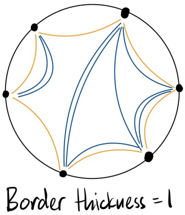
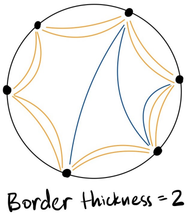


Results

Theorem (Borodin 2023)

Border thickness is an invariant of this group action. When $n = 3$, border thickness is the only invariant so all diagrams with the same border thickness lie in the same orbit.

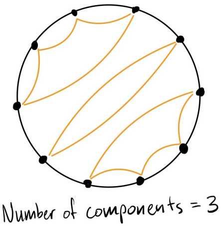




Results

Theorem (Borodin 2023)

The orbits over the set $X(2, 2, \ldots, 2)$ are completely characterized by the number of components. That is, there is exactly one orbit for every possible number of components. In particular, there is a total of $\lfloor n/2 \rfloor$ orbits.



Results

Theorem (Borodin 2023)

The cactus group J_n acts transitively on the set $X(\ell_1, \ell_2, \ldots, \ell_n, \ell_\infty)$ when there exists some $\ell_i = 1$ (or $\ell_\infty = 1$).

Theorem (Borodin 2023)

When the cactus group J_n acts on the set $X(\ell_1, \ell_2, \ldots, \ell_n, \ell_\infty)$, the braid relation $s_{i,i+1}s_{i-1,i}s_{i,i+1} = s_{i-1,i}s_{i,i+1}s_{i-1,i}$ is always satisfied.

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