On the Spum and Sum-Diameter of Paths

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Sum Graphs

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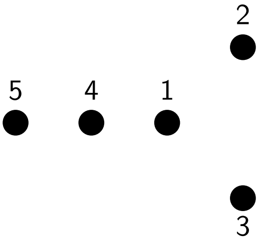

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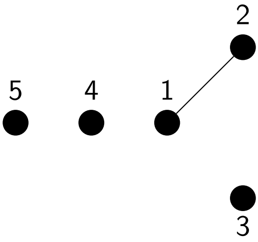

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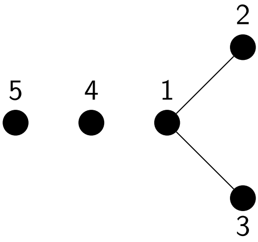

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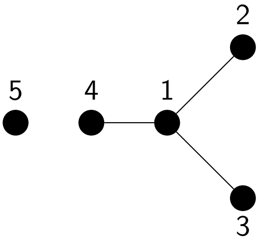

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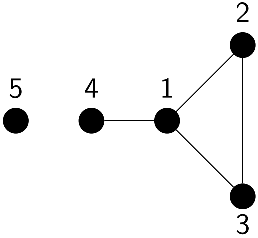

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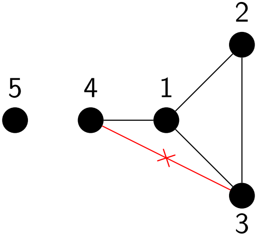

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Sum Graphs and Sum Graph Labelings

Sum Graphs (Harary, '90)

- The *sum graph* $G(V, E)$ *with sum graph labeling* $L \subseteq \mathbb{Z}^+$ is given by $V = L$ and $(u, v) \in E$ if and only if $u + v \in L$.

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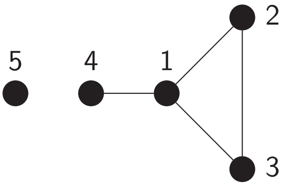
Example: Sum Graph Labeling of G



$L = [1, 2, 3, 4, 5]$ is a sum graph labeling of G

The Existence of a Sum Graph Labeling

Natural Question

- Does every graph have a sum graph labeling?

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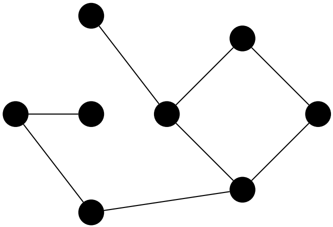

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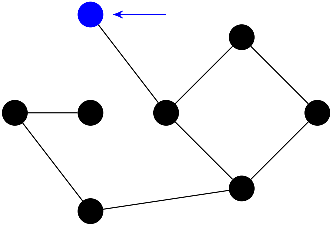

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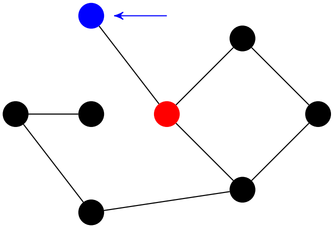

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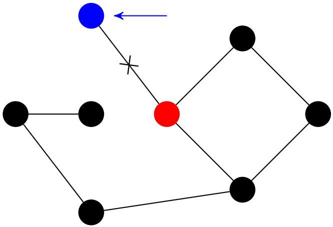

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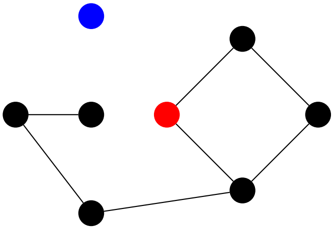

- No connected graph is a sum graph.

Lower Bound on Isolated Vertices

Theorem (Harary, '90)

- For any G, there is a finite $\sigma(G)$ such that $G \cup I_{\geq \sigma(G)}$ is a sum graph.

Lower Bound on Isolated Vertices

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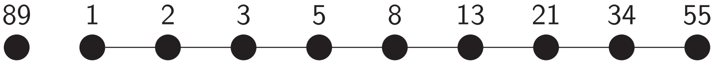
Example: $\sigma(P_9) = 1$



Theorem (Harary, '90)

- It holds that $\sigma(P_n) = 1$.

Upper Bound on Isolated Vertices

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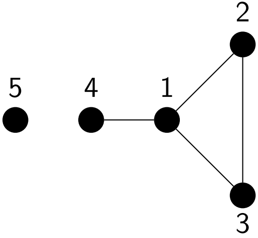

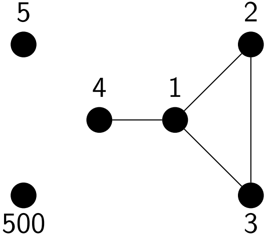

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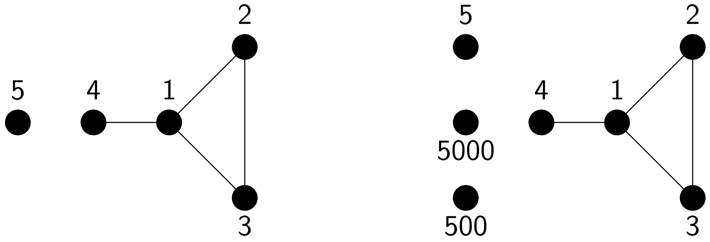
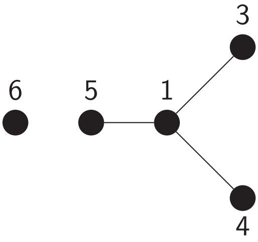
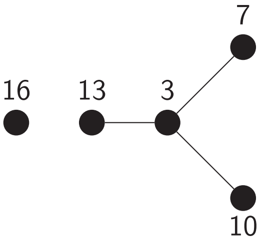


Labelings with the Smallest Range

Motivation: Sum graph labelings are not unique



$L = [1, 3, 4, 5, 6]$



$L = [3, 7, 10, 13, 16]$

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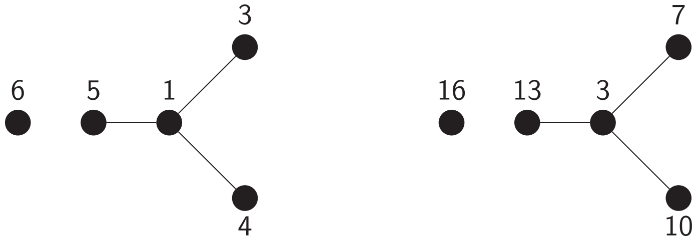

$L = [1, 3, 4, 5, 6]$ $L = [3, 7, 10, 13, 16]$

Natural question

- What is the smallest possible range $(\max - \min)$ of the labels?

$\mathrm{Spum}(G)$

Spum (Goodell et al., '90)

- The minimum $\mathrm{range}(L)$ over all sum graphs $G \cup I_{\sigma(G)}$ with labels L.

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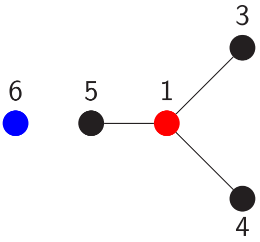
Example: $\mathrm{spum}(G) = 6 - 1 = 5$



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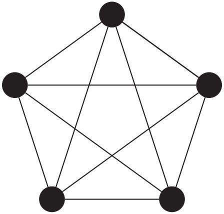
Complete Graphs K_n

Example: K_5



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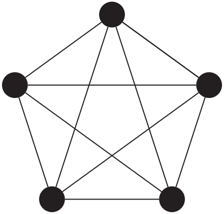


Theorem (Bergstand et al, '89)

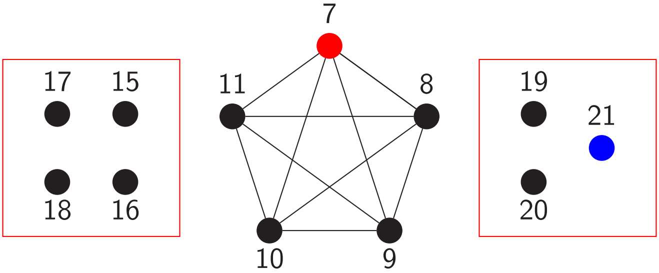
- It holds that $\sigma(K_n)$ is $2n - 3$.

Theorem (Li, '22)

- It holds that $\mathrm{spum}(K_n)$ is $4n - 6$.

Complete Graphs K_n

Example: $\mathrm{spum}(K_5) = 4 \times 5 - 6 = 14$



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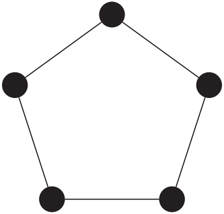

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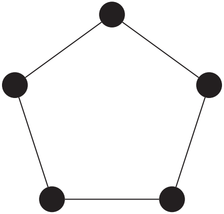


Theorem (Fernau, Ryan, and Sugeng, '08)

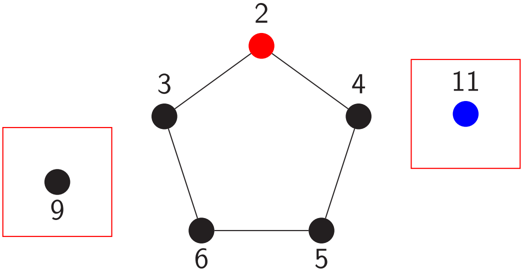
- It holds that $\sigma(C_n) = 2$.

Theorem (Li, '22)

- It holds that $\mathrm{spum}(C_n) = 2n - 1$.

Cycles C_n

Example: $\mathrm{spum}(C_5) = 2 \times 5 - 1 = 9$



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Stars $K_{1,n}$

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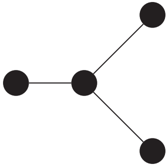

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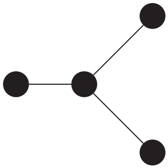

Theorem (Ellingham, '93)

- The sum number of any tree is 1.

Theorem (Singla, Tiwari and Tripathi, '21)

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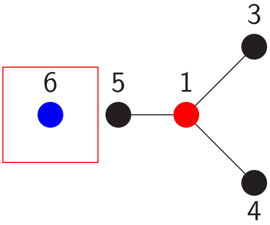

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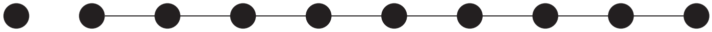
The Sum Number of Paths P_n

Example: P_9



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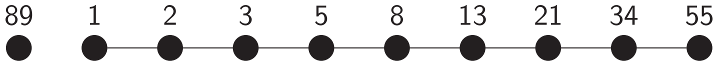

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The Spum of Paths P_n

Theorem (Singla, Tiwari, and Tripathi, '21)

It holds that

$$\mathrm{spum}(P_n) \in \begin{cases} [2n-3, 2n+1] & \text{if } n \geq 9 \text{ is odd} \\ [2n-3, 2n+2] & \text{if } n \geq 9 \text{ is even} \end{cases}.$$

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The Spum of Paths P_n

Theorem (B.C.T., '23)

It holds that

$$\mathrm{spum}(P_n) = \begin{cases} 2n-3 & \text{if } 3 \leq n \leq 6 \\ 2n-2 & \text{if } n = 7 \\ 2n-1 & \text{if } n \geq 8 \text{ is even} \\ 2n+1 & \text{if } n \geq 9 \text{ is odd} \end{cases}.$$

Integral Sum Number

Natural Question

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Integral Spum (Singla, Tiwari, and Tripathi, '21)

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Integral Sum Number

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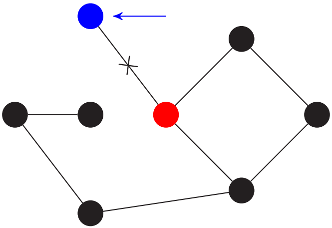
Does our argument for $\sigma(G)$ work for $\zeta(G)$?



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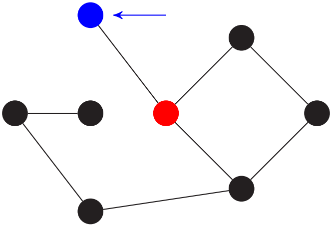
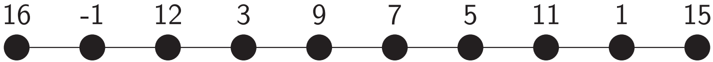


Example: $\zeta(G) = 0$ for P_{10}



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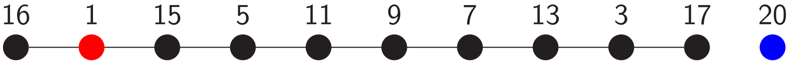

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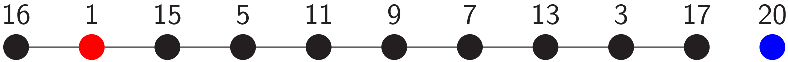
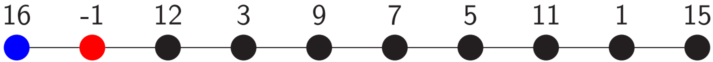


Example: $\mathrm{ispum}(P_{10}) = 16 - (-1) = 17$



Integral Spum of Paths

Theorem (Singla, Tiwari, and Tripathi, '21)

If $n \geq 7$, then $2n - 5 \leq \mathrm{ispum}(P_n) \leq \begin{cases} 2n-3 & \text{if } n \text{ is even} \\ \frac{5}{2}(n-3) & \text{if } n \text{ is odd} \end{cases}$.

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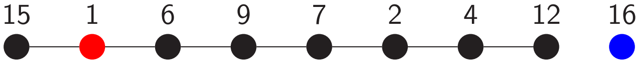
Example: $\mathrm{spum}(P_8) = 16 - 1 = 15$



Example: $\mathrm{sd}(P_8) = 21 - 7 = 14$

Sum-Diameter and Integral Sum-Diameter

Natural Question

- What if we allow an arbitrary number of isolated vertices **and** allow for $L \subseteq \mathbb{Z}$?

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Integral Sum-Diameter (Li, '22)

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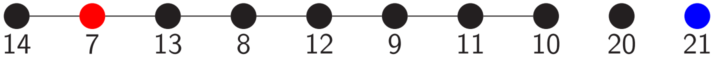
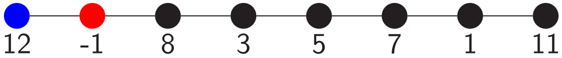
Integral Sum-Diameter (Li, '22)

- The $\mathrm{isd}(G)$ is the minimum $\mathrm{range}(L)$ over all $G \cup I_{\geq \zeta(G)}$ with labels $L \subseteq \mathbb{Z}$.

Example: $\mathrm{sd}(P_8) = 21 - 7 = 14$



Example: $\mathrm{isd}(P_8) = 12 - (-1) = 13$



spum, sd, ispum, and isd

	$L \subseteq \mathbb{Z}^+$	$L \subseteq \mathbb{Z}$
Minimum Number of Isolated Vertices	spum	ispum
Arbitrary Number of Isolated Vertices	sd	isd

Results on Sum-Diameter

Proposition (Li, '22)

If $n \geq 3$, then $2n - 3 \leq \mathrm{sd}(P_n) \leq 2n - 2$.

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Results on Integral Sum-Diameter

Proposition (Li, '22)

If $n \geq 3$, then $2n - 5 \leq \operatorname{isd}(P_n) \leq \begin{cases} 2n-2 & \text{if } n \text{ is odd} \\ 2n-3 & \text{if } n \text{ is even} \end{cases}$.

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Theorem (B.C.T., '23)

If $n \geq 27$, then $\operatorname{isd}(P_n) = \begin{cases} 2n-2 & \text{if } n \text{ is odd} \\ 2n-3 & \text{if } n \text{ is even} \end{cases}.$

Conclusion

Best Known Bounds for $n \geq 27$

	$L \subseteq \mathbb{Z}^+$	$L \subseteq \mathbb{Z}$
Minimum Number of Isolated Vertices	$\mathrm{spum}(P_n) \subseteq \begin{cases} [2n-2, 2n-1] & \text{for even } n \\ [2n-2, 2n+1] & \text{for odd } n \end{cases}$ (Li, '22)	
Arbitrary Number of Isolated Vertices		

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Arbitrary Number of Isolated Vertices	$\mathrm{sd}(P_n) \subseteq [2n-3, 2n-2]$ (Li, '22)	

Conclusion

Best Known Bounds for $n \geq 27$

	$L \subseteq \mathbb{Z}^+$	$L \subseteq \mathbb{Z}$
Minimum Number of Isolated Vertices	$\mathrm{spum}(P_n) \subseteq \begin{cases} [2n-2, 2n-1] & \text{for even } n \\ [2n-2, 2n+1] & \text{for odd } n \end{cases}$ (Li, '22)	$\mathrm{ispum}(P_n) \subseteq \begin{cases} [2n-5, 2n-3] & \text{for even } n \\ [2n-5, \frac{5}{2}(n-3)] & \text{for odd } n \end{cases}$ (Singla, Tiwari, and Tripathi, '21)
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Conclusion

Theorems (B.C.T., '23)

	$L \subseteq \mathbb{Z}^+$	$L \subseteq \mathbb{Z}$
Minimum Number of Isolated Vertices	$\text{spum}(P_n) = \begin{cases} 2n-1 & \text{for even } n \\ 2n+1 & \text{for odd } n \end{cases}$	
Arbitrary Number of Isolated Vertices		

Conclusion

Theorems (B.C.T., '23)

	$L \subseteq \mathbb{Z}^+$	$L \subseteq \mathbb{Z}$
Minimum Number of Isolated Vertices	$\text{spum}(P_n) = \begin{cases} 2n-1 & \text{for even } n \\ 2n+1 & \text{for odd } n \end{cases}$	$\text{ispum}(P_n) \begin{cases} = 2n-3 & \text{for even } n \\ \in [2n-2, \frac{5}{2}(n-3)] & \text{for odd } n \end{cases}$
Arbitrary Number of Isolated Vertices		

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Arbitrary Number of Isolated Vertices	$\mathrm{sd}(P_n) = 2n - 2$	

Conclusion

Theorems (B.C.T., '23)

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Conclusion

Theorems (B.C.T., '23)

	$L \subseteq \mathbb{Z}^+$	$L \subseteq \mathbb{Z}$
Minimum Number of Isolated Vertices	$\mathrm{spum}(P_n) = \begin{cases} 2n - 1 & \text{for even } n \\ 2n + 1 & \text{for odd } n \end{cases}$	$\mathrm{ispum}(P_n) \begin{cases} = 2n - & \text{for even } n \\ \in [2n - 2,\ \frac{5}{2}(n - 3)] & \text{for odd } n \end{cases}$
Arbitrary Number of Isolated Vertices	$\mathrm{sd}(P_n) = 2n - 2$	$\mathrm{isd}(P_n) = \begin{cases} 2n - 3 & \text{for even } n \\ 2n - 2 & \text{for odd } n \end{cases}$

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References

References

- J. Goodell et al. Sum graphs. *unpublished* (1990).
- F. Harary. Sum graphs and difference graphs. *Congr. Numer.* 72 (1990), pp. 101–108.
- F. Harary. Sum graphs over all integers. *Discrete Math.* 124 (1994), pp. 99–105.
- R. Li. The Spum and Sum-Diameter of Graphs: Labelings of Sum Graphs. *Discrete Math.* 345.5 (2022), p. 112806.
- S. Singla, A. Tiwari, and A. Tripathi. Some results on the spum and the integral spum of graphs. *Discrete Math.* 344 (2021), P3.32.