Improved Bounds on Helly Numbers of Exponential Lattices

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Under the Guidance of Travis Dillon
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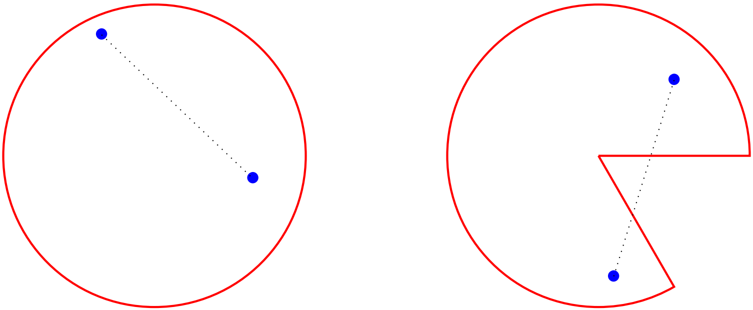
Convexity

Definition

A set $S \subseteq \mathbb{R}^d$ is *convex* if for any u and v in S, every point on the segment between u and v is in S.



Helly's Theorem

Theorem (Helly, 1923)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $d + 1$ or fewer sets in $\mathcal{F}$ have nonempty intersection, then all sets in $\mathcal{F}$ have nonempty intersection.

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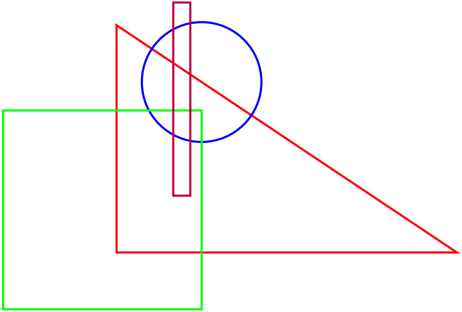
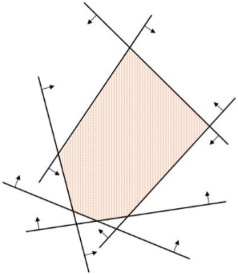
Example



Helly's Theorem

Example (Feasibility of a Linear Program)

To check whether or not n linear contraints can be simultaneously satisfied, it suffices to check whether every $d + 1$ constraints can be simultaneously satisfied.



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[1]Computational Geometry, WS 2007/08, Dr. Thomas Ottmann

Helly's Theorem

Can the bound be improved?

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In particular, can we replace $d+1$ with d?

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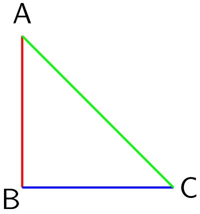

Every 2 edges intersect at a vertex, but not all edges intersect.

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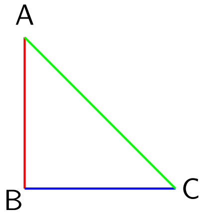

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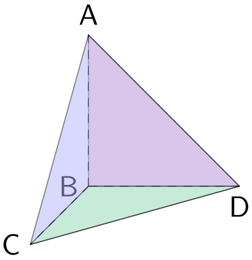

Every 3 faces intersect at a vertex, but not all faces intersect.

Doignon's Theorem

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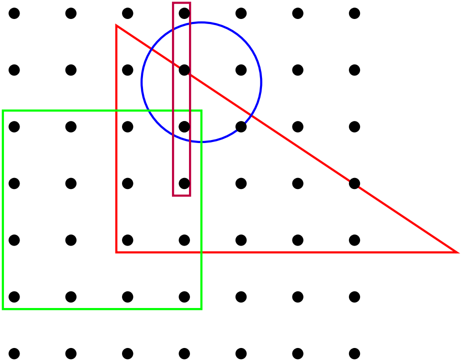
Theorem (Doignon, 1973)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $\mathbf{2^d}$ or fewer sets in $\mathcal{F}$ intersect at a **lattice** *point, then all sets in $\mathcal{F}$ intersect at a* **lattice** *point.*

Example



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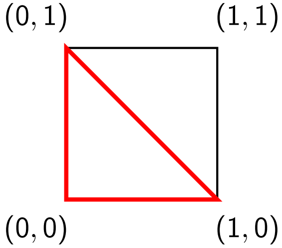

Every 3 triangles of the above form intersect at a lattice point, but all 4 do not.

Doignon's Theorem

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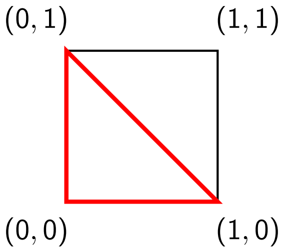

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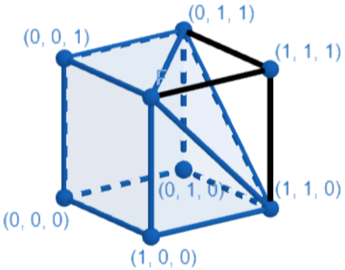


Every 7 polytopes of the above form intersect at a lattice point, but all 8 do not.

Helly Numbers

Definition

Given a set $S \subseteq \mathbb{R}^d$, the *Helly number* of S, denoted $h(S)$, is the smallest h such that the following *Helly-type theorem* holds:

> *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every h or fewer sets in $\mathcal{F}$ intersect at a point in S, then the intersection of all sets in $\mathcal{F}$ contains a point in S.*

If no such h exists, we say $h(S) = \infty$.

Helly Numbers

Theorem (Helly, 1923)

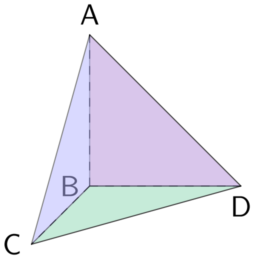
$h(\mathbb{R}^d) = d + 1.$

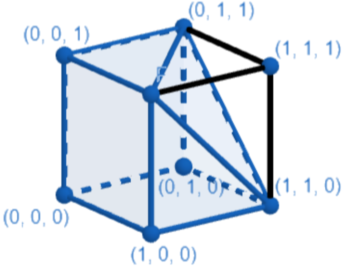
Theorem (Doignon, 1973)

$h(\mathbb{Z}^d) = 2^d.$

Fundamental Results

Recall the examples we used to show that the Helly and Doignon bounds are sharp.





Both constructions involve taking the convex hull of *all but one* vertex of a polytope.

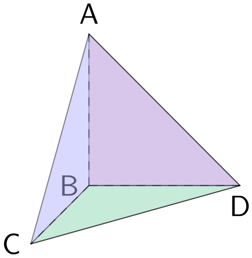
Fundamental Results

Definition

We say $\{x_1, \ldots, x_m\} \in S$ is *intersect-empty* in S if the sets

$$\text{conv}(\{x_1, \ldots, x_m\} \backslash \{x_i\}), \quad i = 1, \ldots, m$$

do not all intersect at a point in S.



Fundamental Results

Theorem

If $S \subseteq \mathbb{R}^d$, then $h(S)$ is equal to the maximum number of vertices of an intersect-empty subset of S.

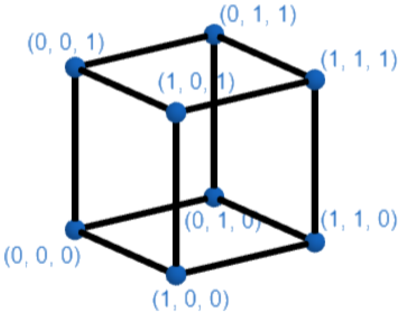
Example

A tetrahedron is a maximal intersect-empty set in $S = \mathbb{R}^3$.

Fundamental Results

Definition

We say a convex set in S is *empty* if the only elements of S in it are its vertices.



Fundamental Results

Theorem

If $S \in \mathbb{R}^d$ and S is discrete[2], then $h(S)$ is equal to the maximum number of vertices of an empty subset of S.

Example

A cube is a maximal empty set in $S = \mathbb{Z}^3$.

[2] Here, we say S is discrete if any bounded region contains finitely many points in S

Exponential Lattices

Question (Dillon,2021)

What is $h(\{2^n : n \in \mathbb{N}\}^2)$?

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Question (Generalization)

Given $\alpha > 1$, *what is* $h(\{\alpha^n : n \in \mathbb{N}\}^2\}$?

Exponential Lattices

Theorem (Ambrus, Balko, Frankl, Jung, and Naszódi, 2023)

Define $L_2(\alpha) = \{\alpha^n : n \in \mathbb{N}\}^2$.

- *If* $\alpha \geq 2$, *then* $h(L_2(\alpha)) = 5$.
- *If* $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$, *then* $h(L_2(\alpha)) = 7$.
- *If* $\alpha \in (1, \frac{1+\sqrt{5}}{2})$, *then* $h(L_2(\alpha)) \leq 3\lceil \log_\alpha(\frac{\alpha}{\alpha-1}) \rceil + 3$.

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Theorem (S.A., 2023)

We have

$$h(L_2(\alpha)) \leq 2\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 3.$$

Acknowledgements

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