Multidisperse RSA and Generalizations

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Gunn High School

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Rényi Parking Process

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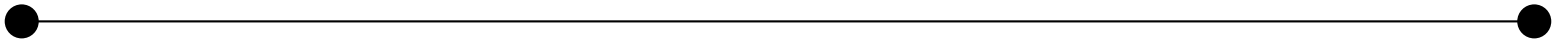
We begin with a length-L road.



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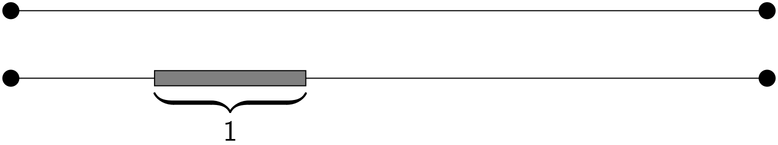

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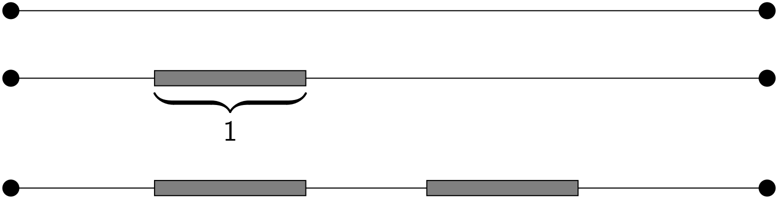

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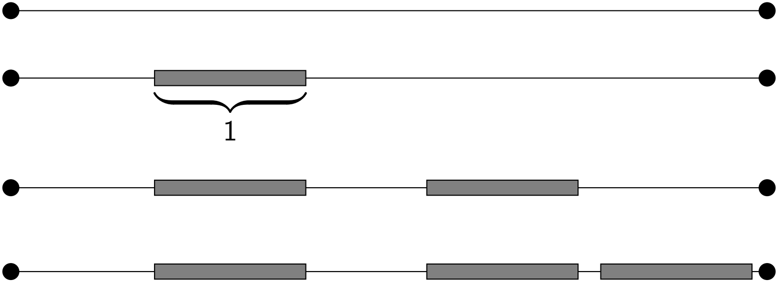

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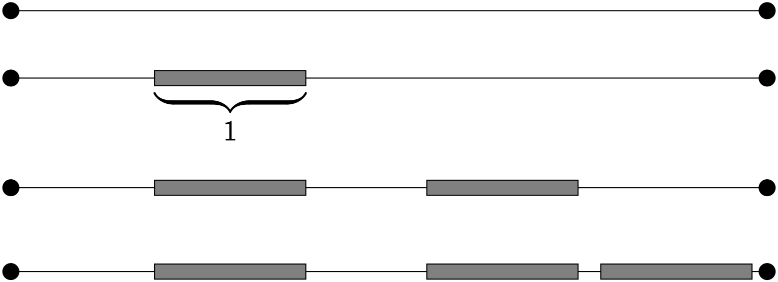

Question

What is the expected number of cars placed at saturation?

Rényi Parking Process

Let $M(L)$ be the expected number of cars on a length-L road.

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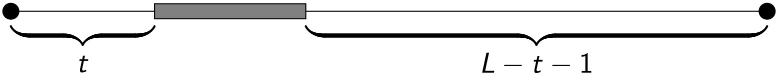

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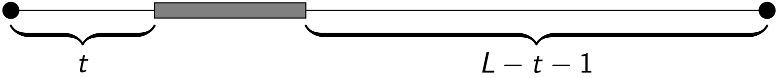

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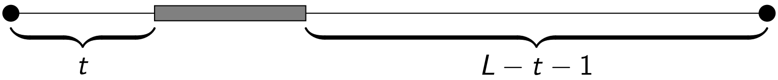

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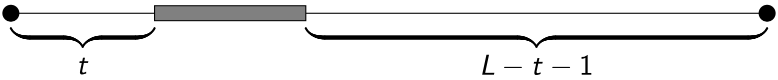

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Step 2: Use Laplace transforms.

Rényi Parking Process

Theorem (Rényi, 1958)

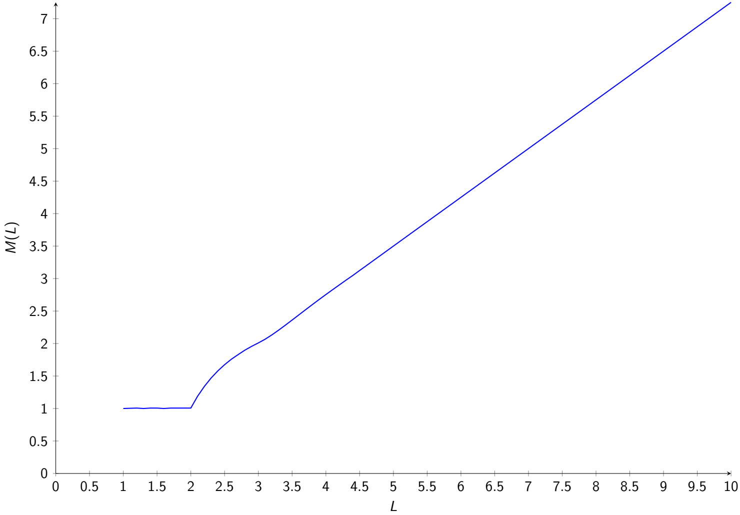
In the Rényi Parking Process, as $L \to \infty$

$$M(L) \sim C_R \cdot L,$$

where $C_R := \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\,\mathrm{d}u\right)\mathrm{d}t \approx 0.747598.$

Rényi Parking Process

$M(L)$ plotted with L:

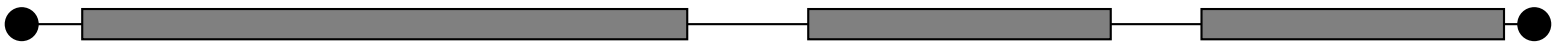


Random Sequential Adsorption

Random Sequential Adsorption (RSA) refers to processes in which 1-dimensional cars are parked onto a road.

Multidisperse RSA

We have n different types of cars with lengths $\ell_1 < \ell_2 < \cdots < \ell_n$ and probabilities $q_1, q_2, \ldots, q_n$.



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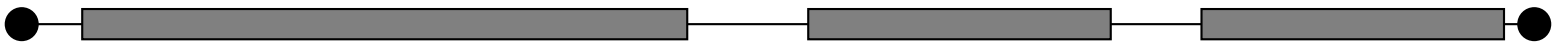

- Subashiev and Luryi (2007) characterized $\lim_{L\to\infty} \frac{M(L)}{L}$ in the $n = 2$ case.
- We characterize $\lim_{L\to\infty} \frac{M(L)}{L}$ in the general $n \geq 3$ case.

Multidisperse RSA Example

When $\ell_1 = 1, \ell_2 = 1.3, \ell_3 = 1.5$ with $q_1 = 0.5, q_2 = 0.3, q_3 = 0.2$,

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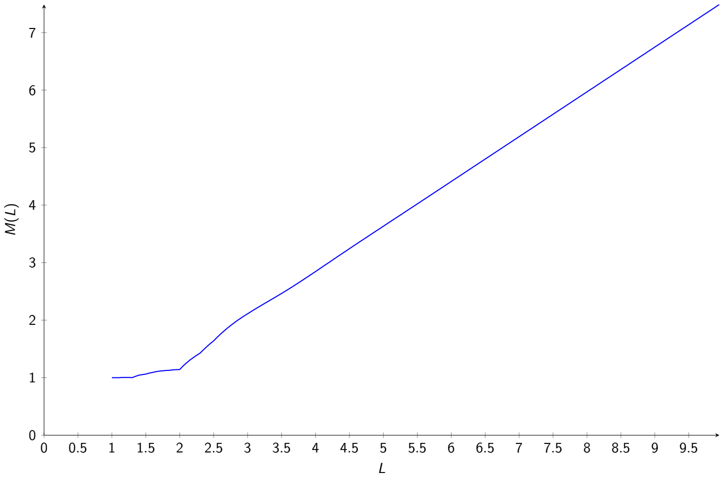

Rényi Parking Process

Multidisperse RSA

General Length Distributions

Power Law RSA

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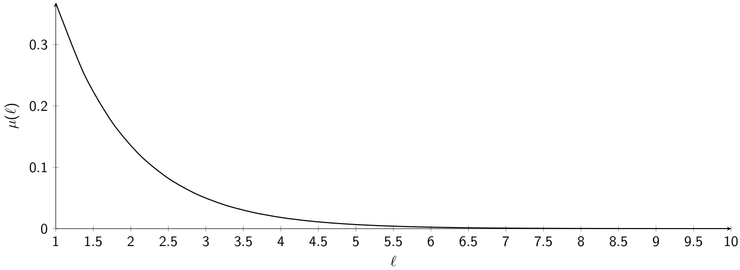

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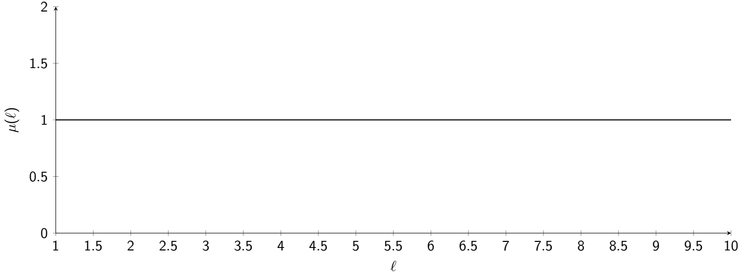

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On a length-L road, convergent distributions weight small cars more, whereas divergent distributions weight large cars more.

Question

Given an length distribution μ, how does $M(L)$ behave?

Convergent LDF

Theorem

When μ is convergent, then under mild conditions[1] *then there is* $\alpha < 1$ *with*

$$M(L) \sim \alpha L.$$

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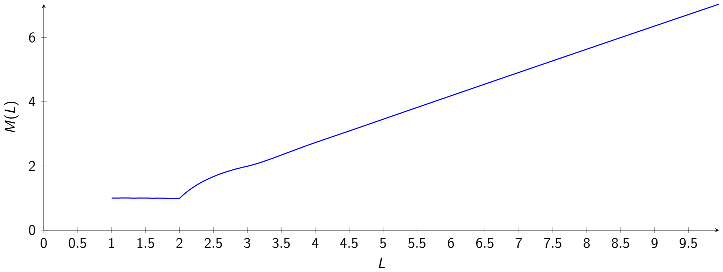

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Divergent LDF

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When μ is divergent, then under mild conditions,[2]

$$M(L) = L - o(L).$$

[2] We require that there exists $\epsilon > 0$ such that for sufficiently large L, $\int_0^L \ell \cdot Z(\ell)\,\mathrm{d}\ell \leq (1-\epsilon) \cdot \int_0^L \frac{L}{2} \cdot Z(\ell)\,\mathrm{d}\ell$,

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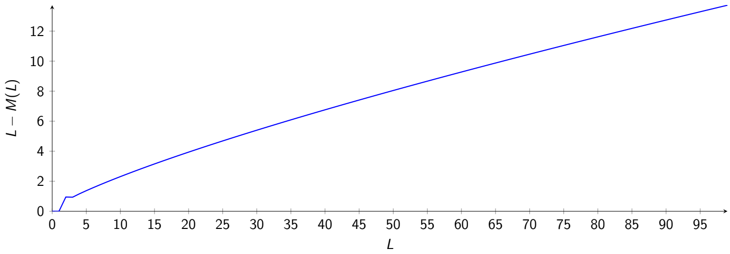

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Conjectures

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$M(L) \sim \alpha L$ for all convergent ldfs, and $M(L) \sim L - o(L)$ for all divergent ldfs.

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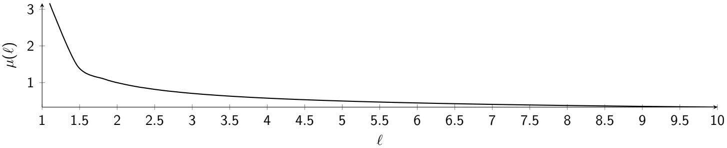

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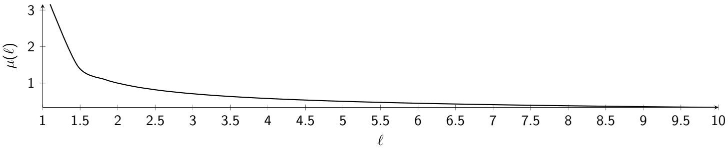

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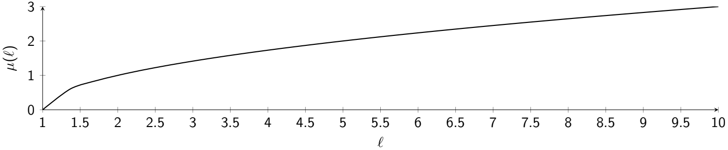

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Theorem

When $\mu(\ell) = (\ell - 1)^2$*, let* $\beta = \frac{\sqrt{33}-5}{2} \approx 0.372$*. Then,*

$$L^{\beta-\epsilon} \ll L - M(L) \leq L^{\beta}.$$

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In general, we proved this for all $\mu(\ell) = (\ell - 1)^p$, with β as the solution to $\Gamma(\beta + p + 3) = 2\Gamma(p + 3)\Gamma(\beta + 1)$.

Acknowledgements

I would like to thank:

- My mentor Nitya Mani for suggesting the topic and for her continuous and invaluable guidance;
- Dr. Khovanova, Dr. Gerovitch, Mr. Dixon, and the rest of the PRIMES organizers for their helpful advice and for making this project possible;
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Appendix: Multidisperse Theorem

Theorem

Given the multidisperse process with lengths $\ell_1, \ldots, \ell_n$ and probabilities $q_1, \ldots, q_n$, define functions P_i and G as

$$P_i(s) := \int_{\ell_n - \ell_i}^{\ell_n} M(L) e^{-sL} \, \mathrm{d}L,$$

$$G(s) := e^{-(\ell_n - \sigma)s} \Big(\sigma + s(\ell_n - \sigma) M(\ell_n) \Big) + 2 s e^{\sigma s} \sum_{i=1}^{n} q_i e^{-\ell_i s} P_i(s).$$

Then,

$$\lim_{L \to \infty} \frac{M(L)}{L} = \int_0^\infty G(t) \exp\left(-2 \sum_{i=1}^{n} q_i \operatorname{Ein}(\ell_i t) \right) \mathrm{d}t.$$