Betti Graphs of Puiseux Monoids

2023 PRIMES October conference

Joshua Jang, Jason Mao, Skyler Mao

(Mentored by Prof. Scott Chapman and Dr. Felix Gotti)

October 14

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Terminology

- We let $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- We let $\mathbb{P}$ denote the set of primes.
- We set $[\![b, c]\!] := \{n \in \mathbb{Z} \mid b \le n \le c\}$.
- For a positive rational q, we let $\mathsf{n}(q)$ and $\mathsf{d}(q)$ be the unique pair of relatively prime positive integers such that $q = \frac{\mathsf{n}(q)}{\mathsf{d}(q)}$.
- For $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $\nu_p(n)$ denote the exponent of the largest power of p dividing n. For $q \in \mathbb{Q}_{>0}$, we set $\nu_p(q) = \nu_p(\mathsf{n}(q)) - \nu_p(\mathsf{d}(q))$.

What is a Puiseux monoid?

Let $M \subseteq \mathbb{Q}_{\geq 0}$. We say $(M, +)$ is a Puiseux monoid if these two conditions hold:

- The set M contains the identity element 0.
- The set M is closed under addition; that is, for all rationals a and b contained in M, their sum $a + b$ is also contained in M.

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Under the operation $+$, the following are examples of Puiseux monoids:

- Naturals (including zero): $\mathbb{N}_0$.
- Nonnegative rationals: $\mathbb{Q}_{\geq 0}$.
- Rationals greater than 1, including zero: $\mathbb{Q}_{>1} \cup \{0\}$.

Atoms of Monoids

Let $(M, +)$ be a Puiseux monoid. We say that a nonzero element $m \in M$ is an atom if whenever we can express $m = a + b$ for $a, b \in M$, we must have $a = 0$ or $b = 0$. Let $\mathcal{A}(M)$ denote the set of atoms.

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Example. The set of all atoms of the Puiseux monoid $M = (\mathbb{N}_0, +)$ is $\mathcal{A}(M) = \{1\}$; observe that 1 is an atom because the only decomposition of 1 is $1 = 0 + 1$, which has a 0 in it.

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Atoms are like the building blocks of our factorizations: our goal is better understand the decomposition of the elements of M into atoms.

Factorizations in Monoid

A factorization of an element $m \in M$ is a formal addition of (not necessarily distinct) atoms $a_1, a_2, \dots a_\ell$ whose sum is m; namely, $m = a_1 + a_2 + \cdots + a_\ell$. We call $\ell \in \mathbb{N}$ the length of the factorization.

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For a set $S = \{a_1, a_2, \ldots, a_n\}$, we write $\langle S \rangle$ to denote the monoid consisting of all linear combinations of the elements of S. The same definition applies when S is an infinite set. For example, $\langle 2, 3 \rangle := \{2x + 3y \mid x, y \in \mathbb{N}_0\}$. Note that $\mathcal{A}(M) \subseteq S$.

Examples of Factorization

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- $4 = 2 + 2$,
- $4 = \frac{4}{3} + \frac{4}{3} + \frac{4}{3}$,
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We see that the element 4 in fact has infinitely many factorizations.

Valuation Monoids

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Example. The Puiseux monoid $M = (\mathbb{Q}_{\geq 0}, +)$ is a valuation monoid, since for any $a, b \in \mathbb{Q}_{\geq 0}$, we choose $c = |a - b| \in \mathbb{Q}_{\geq 0}$, for which either $a = b + c$ or $b = a + c$ must hold.

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Example. The Puiseux monoid $M = (\mathbb{Q}_{>1} \cup \{0\}, +)$ is not a valuation monoid, since $\frac{4}{3}$ and $\frac{5}{3}$ are elements of M, yet $c = \frac{5}{3} - \frac{4}{3}$ is not an element of M.

Antimatter Monoids

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Example. The Puiseux monoid $M = \langle \frac{1}{2^k} \mid k \in \mathbb{N}_0 \rangle$ is antimatter. This is because $\frac{1}{2^k} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}}$, so $\frac{1}{2^k}$ is not an atom for all $k \in \mathbb{N}_0$. However, $\mathcal{A}(M) \subseteq \{\frac{1}{2^k} \mid k \in \mathbb{N}_0\}$, so it follows that $\mathcal{A}(M)$ must be empty.

Motivating Results around Puiseux Monoids

1. Grams used Puiseux monoids to disprove Cohn's conjecture that any atomic domain must satisfy the ACCP.
   - A. Grams: *Atomic domains and the ascending chain condition for principal ideals*, Math. Proc. Cambridge Philos. Soc. **75** (1974) 321–329.
2. Anderson, Anderson, and Zafrullah used Puiseux monoids to find a BFD whose integral closure is not a BFD.
   - D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Factorizations in integral domains*, J. Pure Appl. Algebra **69** (1990) 1–19.
3. Gotti and Li used Puiseux monoids to construct an atomic integral domain whose polynomial ring is not atomic.
   - F. Gotti and B. Li: *Divisibility in rings of integer-valued polynomials*, New York J. Math. **28** (2022) 117–139.

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Betti Graphs and Betti Elements

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Given a Puiseux monoid $(M, +)$, we say that an element $m \in M$ is a Betti element if its Betti graph ∇_m is disconnected; that is, there exist two vertices in ∇_m not connected by a path of edges.

Examples of Betti Elements

Example. Consider the Puiseux monoid $M = \langle 2, 3 \rangle$, the set of all nonnegative integers excluding 1.

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These factorizations do not share an atom, so the graph of ∇_6 consists of two vertices with no edge between them.

Since ∇_6 is a disconnected graph, we see that 6 is a Betti element.

Examples of Betti Elements

Example. For Puiseux monoid $N = \langle 5, 7, 17, 23 \rangle$, we have that 40 is not Betti element, whereas 46 is. The notation (a, b, c, d) represents the factorization $a \cdot 5 + b \cdot 7 + c \cdot 17 + d \cdot 23$.

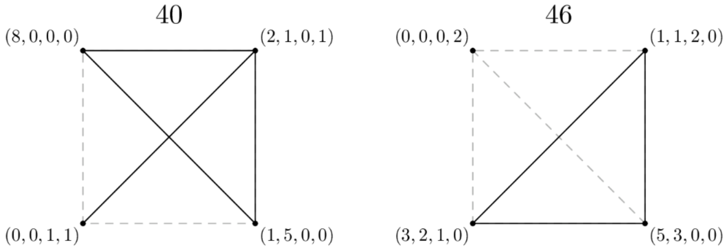

Figure: The figure shows the Betti graph of $40 \notin \text{Betti}(N)$ on the left and that of $46 \in \text{Betti}(N)$ on the right.

Grams Monoid Example

Let $(p_n)_{n \geq 0}$ be the strictly increasing sequence of odd primes. We define the monoid

$$M := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle$$

to be the Grams' monoid, used in the construction of an atomic ring that does not satisfy the ascending chain condition of principal ideals. Its Betti elements are $\{\frac{1}{2^n} \mid n \in \mathbb{N}_0\}$.

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Atomization

Let $(q_n)_{n\geq 1}$ be a sequence of rationals, and let $(p_n)_{n\geq 1}$ be a sequence of pairwise distinct primes such that

$$\gcd(p_i, \mathsf{n}(q_i)) = \gcd(p_i, \mathsf{d}(q_j)) = 1$$

for all $i, j \in \mathbb{N}$. We say that

$$M := \left\langle \frac{q_n}{p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is the Puiseux monoid of $(q_n)_{n\geq 1}$ atomized at $(p_n)_{n\geq 1}$.

Atomization can be used to construct monoids with desired properties.

Canonical Decomposition

Let M be the Puiseux monoid of $(q_n)_{n\geq 1}$ atomized at $(p_n)_{n\geq 1}$, for suitable rationals $(q_n)_{n\geq 1}$ and primes $(p_n)_{n\geq 1}$. Every element $q \in M$ has a unique canonical decomposition

$$q = n_q + \sum_{n\in\mathbb{N}} c_n \frac{q_n}{p_n},$$

where $n_q \in \langle q_n \mid n \in \mathbb{N}\rangle$ and $c_n \in [\![0, p-1]\!]$.

This is an interesting property of atomized monoids that is a key driver behind our results.

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Results on Betti elements

Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)

Let M be the Puiseux monoid resulting from atomizing $(q_n)_{n\geq 1}$ at $(p_n)_{n\geq 1}$. Then the following hold:

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1. *For every $j \in \mathbb{N}$, the factorization $p_j \frac{q_j}{p_j}$ of q_j is an isolated vertex in the Betti graph ∇_{q_j}.*
2. $\mathsf{Betti}(M) \subseteq \langle q_n \mid n \in \mathbb{N} \rangle$.
3. $\{q_n \mid n \in \mathbb{N}\} \subseteq \mathsf{Betti}(M)$ *if* $\langle q_n \mid n \in \mathbb{N} \rangle$ *is an antimatter monoid.*

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1. *For every $j \in \mathbb{N}$, the factorization $p_j \frac{q_j}{p_j}$ of q_j is an isolated vertex in the Betti graph ∇_{q_j}.*
2. $\mathrm{Betti}(M) \subseteq \langle q_n \mid n \in \mathbb{N} \rangle$.
3. $\{q_n \mid n \in \mathbb{N}\} \subseteq \mathrm{Betti}(M)$ *if* $\langle q_n \mid n \in \mathbb{N} \rangle$ *is an antimatter monoid.*
4. $\mathrm{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\}$ *if* $\langle q_n \mid n \in \mathbb{N} \rangle$ *is a valuation monoid.*

Monoids with any number of Betti elements

Here is an application of our results:

Theorem (Chapman, Gotti, Jang, Mao, Mao, 2023)

For each $k \in \mathbb{N}$, we can construct a Puiseux monoid with exactly k Betti elements.

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Sketch of proof. Consider the Puiseux monoid

$$M := \left\langle \frac{1}{p_1}, \frac{2}{p_2}, \dots \frac{k}{p_k}, \frac{1}{p_{k+1}}, \dots, \frac{k}{p_{2k}}, \dots \right\rangle,$$

where $(p_n)_{n \geq 1}$ is an increasing sequence of primes with $p_1 > k$. From the previous result, we can conclude that $\text{Betti}(M) = [\![1, k]\!]$.

Open Research Questions

1. Suppose M is an atomic Puiseux monoid that does not satisfy the ACCP; that is, there exists an infinite sequence of elements $a_1, a_2, a_3, \dots$ of M such that for all integers $i \geq 1$, there exists some nonzero $d_i \in M$ satisfying $a_i = a_{i+1} + d_i$.

   Must M necessarily have infinitely many Betti elements?

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Acknowledgements

- We would like to express our deep gratitude to our PRIMES mentors Prof. Scott Chapman and Dr. Felix Gotti for introducing us to this area of research and for providing this research project and valuable guidance during its completion, as well as Dr. Harold Polo for his helpful comments and feedback.

- We would also like to thank the PRIMES-USA Program for providing us with this opportunity for research.

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Thank you!