Differential Geometry of Curves and Surfaces

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Regular Curves

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$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$$

- Curvature

$$k(s) = |\alpha''(s)|$$

The Isoperimetric Inequality

Let C be a simple closed plane curve with length l, and let A be the area of the region bounded by C. Then $l^2 \geq 4\pi A$, and equality holds if and only if C is a circle.

The Four-Vertex Theorem

A simple closed convex curve has at least four points where $k'(t) = 0$.

Properties of Plane Curves

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Cauchy Crofton Formula

Let C be a regular plane curve with length l. The measure of the set of straight lines (counted with multiplicities) which meet C is equal to $2l$.

Regular Surfaces

Parameterization: $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)),\ (u,v) \in U$, where is an open subset of $\mathbb{R}^2$.

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Definition 2

A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood $V \subset \mathbb{R}^3$ and a parameterization $\mathbf{x} : U \to V \cap S$ such that for each $q \in U$, the differential map $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

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The First Fundamental Form

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Definition 3

Call the following the *area* of bounded region $R \in S$, where S is a regular surface:

$$\iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v| du dv = \iint_Q \sqrt{EG - F^2} du dv, \quad Q = \mathbf{x}^{-1}(R).$$

Second Fundamental Form

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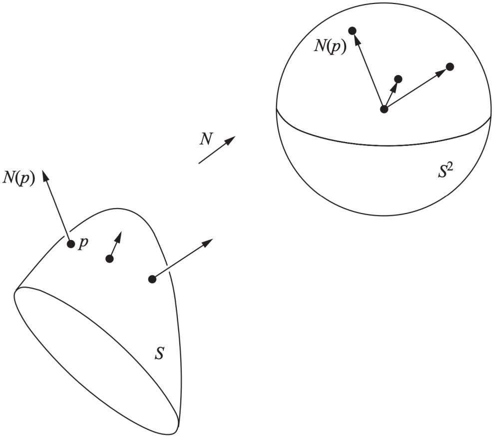

Principal Curvatures

Definition 5

Call $k_1 \geq k_2$ be the principal curvatures of $p \in S$ if $-k_1, -k_2$ are the eigenvalues of dN_p.

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dN_p can be expressed by the first and second fundamental forms.

Gaussian Curvature & Mean Curvature

Definition 6

Define the Gaussian curvature K and the mean curvature H at $q \in S$ as

$$K = k_1 k_2 = \frac{eg - f^2}{EG - F^2}$$

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.$$

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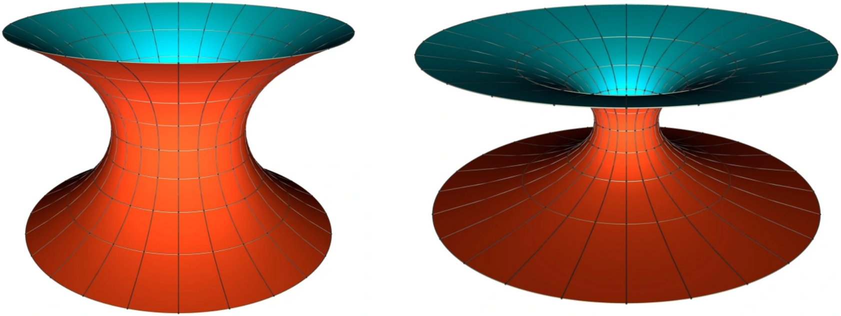
Definition 7

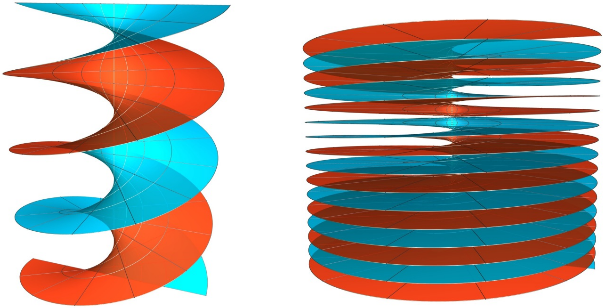
A regular surface is called minimal if $H \equiv 0$.

Minimal Surfaces - Catenoid



Matthias Weber
https://minimalsurfaces.blog/author/matthiasweber64/

Minimal Surfaces - Helicoid



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Definition 8

An isometry is a diffeomorphism $\varphi : S \to \overline{S}$ such that for all $p \in S$ and all $w_1, w_2 \in T_p(S)$, we have

$$\langle w_1, w_2 \rangle = \langle d\varphi_p(w_1), d\varphi(w_2) \rangle$$

The surfaces S and $\overline{S}$ are said to be isometric.

Gauss Theorem

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$$K = \frac{\det \begin{pmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{pmatrix} - \det \begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}}{(EG - F^2)^2}$$

Covariant Derivative

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Definition 10

Consider curve α such that $\alpha(0) = p$ and $\alpha'(0) = y \in T_p(S)$. Let $w(t), t \in (-\epsilon, \epsilon)$ be the restriction of differentiable vector field w to α. The normal projection vector of $(dw/dt)(0)$ onto $T_p(S)$ is the *covariant derivative* at p of w relative to vector y, denoted by $(Dw/dt)(0)$.

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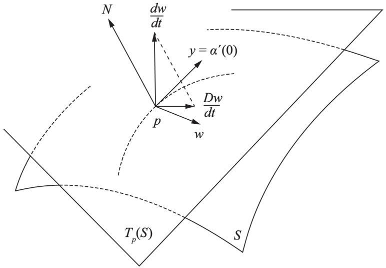


Parallel Transport

Definition 11

Let $\alpha : I \to S$ be a parameterized curve and $w_0 \in T_{\alpha(t_0)}(S), t_0 \in I$. Let w be the vector field along α, such that $w(t_0) = w_0$ and $(Dw/dt) \equiv 0$. The vector $w(t_1), t_1 \in I$, is called the parallel transport of w_0 along α at the point t_1.

Definition 12

A nonconstant curve $\gamma : I \to S$ is a parameterized *geodesic* if

$$\frac{D\gamma'(t)}{dt} \equiv 0, \ \ t \in I.$$

For any parameterized curve $\alpha'(s)$ in a neighborhood of p, the *geodesic curvature* is $k_g(s) := |D\alpha'(s)/ds|$.

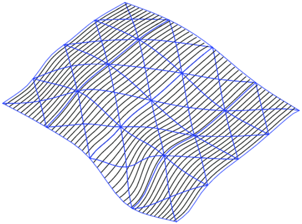
Global Gauss-Bonnet Theorem

Let $R \subset S$ be a regular region and let $C_1, \ldots, C_n$ be the closed, simple, piecewise regular curves which form the boundary of R. Let $\theta_1, \ldots, \theta_p$ be the set of all external angles of boundary. Then,

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)ds + \iint_R K d\sigma + \sum_{l=1}^{p} \theta_l = 2\pi\chi(R),$$

where s is the arc length of C_i and integration over C_i takes the sum of integrals over each regular arc of C_i.

The Euler-Poincaré Characteristic



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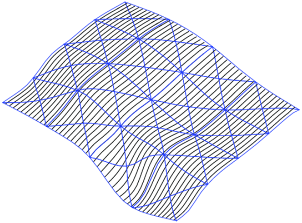

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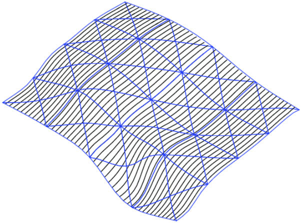

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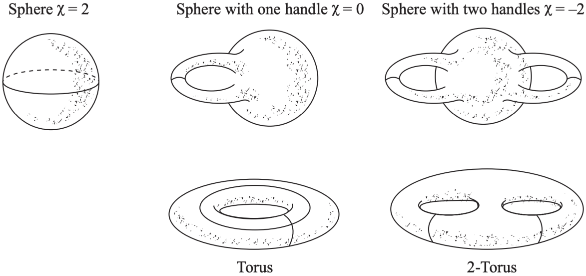

Applications of the Gauss-Bonnet Theorem

- A compact surface of positive curvature is homeomorphic to a sphere.

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- A compact surface of positive curvature is homeomorphic to a sphere.
- The sum of the interior angles of a geodesic triangle is
  1. Equal to π if $K = 0$.
  2. Greater than π if $K > 0$.
  3. Smaller than π if $K < 0$.

Applications of the Gauss-Bonnet Theorem

A *singular point* of a differentiable vector field v on S: $v(p) = 0$. Let ϕ be the angle formed by $\mathbf{x}_u$ and v along a closed curve with p as the only singular point.

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$$2\pi I = \phi(l) - \phi(0) = \int_0^l \frac{d\varphi}{dt} dt$$

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Poincaré Theorem

The sum of the indices of a differentiable vector field v with isolated singular points on a compact surface S is equal to the Euler-Poincaré characteristic of S.

Rigidity of a Sphere

Theorem (Liebmann(1899), later Hilbert & Chern)

Let S be a compact, connected, regular surface with constant Gaussian curvature K. Then S is a sphere.

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Let S be a compact, connected, regular surface with constant Gaussian curvature K. Then S is a sphere.

Theorem (Hilbert & Chern)

Let S be a regular, compact, and connected surface with Gaussian curvature $K > 0$ and constant mean curvature H. Then S is a sphere.

Rigidity of a Sphere

Theorem (Hilbert & Chern)

Let S be a regular, compact, and connected surface of positive Gaussian curvature. If there exists a relation $k_2 = f(k_1)$ in S, where f is a decreasing function of k_1, $k_1 \geq k_2$, then S is a sphere.

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Theorem (Hopf)

A regular surface of constant mean curvature that is homeomorphic to a sphere is a sphere.

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Theorem (Alexandrov)

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Variation of Curves

Definition 13

Let $\alpha(s) : [0, l] \to S$ be a regular parametrized curve. A variation of α is a differentiable map $h : [0, l] \times (\epsilon, \epsilon) \subset \mathbb{R}^2 \to S$ such that

$$h(s, 0) = \alpha(s), s \in (0, l].$$

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$V(s) = (\partial h/\partial t)(s, 0)$, $s \in (0, l]$ is called the variational vector field of h.

1st Variation of Arc Length

Definition 14

Let $h : [0, l] \times (-\epsilon, \epsilon)$ be a proper variation of the curve $\alpha : [0, l] \to S$ and let $V(s)$ be the variational vector field of h. Then

$$L'(0) = \int_0^l \langle A(s), V(s) \rangle \mathrm{d}s,$$

where $A(s) = (D/\partial s)(\partial h/\partial s)(s, 0)$.

2nd Variation of Arc Length

Proposition 2

Let $h : [0, l] \times (\epsilon, \epsilon) \to S$ be a proper variation of a geodesic $\gamma : [0, l] \to S$ such that $\langle V(s), \gamma'(s) \rangle = 0, \ s \in [0, l]$. Let $V(s)$ be the variational vector field of h. Then

$$L''(0) = \int_0^l \left(\left| \frac{D}{\partial s} V(s) \right|^2 - K(s) |V(s)|^2 \right) \mathrm{d}s,$$

where $K(s) = K(s, 0)$ is the Gaussian curvature of S at $\gamma(s) = h(s, 0)$.

Bonnet's Theorem

Theorem (Bonnet)

Let the Gaussian curvature K of a complete surface S satisfy the condition

$$K \geq \delta > 0.$$

Then S is compact and the diameter ρ of S satisfies the inequality

$$\rho \leq \frac{\pi}{\sqrt{\delta}}.$$

Definition 15

The total curvature of a parametrized regular curve α with arc length l and parametrized with respect to arc length is defined as

$$\int_0^l |k(s)| \mathrm{d}s.$$

Fary-Milnor Theorem

The total curvature of a knotted simple closed curve is greater than 4π.

Bibliography

Differential Geometry of Curves and Surfaces: Revised & Updated Second Edition by Manfredo P. Do Carmo

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Thank you for listening!