GONALITY SEQUENCES OF MULTIPARTITE GRAPHS

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Abstract. In this paper, we deal with a particular sequence associated with a graph, the gonality sequence. This gonality sequence is a part of a larger topic of the chip-firing game on a graph G. The gonality sequence of a graph measures how much the degree of a divisor on that graph needs to change in order to increase its rank. The portions of the gonality sequence are known for when the input is greater than the genus. However, there has been little work done to find the first terms of the gonality sequence. In this paper, we partially compute the first terms of the gonality sequence for some complete multipartite graphs. In particular, the ones with all but one partite class having one vertex are analyzed, and here we present some results and further conjectures.

1. Introduction

The notion of a chip-firing game was first introduced concurrently by many independent researchers, including Spencer in [13], Bitar and Goles in [4], and Bak, Tang, and Wiesenfeld in [2]. This first notion of a chip-firing game, known as either the *parallel chip-firing game* or *abelian sandpile model*, was an autonomous game in which vertices in a graph $G(V, E)$ are assigned a nonnegative number of chips and *fire*, or send one chip to each of its neighbors whenever it is able to. Later, Baker and Norine in [3] extended this notion to another game, no longer autonomous, in which each vertex could fire at any time, controlled by an outside entity, and negative chip values were allowed.

This view of the game is much more analogous to real-world examples of similar phenomena. One such example is a chain of banks, with each having an attached profit margin, which would correspond to the chip value on the vertices. The branches of banks additionally have a connection system where some branches are connected to other branches. Through this connection system, the branches are able to send money amongst themselves. However, there is a caveat in this system. This caveat is that when a branch wants to send money to another branch, it must send the same amount through each of its connections. Through this and similar examples, one can see this non-autonomous game is a natural extension of the original game and merits studying.

This view of the chip-firing game introduced the notion of *divisors* in the game, which represents a configuration of chips on the vertices. These divisors defined a linear system

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on the graph similar to those on algebraic curves, which allowed for theorems from algebraic geometry to carry over to this combinatorial setting. Some examples of this include the Riemann-Roch theorem in [3] and Clifford's theorem in [6]. The definition of the gonality sequence also comes from this connection in the linear systems, and is the corresponding definition for most linear systems. We refer to Section 2 for further details.

Cools and Panizzut classify the exact gonality sequence for complete graphs and complete bipartite graphs in [5]. This paper continues on that work, seeking to classify the exact gonality sequence of complete multipartite graphs, the natural extension of Cools and Panizzut's work. In particular, this paper focuses on multipartite graphs with exactly one partite class with size larger than 1. This paper classifies exactly the gonality sequence for such complete tripartite graphs as well as the majority of the gonality sequence for such complete 4-partite graphs. This paper also provides some partial conjectures and progress in further cases.

In Section 2, we discuss the necessary background definitions and information to understand the problem and begin to dissect the problem. Importantly, we present three of the most overarching theorems and algorithms used in the computation process, namely, Dhar's burning algorithm, the Riemann-Roch theorem for graphs, and Clifford's theorem. In Section 3, we go into more specifics as to the exact processes and methodology used in the computation process, both using combinatorial approaches as well as using computers to help aid in forming conjectures. The basic premise for this methodology is to make use of the great power that Riemann-Roch provides for bounding terms. In fact, Riemann-Roch already gives the terms of the sequence at indicies at least the genus of the graph. Therefore, there is only a section at the beginning of the gonality sequence that requires true computation. The idea is then to combinatorially bound the first few terms of the section, then use Riemann-Roch to provide bounds for the last few terms of the section, or vice versa, which narrows the unknown section. This, along with using computer modeling to generate conjectures, is the basic idea of this section. We conclude the section with the application to the tripartite case and classify the exact gonality sequence for complete tripartite graphs with at most two partite classes of size 1. In Section 4, we discuss the application to the 4-partite case and classify the majority of the gonality sequence for these graphs. We also conclude the section with a discussion of partial results and conjectures in other multipartite graphs with exactly one partite class of size greater than 1.

2. Background

We now define the key components necessary for the paper. Let $G(V, E)$ be a connected, undirected graph with the set of vertices V and set of edges E. In the chip-firing game on G, we assign each vertex in the graph an integer number of chips, and we call this assignment a *divisor*. We further define the *degree* of a divisor to be the total number of chips across all vertices. The chip-firing game centers around a move: a vertex *firing*. In this move, we select a vertex to *fire*, and the chosen vertex gives away one chip to each of its neighbors. In fact, we can also define the opposite operation, *anti-firing*, where we do the exact opposite: each of the neighbors gives a chip to the chosen vertex. We say that two divisors are equivalent if and only if one can be reached by another using chip-firing moves. In addition to these chip-firing moves, we define a binary operation on divisors, defining adding two divisors on the same graph to be adding the number of chips on each corresponding vertex to get a new divisor.

We define the Jacobian of a graph by focusing on degree-0 divisors. Since the sum of two degree-0 divisors is another degree-0 divisor, and the divisor with 0 chips at every vertex is the identity element. Since there are also inverses, the degree 0 divisors form a group under this addition operation. However, the notion of two divisors being equivalent under chip-firing creates equivalence classes among the degree 0 divisors. These equivalence classes are what form the Jacobian group of the graph.

Given a graph, there is a second way to define the Jacobian of the graph using only linear algebra. Given the graph, $G(V,E)$, and an ordering of the vertices, namely v_1, v_2, …, v_n, recall that the degree matrix, denoted D, of the graph is the matrix that has the degree of v_i at its ith entry along the main diagonal and is 0 everywhere else. Again recall that the adjacency matrix, denoted A, of the graph is the matrix that has the number of edges connecting v_i and v_j at the entry in column i and row j. Further recall that the Laplacian matrix is the difference of these two matrices, $D-A$. The Jacobian is then the torsion portion of the cokernel of the Laplacian. To find the structure of the Jacobian easily, we can use the Smith Normal Form of the Laplacian. The Smith Normal Form of the Laplacian will have the form of a matrix with only entries along the diagonal, with these entries being $s_1|s_2|\cdots|s_L$, where L is the rank of the Laplacian. The Jacobian is then the group $(\mathbb{Z}/s_1\mathbb{Z})\oplus(\mathbb{Z}/s_2\mathbb{Z})\oplus\cdots\oplus(\mathbb{Z}/s_L\mathbb{Z})$. This method gives us a more reasonable way to compute the Jacobian of any given graph without needing to resort to complex computations with chip-firing.

This paper pertains to the gonality sequence of a given graph. To define the gonality sequence, we need a few terms. We call a divisor *effective* if all the vertices have a nonnegative number of chips on them. We denote the set of effective divisors with degree r as eff_r. Furthermore, we say that the set of effective divisors that are equivalent to a given divisor D is called the *linear system* of D, and is denoted as $|D|$. Now, we define the *rank* of a divisor D on a given graph G, denoted $\text{rk}_G(D)$, to be -1 if $|D|=\emptyset$, otherwise $\text{rk}_G(D)=\max(r\in\mathbb{Z}_{\geq 0}:|D-E|\neq\emptyset,$ for all $E\in\text{eff}_r)$. In other words, the rank is the largest degree r such that subtracting all effective divisors of degree r from D results in a divisor that is equivalent to an effective divisor. The gonality sequence is then as follows: the rth term of the gonality sequence, where $r\geq 0$, is the minimum degree such that there is a divisor with that degree with rank r, which is denoted by $\text{gon}_r(G)$. In terms of the bank analogy, this is equivalent to knowing that r dollars will be withdrawn across the banks in the future, and being asked to best distribute money among the banks so that all the banks can be not in debt both today and in the future.

We now discuss a key component of the computational side of this paper, Dhar's burning algorithm. The aim of the algorithm is to tell whether a given divisor on a graph is equivalent to an effective divisor or not. We first provide a few key definitions. Fixing a vertex v in the graph, we call a given divisor *effective away from* v if any other vertex on the graph has a nonnegative number of chips. Furthermore, we call a divisor v*-reduced* if it is effective away from v, and for any subset of vertices not containing v, firing all of the vertices in the subset once will turn the divisor into one that is not effective away from v. The crux of the algorithm is the following lemma.

Lemma 2.1 ([8], [9])**.** *Given a graph $G(V, E)$, fix a vertex $v \in V$. Then, any divisor is equivalent to a v-reduced divisor under chip-firing.*

The algorithm, given a vertex v and a divisor D, finds this v-reduced divisor, and it is a simple consequence that the chip count at v is nonnegative in this v-reduced divisor if and only if D is equivalent to an effective divisor. The algorithm works as follows.

1. First find a divisor equivalent to D that is effective away from v. The easiest way to do this is as follows:
   (a) Find the size of the Jacobian, and denote it as m.
   (b) For every vertex with a negative number of chips that is not v, successively add m chips to it and take m chips away from v. The reason why this works is because the order of any element in a finite abelian group divides the size of the group.
2. Burn the vertex v.
3. Iterate through the remaining vertices and burn any vertex that has strictly more burnt neighbors than number of chips. Repeat until no more vertices burn.
4. If the set of unburnt vertices is nonempty, fire each of the vertices in this set, and restart the process. If the set of unburnt vertices is empty, the v-reduced divisor is found.

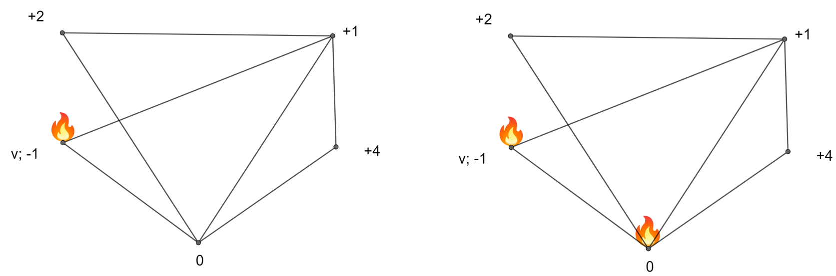
To better visualize how this algorithm works, we provide an example iteration of steps 2, 3, and 4 in Figure 1 and Figure 2.

This algorithm is the central algorithm for the computational testing, as it can in a reasonably short time and effort, determine whether the linear system of a given divisor D is empty or not.

Now, we introduce some more useful terminology and a couple of useful theorems.

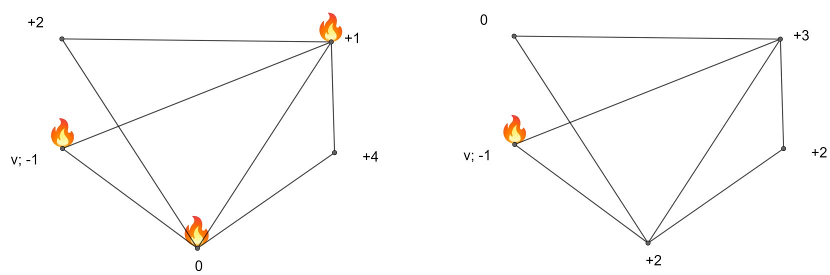
Definition 2.2. For a graph G, with set of vertices V and set of edges E, define the genus g of the graph to be $g = |E| - |V| + 1$.

Definition 2.3. We call the canonical divisor on a graph G, denoted as $\mathscr{K}$, as the divisor where at a vertex V of G, the number of chips at V is $\deg(V) - 2$.



(A) Step 2 (B) Step 3

FIGURE 1. Steps 2 and 3 of Dhar's Visualized



(A) Step 3 (B) Step 4

FIGURE 2. Steps 3 and 4 of Dhar's Visualized

Definition 2.4. Given a graph G and a divisor D on G, the support of D, denoted $\operatorname{supp}(D)$, is the set of vertices of G such that D has a positive number of chips on these vertices.

The following theorem is useful for producing upper bounds on certain portions of the gonality sequences of a graph G.

Theorem 2.5 (Riemann-Roch for graphs, [3])**.** *For any divisor D on a graph G, we have that* $\operatorname{rk}_G(D) - \operatorname{rk}_G(\mathscr{K} - D) = \deg(D) - (g-1)$.

The following theorem, paired with Theorem 2.5, is useful in eliminating lots of possibilities for the gonality sequence for graphs.

Theorem 2.6 (Clifford's theorem, [6])**.** *If a graph G has genus $g \geq 2$, then the r-th term of the gonality sequence of G is such that* $\operatorname{gon}_r G \geq 2r$ *for* $1 \leq r \leq g$.

3. Methodology for Calculation

Many of the results and conjectures that we will present in this section come from extensive testing using computers and Dhar's burning algorithm. The general premise of finding the r-th term in the gonality sequence of a graph G with n vertices is to first guess what the term would be, say this guess is k. Then, we go through all possible effective divisors of degree k, and for each divisor, subsequently subtract off each effective divisor of degree r. We then test whether these divisors are equivalent to an effective divisor using Dhar's burning algorithm, and if one of the original divisors of degree k produces all divisors equivalent to an effective divisor, we say that $k \geq \text{gon}_r(G)$. Otherwise, $k < \text{gon}_r(G)$. Due to the fact that generating all effective divisors of a given degree is equivalent to generating all partitions of the degree into n parts, this gets much slower as n gets larger. Thus, we cannot test many graphs.

Now, there is a step in the above that is very hard to do if we do not know the gonality number, which is guessing k. In addition to Theorem 2.6, there are a few lemmas that help with this.

Lemma 3.1. *The gonality sequence is strictly increasing, that is,* $\text{gon}_r(G) < \text{gon}_{r+1}(G)$.

Proof. It is clear that the gonality sequence is nondecreasing. To prove that it is strictly increasing, assume for contradiction that at some point, $\text{gon}_r(G) = \text{gon}_{r+1}(G)$. Then, let D be a divisor of degree $\text{gon}_{r+1}(G)$ that has rank $r+1$. Take a vertex of G that is in the support of D and take a chip away from that vertex. Then, we have a divisor of degree $\text{gon}_{r+1}(G) - 1$ with rank at most r, which contradicts the minimality of $\text{gon}_r(G)$. Thus, the gonality sequence must be strictly increasing. □

Lemma 3.2. *The gonality sequence also satisfies* $\text{gon}_{r+1}(G) \leq \text{gon}_1(G) + \text{gon}_r(G)$.

Proof. It is clear that for two divisors, D_1 and D_2, $\text{rk}_G(D_1) + \text{rk}_G(D_2) \leq \text{rk}_G(D_1 + D_2)$. The result follows. □

These two lemmas provide a nice bounding range for the possibilities of k in the numerical calculations, which is very nice. Now, we apply this to our specific set of graphs, which is multipartite graphs. In fact, we narrow this set of graphs even further, down to multipartite classes with exactly one partite class with size greater than one. We define the following notation:

Definition 3.3. For a multipartite graph $K_{n,1,\ldots,1}$, where there are m partite classes of size 1, we denote this by $K_{n,(m)}$.

In Lemma 3.2, we also note there is a $\text{gon}_1 G$ term, which is not helpful if we cannot bound $\text{gon}_1 G$ by itself. Therefore, the following lemma comes in useful in this fact.

Lemma 3.4 ([1]). *The first term of the gonality sequence,* $\mathrm{gon}_1(G)$*, is at most the edge connectivity of* G*. The edge connectivity is the minimum number of edges one can remove from the graph to disconnect the graph.*

In particular, for our cases of $K_{n,(m)}$, we have the following:

Lemma 3.5. *Denote the complete* $(m+1)$*-partite graph with one partition having* n *vertices and the other* m *having* 1 *vertex as* $K_{n,(m)}$*. Then, the edge connectivity, or the minimum number of edges needed to remove so that the graph is disconnected, is* m*.*

Proof. The edge connectivity of a graph is equivalent to finding the number of edges across a given bipartition of a graph, and then taking the minimum of this number across all bipartitions. Thus, we can do casework on the bipartition. There are two groups of vertices in a $K_{n,(m)}$, namely, the group of n vertices, and the group of m vertices. Then, let us say that our bipartition contains a vertices from the first group, and b vertices from the second group. Further assume that $b \leq \frac{n}{2}$, as at least one of the halves of the bipartition must satisfy this. Then, we have that the number of edges between the two halves of the bipartition is $a(m-b)+b(m-b)+b(n-a) = am+bm+bn-2ab-b^2 = a(m-2b)+bm+bn-b^2$. Thus, for a given b, as $m-2b$ is positive, in order to minimize this expression, it is equivalent to minimizing a. Now, if $b \neq 0$, then this is achieved when $a=0$, and the expression becomes $bm+bn-b^2 \geq bm \geq m$, as both b and $n-b$ are nonnegative. Further, if $b=0$, then the minimum a is 1, since it is impossible to take a partition where a and b are both 0. This gives the expression as m, and therefore the edge connectivity is m, as desired. □

Furthermore, we claim that the first term of the gonality sequence for $G = K_{n,(m)}$ is exactly m. In order to show this, we simply take the divisor with m chips on one of the vertices in the partite set of size n. Note that by firing at this vertex and anti-firing at another of the vertices in the partite set, we can move these m chips between the vertices in this partite set. This takes care of subtracting any divisor of degree 1 with the chip in this partite set. Meanwhile, if the chip is in one of the m other partite sets, we simply fire at the original vertex, and this takes care of that case as well. Thus, since the edge connectivity is m and there is a divisor of degree m that works for $\mathrm{gon}_1(G)$, $\mathrm{gon}_1(G) = m$.

Now, we have an additional problem with the algorithm above. It is unclear which r we stop running the algorithm at. Thus, the following lemmas are useful.

Lemma 3.6. *Any divisor of degree at least* g *is equivalent to an effective divisor.*

Proof. This is clear from the statement of Riemann-Roch. □

Lemma 3.7. *Given a graph* G *with* $g \geq 2$*, for any* $r \geq g$*,* $\mathrm{gon}_r(G) = r+g$*.*

Proof. At $r=g$, Lemma 3.6 and Theorem 2.6 combine to prove that $\mathrm{gon}_g(G) = 2g$. After this, the bounds given by Lemma 3.1 and Lemma 3.6 give the result exactly. □

Now, with the intricacies of the computation methods worked out, we first present an example of the calculation for the tripartite case. A more general version where the first gonality number is 2 appears in Proposition 3.6 of [1].

Theorem 3.8 ($K_{n,1,1}$, [1])**.** *If $G = K_{n,1,1}$ is a complete multipartite graph, then the gonality sequence is exactly as follows:*

$$\operatorname{gon}_r(G) = \begin{cases} 2r & r \leq n \\ r+n & r > n. \end{cases}$$

Proof. We have a $K_{n,1,1}$, and computing, we get $g = n$. Additionally, we have that the first term of the gonality sequence is 2, and therefore from iterating Lemma 3.2 multiple times, we get the bound $\operatorname{gon}_r K_{n,(2)} \leq 2r$. However, for $1 \leq r \leq g = n$, Theorem 2.6 also gives us that $\operatorname{gon}_r K_{n,(2)} \geq 2r$. Therefore, we must have that for $1 \leq r \leq g = n$ that $\operatorname{gon}_r K_{n,(2)} = 2r$, and Lemma 3.7 gives us that for $r \geq g = n$, $\operatorname{gon}_r K_{n,(2)} = r + g = r + n$. □

4. The $m = 3$ case

Note that by computation, for a $K_{n,1,1,1}$, $g = 2n+1$ and the first gonality is again 3. From the computational data and analyzing patterns, we conjecture the following for the gonality sequence of G.

Conjecture 4.1. *If $G = K_{n,1,1,1}$ is a complete multipartite graph, the gonality sequence is exactly as follows:*

$$\operatorname{gon}_r(G) = \begin{cases} 3r & r \leq \frac{g}{3} = \frac{2n+1}{3} \\ \lfloor \frac{3}{2}r + \frac{g}{2} \rfloor & \frac{2n+1}{3} < r \leq g = 2n+1 \\ r+g & 2n+1 < r. \end{cases}$$

In order to understand the partial progress for this problem, we need to define some more terminology, in particular what a rank-determining set is.

Definition 4.2. Let A be a subset of the vertices of G. Then, the A-rank of a divisor D is defined the same way as the usual rank, except that there is an additional condition on E: the support of E must be contained in A.

Definition 4.3. A subset A of the vertices of G is known as rank-determining if the A-rank for any divisor is the same as the rank of the divisor.

Now, this is not too useful in practice, as it is hard to check that A-rank and determine whether it is the same or not for all divisors. Thus, we use the following definition and lemma:

Lemma 4.4 ([10], [12])**.** *If a subset A of the vertices of G is such that for any divisor D with A contained in the support and vertex v, there is a divisor D' equivalent to D that contains v in its support, then A is rank-determining.*

Through this definition of a rank-determining set, we can much more easily find and make full use of the special rank-determining set on the $K_{n,(3)}$.

Let $G = K_{n,1,1,1}$. Label the n vertices in one partite class by $u_1, u_2, \ldots, u_n$, and the three remaining vertices by v_1, v_2, v_3.

Observation 4.5. *Note that we can always move multiples of* 3 *chips between the* u_i*'s by performing a firing and then an anti firing move, and we can move multiples of* $n+3$ *chips between* v_i*'s by the same method.*

Observation 4.6. *The set* $\{v_1, v_2, v_3\}$ *is rank determining. This is because, given an effective divisor whose support contains this set, we can perform an anti-fire at any* u_i *to get an equivalent effective divisor for which* u_i *is in the support.*

Observation 4.7. *Every divisor of rank at least* n *on* G *is equivalent to an effective divisor whose support contains every* u_i*. This is because we can subtract off the divisor* $u_1+u_2+\cdots+u_n$ *which has degree* n*, and the remainder must be equivalent to something effective by the rank condition.*

Now, with the rank-determining set found, we are able to do a lot of combinatorial bounding on certain terms of the gonality sequence.

Theorem 4.8. *Suppose* D *is a divisor of rank* r *on* G *that contains each* u_i *in its support. Then* $D + v_1 + v_2 + v_3$ *is a divisor of rank at least* $r + 2$ *on* G*.*

Proof. By Observation 4.6, we need only consider subtracting off effective divisors with support contained in $\{v_1, v_2, v_3\}$. Suppose E is an effective divisor of degree $n + 2$ that has support on at least two vertices. Then $v_1+v_2+v_3-E$ can be written as E_1-E_2 for some effective divisors E_1, E_2 where E_2 has degree at most n. So $D+(v_1+v_2+v_3-E)$ is equivalent to an effective divisor, since we assumed D has rank at least n.

Otherwise suppose E has support only on one vertex, i.e., $E = (r + 2)v_i$. If $r > n$, then by Observation 4.5 E is equivalent to an effective divisor of the type handled above, so we can assume $r = n$. By Observation 4.7, D is equivalent to an effective divisor D' whose support contains every u_i. Thus $(D' - E) \sim (D - E)$ and $D' + v_1 + v_2 + v_3$ contains every vertex in its support, so we can perform an anti-fire at v_i to get an effective divisor. □

Corollary 4.9. *Suppose* D *is a divisor of rank* $r \geq n$ *on* G*. Then* $D + v_1 + v_2 + v_3$ *is a divisor of rank at least* $r + 2$ *on* G*.*

Proof. By Observation 4.7, D is equivalent to an effective divisor D' whose support contains every u_i. So we can apply Theorem 4.8 to D'. □

Corollary 4.10. *We have* $\mathrm{gon}_r(G) + 3 \geq \mathrm{gon}_{r+2}(G)$ *for* $r \geq n$*.*

Lemma 4.11. *The divisor* $D_{2k} = \sum_{i=1}^{n} u_i + k\sum_{j=1}^{3} v_j$ *has rank at least* $2k$*.*

Proof. We first check this for $k = 1$. If 2 chips are subtracted from any vertex, then we can perform an anti-fire at this vertex since the divisor is positive elsewhere. Otherwise, we subtract off at most 1 for any vertex, in which case the remainder is effective. So indeed, the rank is at least 2.

By Theorem 4.8, it follows that $D_{k+1} = D_k + v_1 + v_2 + v_3$ has rank at least $r(D_k) + 2$. So the result is obtained via induction. □

Lemma 4.12. *The divisor* $D_{2k+1} = \sum_{i=2}^{n} u_i + (k+1)\sum_{j=1}^{3} v_j = \sum_{i=2}^{n} u_i + 3u_1 + k\sum_{j=1}^{3} v_j$ *has rank at least* $2k + 1$*.*

Proof. We first check this for $k = 1$, restricting to effective divisors on $\{v_1, v_2, v_3\}$ by Observation 4.6. If a 3 is subtracted from any vertex v_j, then we can perform an anti-fire at u_1 and then an anti-fire at v_j since the divisor is positive elsewhere. Otherwise, we subtract off at most 2 for each v_j, in which case the remainder is effective. So indeed, the rank is at least 3.

By Theorem 4.8, it follows that $D_{k+1} = D_k + v_1 + v_2 + v_3$ has rank at least $r(D_k) + 2$. So the result is obtained via induction. □

Theorem 4.13. *We have* $\mathrm{gon}_{2k} \le n + 3k$ *and* $\mathrm{gon}_{2k+1} \le n + 3k + 2$*. Moreover, these inequalities are equalities whenever* $2k \ge n$ *or* $2k + 1 \ge n$*, respectively.*

Proof. The upper bounds are obtained directly from Lemma 4.11 and Lemma 4.12. Since we know $\mathrm{gon}_{g+m} = 2g + m$ for $m \ge 0$, we can use Corollary 4.9 to show the corresponding lower bounds when the rank is at least n. □

Importantly, this last result extends the known region of the conjecture, now to all $r \ge n$ rather than $r \ge g = 2n+1$. Now, we apply the following, slightly modified version of Riemann-Roch that can be easily derived from the expression in Theorem 2.5.

Lemma 4.14 (Modified Riemann-Roch)**.** *For any rank* r*, we have:*
$\mathrm{gon}_r + \mathrm{gon}_{r+(g-1)-\mathrm{gon}_r} = 2g - 2$.

Theorem 4.15. *For* $r \le \frac{n}{2} = \frac{g-1}{4}$*, we have* $\mathrm{gon}_r = 3r$ *exactly.*

Proof. Applying Lemma 4.14 to the equality case for even rank, we have $\mathrm{gon}_{k+g-1-\mathrm{gon}_k} = 2g - 2 - (n + 3k)$, or $\mathrm{gon}_{n-k} = 3n - 3k$ for $k \ge \frac{n}{2}$. □

Now, this narrows down the unkown part of the gonality sequence further to the section between $\frac{g}{4} \le r \le \frac{g}{2}$. Importantly, one conjecture that is useful seems to be Question 4.5 in [1]. Moreover, the authors note that the analogous question is solved as affirmative for smooth curves, however remains open in terms of graphs.

Question 4.16. *If there exist* r *and* s *such that* $\mathrm{gon}_r + \mathrm{gon}_s = \mathrm{gon}_{r+s}$*, is it true that* $\mathrm{gon}_k = k\,\mathrm{gon}_1$ *for all* $1 \le k \le r + s$*?*

This would simplify and tighten the bounds on the current unknown section in our study on the $m = 3$ case, and given that the analogous question is affirmative for smooth curves, it is reasonable that it would be as well for graphs. Figure 3 depicts the total progress on the conjecture, with the middle section still unknown, however, there are some bounds for the middle section in terms of Lemma 3.1 and Lemma 3.2. The solid lines are the known parts and the dotted and shaded sections represent the bounds.

The $m \geq 4$ case is vastly harder due to the increased computational demand required. We have made very little headway in this section, mostly due to the vastly larger amounts of time it takes to run code. We did have a somewhat weak proposition, which is motivated by the previous results. Note that for both $m = 2$ and $m = 3$, the gonality sequences are formed by lines of slope m, then $\frac{m}{3}$, and so on until $\frac{m}{m} = 1$. However, this proposition has broken down quickly with some of the smaller computations in $m = 4$. It seems that the last two slopes of $\frac{m}{m-1}$ and 1 are indeed true as well as the starting slope of m, however, the others do not seem to hold.

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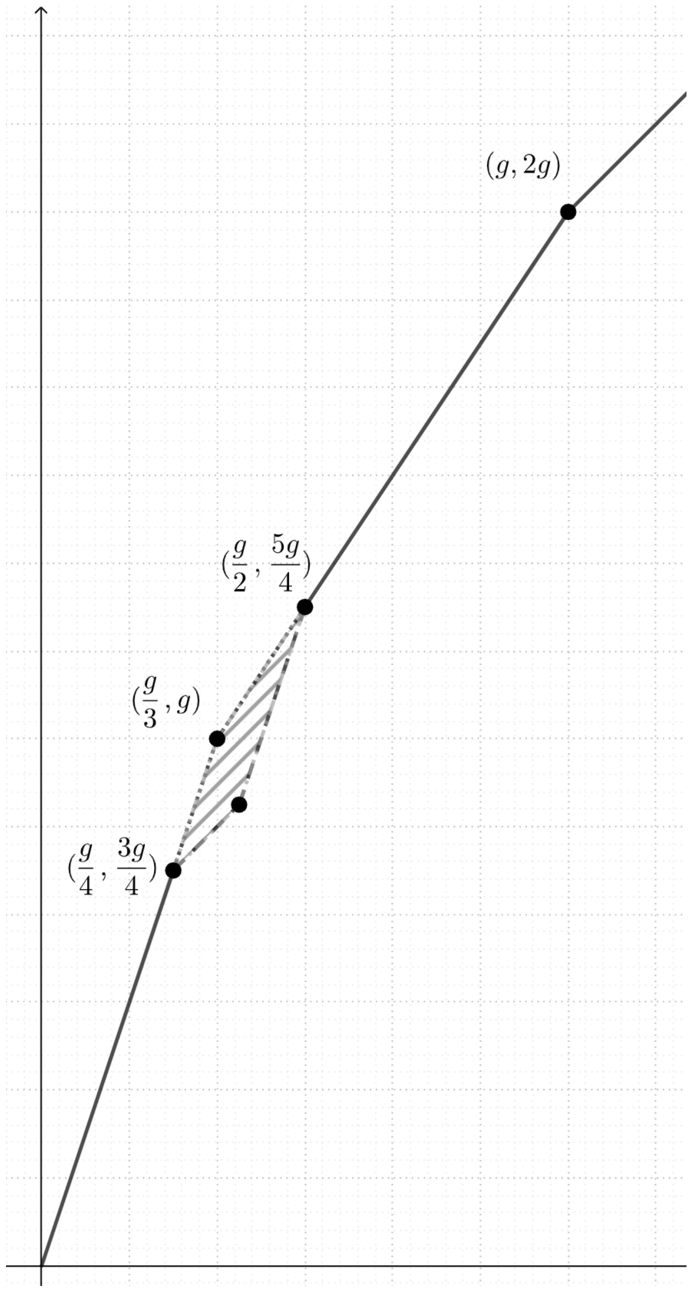

FIGURE 3. A depiction of the progress made

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