On the Hilbert Series of the Rational Cherednik Algebra in Type A_n in Characteristic p

Annie Wang

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Abstract

We study the polynomial representation of the rational Cherednik algebra of type A in characteristic $p = 3$ for p dividing $n - 2$, some parameter $t = 0$, and generic parameter c. We describe all the polynomials in the maximal proper graded submodule $\ker \mathcal{B}$, which is the kernel of the contravariant form $\mathcal{B}$, and we use this to find the Hilbert series of the irreducible quotient for the polynomial representation. We proceed degree by degree to explicitly determine the Hilbert series and work towards proving Etingof and Rains's conjecture in the case that $p = 3$, $t = 0$, and $n = kp + 2$.

Contents

1 Introduction

In 1993, Cherednik introduced Cherednik algebras, also called Double Affine Hecke Algebras, to use in his proof of Macdonald's conjectures about orthogonal polynomials for root systems in [Che93]. Since then, rational Cherednik algebras have been useful in many applications, ranging from math like topology and elliptic curve theory from [Che05] to physics models like quantum Calogero-Moser systems seen in [Eti07]. Calogero-Moser systems are important and connect to mathematical fields including algebraic geometry and deformation theory as well as physics.

Frobenius first introduced representation theory in 1896, which is explained more in [Con98]. Representation theory studies symmetry in linear spaces and makes an abstract object more concrete by studying the ways it acts on vector spaces. As with any abstract object, we use representation theory to better understand the structure of rational Cherednik algebras. The representation theory of rational Cherednik algebras over fields of characteristic 0 has been studied extensively; thus we proceed by considering only those in positive characteristic.

This paper focuses on studying the rational Cherednik algebra of type A. This restricts our general rational Cherednik for the symmetric group S_n. The rational Cherednik algebra of type A has parameters t, c.

This paper explores the representation theory of the rational Cherednik algebra of type A in characteristic $p = 3$ for $p|n-2$, $t = 0$, and generic parameter c. More specifically, we want to determine the Hilbert series for the irreducible quotient of the polynomial representation. Because the polynomial representation is so big, it is more helpful to study its irreducible quotient. The Hilbert series and Hilbert polynomial help us understand the dimension of the irreducible quotient for each degree.

In Section 2, we first define some preliminary terms including the polynomial representation of the rational Cherednik algebra, and we provide a proposition that simplifies the work in studying rational Cherednik algebras. Then in Section 3, we state an alternate definition of the polynomial representation through Verma modules and define the contravariant form $\mathcal{B}$. We also introduce the irreducible quotient of the Verma module, define its Hilbert series, and state a conjecture about the Hilbert series of the irreducible quotient of the polynomial representation. In Section 4, we summarize previous results. Our work follows after [DS16], which studied the case for characteristic p such that $p|n$, and [CK21], which explored the case for which $p|n-1$. And in Section 5, we prove main results and explicitly analyze the case in which $p|n-2$ for $p = 3$ and $t = 0$. Finally, in Section 6, we consider future directions and possible extensions of our results.

Throughout Sections 2 and 3, we continuously reference the approach and definitions from [CK21].

2 Preliminaries

With some motivation to study the representation theory of rational Cherednik algebras, we begin with preliminary definitions.

We fix an algebraically closed field $\Bbbk$ of positive characteristic p for some prime p. Also fix $t, c \in \Bbbk$ and some integer $n > 1$.

Consider the symmetric group S_n, where σ_{ij} swaps elements i and j. Let V be the permutation representation of S_n with basis $\{y_1, \ldots, y_n\}$, and let its dual space V^* have dual basis $\{x_1, \ldots, x_n\}$. Then we have the subrepresentation

$$\mathfrak{h} = \operatorname{span}(\{y_i - y_j\}) = \left\{\sum \lambda_i y_i : \sum \lambda_i = 0\right\},$$

which we call the standard representation. Its dual is

$$\mathfrak{h}^* = V^*/(x_1 + \cdots + x_n = 0).$$

Denote $[x, y] = xy - yx$ as the commutator of x and y.

Definition 1 (Rational Cherednik Algebra of Type A)**.** A rational Cherednik algebra of type A, denoted $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$, is the quotient of $\Bbbk S_n \ltimes T(\mathfrak{h} \oplus \mathfrak{h}*)$ by the following four relations:

$$[x_i, x_j] = 0,$$

$$[y_i - y_j, y_\ell - y_k] = 0,$$

$$[y_i - y_j, x_i] = t - c\sigma_{ij} - c\sum_{k \neq j} \sigma_{ik},$$

$$[y_i - y_j, x_k] = c\sigma_{ik} - c\sigma_{jk},$$

where $T(\mathfrak{h} \oplus \mathfrak{h}*)$ is the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$.

For the rest of the paper, we use $\mathcal{H}_{t,c}$ to denote this rational Cherednik algebra when clear.

Now consider $S\mathfrak{h}^*$, the symmetric algebra of $\mathfrak{h}^*$, which has the structure given by

$$\Bbbk[x_1, \ldots, x_n]/(x_1 + \cdots + x_n).$$

We define the Dunkl operators that act on $S\mathfrak{h}^*$ as follows.

Definition 2 (Dunkl Operator)**.** A Dunkl operator is defined by

$$D_{y_i} = t\partial_{x_i} - c\sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k}$$

for parameters t, c.

We can define a representation of $\mathcal{H}_{t,c}$ via the following actions: $x_i \mapsto x_i$ acts by multiplication, $\sigma \mapsto \sigma$ acts by permuting the x_i, and $y_i - y_j \mapsto D_{y_i - y_j}$ acts by $D_{y_i - y_j} = D_{y_i} - D_{y_j}$. Refer to Section 2.5 from [EM10] for a proof. Because these operators satisfy the relations from Definition 1, we have a representation of $\mathcal{H}_{t,c}$. We call this the polynomial representation of $\mathcal{H}_{t,c}$.

In this section, we also have the following proposition from [CK21], which generalizes the PBW theorem and reduces our work in studying rational Cherednik algebras.

Proposition 2.1. *We have*

$$\mathcal{H}_{t,c}(S_n, \mathfrak{h}) \cong \mathcal{H}_{at,ac}(S_n, \mathfrak{h})$$

for any $a \in \Bbbk^\times$.

This proposition implies that we only need to study the cases for $t = 0$ and $t = 1$, as everything else follows these cases.

3 Verma Modules and Hilbert Series

We have seen that $S\mathfrak{h}^*$ is a polynomial representation of $\mathcal{H}_{t,c}$. In this section, we give an alternative definition of this representation and its irreducible quotient using Verma modules.

Definition 3 (Verma Module)**.** The Verma module is

$$\mathcal{M}_{t,c}(S_n, \mathfrak{h}, \Bbbk) = \mathcal{H}_{t,c}(S_n, \mathfrak{h}) \otimes_{\Bbbk S_n \ltimes S\mathfrak{h}} \Bbbk.$$

We just call this $\mathcal{M}_{t,c}(\mathfrak{h})$, or if the vector space is clear, $\mathcal{M}_{t,c}$. Note that $\mathcal{M}_{t,c}$ is isomorphic to $S\mathfrak{h}^*$ as graded vector spaces and as representations.

Definition 4 (Contravariant Form)**.** The contravariant form $\mathcal{B} : \mathcal{M}_{t,c}(S_n, \mathfrak{h}, \Bbbk) \times \mathcal{M}_{t,c}(S_n, \mathfrak{h}^*, \Bbbk) \to \Bbbk$ is bilinear and satisfies the following properties:

1. For $\sigma \in S_n$, $f \in \mathcal{M}_{t,c}(\mathfrak{h})$, $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)$ we have $\mathcal{B}(\sigma f, \sigma q) = \mathcal{B}(f, q)$.
2. For $x \in \mathfrak{h}^*$, $f \in \mathcal{M}_{t,c}(\mathfrak{h})$, $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)$, we have $\mathcal{B}(xf, q) = \mathcal{B}(f, D_x(q))$.
3. For $y \in \mathfrak{h}$, $f \in \mathcal{M}_{t,c}(\mathfrak{h})$, $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)$, we have $\mathcal{B}(f, yq) = \mathcal{B}(D_y(f), q)$.
4. If $f \in \mathcal{M}_{t,c}(\mathfrak{h})_i$ and $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)_j$ for $i \neq j$, then $B(f, q) = 0$. In other words, the contravariant form gives 0 for elements of different degrees.
5. If $f \in \mathcal{M}_{t,c}(\mathfrak{h})_0$ and $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)_0$, then $B(f, q) = f \cdot q$.

We can also see that $\mathcal{B}$ defines a bilinear map $\mathcal{B} : S\mathfrak{h} \times S\mathfrak{h}^* \to \Bbbk$ which satisfies $\mathcal{B}(1, 1) = 1$, $\mathcal{B}(1, x_i) = 0$, and $\mathcal{B}(f(y), q(x)) = \mathcal{B}(1, D_{f(y)}(q(x))) = [x^0]f(D_y)q(x)$ where $[x^0]f(D_y)q(x)$ gives the constant term of $f(D_y)q(x)$. More details about the contravariant form can be found in Section 3.12 of [EM10].

Now, using Definition 3 and Definition 4, we have the following representation of $\mathcal{H}_{t,c}$.

Definition 5. Define $\mathcal{L}_{t,c} = \mathcal{M}_{t,c}/\ker \mathcal{B}$, where $\ker \mathcal{B} = \{x \in S\mathfrak{h}^* | \mathcal{B}(y, x) = 0 \text{ for all } y \in \mathfrak{h}\}$.

The representation $\mathcal{L}_{t,c}$ is an irreducible quotient of $\mathcal{M}_{t,c}$. Our goal is to compute its Hilbert series, which is defined as follows.

Definition 6 (Hilbert Series)**.** Consider an $\mathbb{N}$-graded module M. To this, we can associate the Hilbert series

$$h_M(z) = \sum_{i \geq 0} \dim M[i] z^i,$$

where $M[i]$ denotes the ith graded component of M.

Since the quotient $\mathcal{L}_{t,c}$ is graded like $\mathcal{M}_{t,c}$, its Hilbert series is

$$h_{\mathcal{L}_{t,c}}(z) = \sum_{i \geq 0} \dim \mathcal{L}_{t,c}[i] z^i.$$

Etingof and Rains presented a conjecture in the general case. First we introduce the following notation:

$$\begin{aligned} [k]_z &= \frac{1-z^k}{1-z}, \\ [k]_z! &= [k]_z[k-1]_z \cdots [1]_z, \\ Q_r(n,z) &= \binom{n-1}{r-1} z^{r+1} + \sum_{i=0}^{r} \binom{n-r-2+i}{i} z^i. \end{aligned}$$

Then we state the conjecture by Etingof and Rains from [CK21].

Conjecture 1 (Etingof, Rains)**.** Let $n = kp + r$ such that $0 \leq r < p$. The Hilbert series for $\mathcal{L}_{t,c}$ with some $c \in \Bbbk$ is

$$h_{\mathcal{L}_{0,c}}(z) = [r]_z![p]_z Q_r(n,z) \quad \text{for } t = 0$$

and

$$h_{\mathcal{L}_{1,c}}(z) = [p]_z^{n-1}[r]_{z^p}![p]_{z^p}! Q_r(n, z^p) \quad \text{for } t = 1.$$

Now consider the following definition of a singular polynomial.

Definition 7 (Singular Polynomial)**.** A singular polynomial is a polynomial $f \in S\mathfrak{h}^*$ such that $D_{y_i - y_j} f = 0$ for all i, j.

The singular polynomials generate a submodule that lies in $\ker \mathcal{B}$. Thus in analyzing the singular polynomials, we would understand the generators for $\ker \mathcal{B}$ in degree i for all i and therefore also understand the $\mathcal{L}_{t,c}[i]$ for computing the Hilbert series.

The following lemma helps our work as the degree increases. One direction is from [CK21].

Lemma 3.1. *For a fixed $f \in S\mathfrak{h}^*$ with no constant term, $D_{y_i - y_j} f \in \ker \mathcal{B}$ for all i, j, if and only if $f \in \ker \mathcal{B}$.*

Proof. For the first direction, we want to show that $\mathcal{B}(y, f) = 0$ for all $y \in S\mathfrak{h}$ given that $B(z, D_{y_i - y_j} f) = 0$ for all $z \in S\mathfrak{h}$. Because $y \in S\mathfrak{h}$, there exist polynomials $t_{ij} \in \Bbbk[y_1, \ldots, y_n]$ such that $y = c + \sum_{i,j} (y_i - y_j) t_{ij}$ for some $c \in \Bbbk$. We can let $z = t_{ij}$. By linearity of $\mathcal{B}$ and Definition 4, we have $\mathcal{B}(y, f) = \mathcal{B}(c, f) + \sum_{i,j} \mathcal{B}((y_i - y_j) t_{ij}, f) = 0 + \sum_{i,j} \mathcal{B}(t_{ij}, D_{y_i - y_j} f) = \sum_{i,j} \mathcal{B}(z, D_{y_i - y_j} f) = \sum_{i,j} 0 = 0$. Thus $f \in \ker \mathcal{B}$.

For the second direction, we want to show that $\mathcal{B}(z, D_{y_i - y_j} f) = 0$ for all $z \in S\mathfrak{h}$ given that $\mathcal{B}(y, f) = 0$ for all $y \in S\mathfrak{h}$. For all $z \in S\mathfrak{h}$, let $y = (y_i - y_j) z \in S\mathfrak{h}$. By Definition 4, we have $\mathcal{B}(z, D_{y_i - y_j} f) = \mathcal{B}((y_i - y_j) z, f) = B(y, f) = 0$ for all i, j, so $D_{y_i - y_j} f \in \ker \mathcal{B}$, and we are done. $\square$

This next lemma extends the previous lemma into a more specific direction.

Lemma 3.2. *Let $p = \{p_1, \ldots, p_n\} \subset \mathcal{M}_{t,c}[d]$ be a linearly independent set of polynomials so that $p_k \notin \ker \mathcal{B}$ for all k. Then if for some Dunkl operator $D_{y_i - y_j}$, the span of the $D_{y_i - y_j} p_k$ has trivial intersection with $\ker \mathcal{B}[d-1]$, then the span of p has trivial intersection with $\ker \mathcal{B}[d]$.*

Proof. We know that $p_k \notin \ker \mathcal{B}[d]$, so by Lemma 3.1, $D_{y_i - y_j} p_k \notin \ker \mathcal{B}[d-1]$ for all $k = 1, \ldots, n$. By assumption, we must have $\sum_{k=1}^{n} D_{y_i-y_j} a_k p_k = \sum_{k=1}^{n} a_k D_{y_i-y_j} p_k \notin \ker \mathcal{B}[d-1]$ for any constants a_k. By Lemma 3.1, this implies that $\sum_{k=1}^{n} a_k p_k \notin \ker \mathcal{B}[d]$ for all a_k, so any linear combination of the p_k does not lie in $\ker \mathcal{B}[d]$. $\square$

In the next section, we briefly describe the Hilbert series for a few known cases, all of which satisfy Conjecture 1.

4 Summary of Known Cases

The following cases have been studied and have known Hilbert series:

- $p|n$,
- $p|n-1$ for $t = 0$,
- $p|n-1$ for $t = 1$, $p = 2$.

For the first case, where $p|n$, Devadas and Sun showed in [DS16] that the Hilbert series of $\mathcal{L}_{1,c}$ is

$$h(z) = \left(\frac{1 - z^p}{1 - z} \right)^{n-1}.$$

For the second case, where $p|n-1$ and $t = 0$, Cai and Kalinov proved in [CK21] that the Hilbert series of $\mathcal{L}_{0,c}$ over a field with prime characteristic p is

$$h(z) = \left(\frac{1 - z^p}{1 - z} \right) (1 + (n-2)z + z^2).$$

And finally, for the third case, where $p|n-1$ and $t = 1$, $p = 2$, Cai and Kalinov showed in [CK21] that the Hilbert series of $\mathcal{L}_{1,c}$ over a field with characteristic 2 is

$$h(z) = (1 + z^2)(1 + z)^{n-1}(1 + (n-2)z^2 + z^4),$$

or alternatively written as

$$h(z) = (1 + z)^{n-1}(1 + (n-1)z^2 + (n-1)z^4 + z^6).$$

5 The $p|n-2$ Case

Our approach in computing the Hilbert polynomial is to analyze $\mathcal{M}_{t,c}[d]$ for each degree d starting from 0, and for each degree, we look for polynomials that are in or not in $\ker \mathcal{B}$. In other words, we find polynomials that are singular or not. Consider the basis of $\mathcal{M}_{t,c}$ that consists of elements of $\mathbb{F}_p[x_1, \ldots, x_n]$.

In this section, we consider the case for which $p = 3$, $t = 0$, and $p|n-2$.

Because $t = 0$, the Dunkl operator is

$$D_{y_i - y_j} = -c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} + c \sum_{l \neq j} \frac{1 - \sigma_{jl}}{x_j - x_\ell},$$

but c does not matter in this case, so let $c = 1$.

Conjecture 1 implies that the Hilbert polynomial for $p = 3$ and $p|n-2$ is

$$\begin{aligned}
[2]_z![3]_z Q_r(n,z) &= \frac{1-z^2}{1-z}\cdot\frac{1-z}{1-z}\cdot\frac{1-z^3}{1-z}\cdot Q_r(n,z)\\
&= (1+z)(1+z+z^2)(1+(n-3)z+\binom{n-2}{2}z^2+(n-1)z^3)\\
&= 1+(n-1)z+\frac{n^2-n-2}{2}z^2+(n^2-2n)z^3+(n^2-2n+1)z^4\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{n^2-n+2}{2}z^5+(n-1)z^6.
\end{aligned}$$

In the next subsections, we work degree by degree to show that the conjecture holds. We know the dimension for $\mathcal{L}_{0,c}[0]$ of degree 0 is 1 because all elements are constant, so the next step is to compute the dimension of $\mathcal{L}_{0,c}[1]$ of degree 1.

5.1 Degree 1

Theorem 5.1. *The dimension of $\mathcal{L}_{0,c}[1]$ is $n-1$.*

Proof. The characteristic p is odd, which implies that $x_n = -x_1 - \cdots - x_{n-1}$, thus the basis of $\mathcal{M}_{0,c}[1]$ consists of $x_1, \ldots, x_{n-1}$.

Suppose that there is a polynomial $f = \sum_{i<n} a_i x_i$ that is singular, which gives $D_{y_i - y_j} f = 0$ for all i, j. We want to show that $f = 0$. To do this, we look at how the Dunkl operator acts on x_i, so we compute $D_{y_i}(x_i)$ and $D_{y_i}(x_j)$ for $j \neq i$ to get

$$\begin{aligned}
D_{y_i}(x_i) &= \left(-\sum_{k\neq i}\frac{1-\sigma_{ik}}{x_i-x_k}\right)(x_i)\\
&= -\sum_{k\neq i}\frac{x_i-x_k}{x_i-x_k}\\
&= -(n-1),
\end{aligned}$$

while

$$\begin{aligned}
D_{y_i}(x_j) &= \left(-\sum_{k\neq i}\frac{1-\sigma_{ik}}{x_i-x_k}\right)(x_j)\\
&= -\left(\frac{x_j-x_j}{x_i-x_k}+\cdots+\frac{x_j-x_i}{x_i-x_k}+\cdots+\frac{x_j-x_j}{x_i-x_k}\right)\\
&= -(0+\cdots+(-1)+\cdots+0)\\
&= 1.
\end{aligned}$$

We can use the fact that $D_{y_i}(x_i) = -(n-1)$ and $D_{y_i}(x_j) = 1$ when considering $D_{y_n - y_i} f$. Because we assumed that f is singular, we already knew $D_{y_n - y_i} f$ must be 0, but we can now also explicitly compute the result. We have

$$\begin{aligned} D_{y_n} f &= D_{y_n}\left(\sum_{i<n} a_i x_i\right) \\ &= \sum_{i<n} a_i \end{aligned}$$

since f consists of only x_i terms where $i \neq n$. We also have

$$\begin{aligned} -D_{y_i} f &= -D_{y_i}\left(\sum_{i<n} a_i x_i\right) \\ &= (n-1)a_i - \sum_{j\neq i} a_j. \end{aligned}$$

Thus adding the results gives

$$D_{y_n - y_i} f = n a_i = 2a_i$$

since $p|n-2$. Finally, we set this equal to 0 since we assumed that f is a singular polynomial, giving us $2a_i = 0$. Because the characteristic p is 3, this implies that $a_i = 0$ for all i, so $f = 0$.

Therefore there are no nontrivial singular polynomials, which means that the Dunkl operators act nontrivially on all x_i, and the basis for $\mathcal{L}_{0,c}[1]$ is $x_1, \ldots, x_{n-1}$. So the dimension of $\mathcal{L}_{0,c}[1]$ is $n-1$. $\square$

5.2 Degree 2

Theorem 5.2. *The dimension of $\mathcal{L}_{0,c}[2]$ is $\frac{n^2-n-2}{2}$.*

Proof. Following our strategy, we consider $\mathcal{M}_{0,c}[2]$, which has a basis given by all $x_i^2, x_i x_j$ for $i, j < n$. The dimension of this basis is $n - 1 + \binom{n-1}{2} = \frac{n^2-n}{2}$. We claim that the kernel of $\mathcal{B}$ has dimension 1.

We compute the action of the Dunkl operators on each degree 2 monomial as follows:

- $D_{y_i - y_j}(x_k x_\ell) = 0$.
- $D_{y_i - y_j}(x_i x_\ell) = -x_i - x_\ell$ and likewise for $x_j x_\ell$.
- $D_{y_i - y_j}(x_k^2) = x_i - x_j$.
- $D_{y_i - y_j}(x_i^2) = -(x_i + x_j)$ and likewise for x_j^2.

Using these computations, we can show the following lemma:

Lemma 5.3. *The polynomial*

$$h := \sum_{\substack{i \leq j \\ i,j<n}} x_i x_j$$

is singular, and all singular polynomials are multiples of h.

Proof. Suppose

$$g_0 := \sum_{\substack{i\le j\\ i,j<n}} a_{ij}x_ix_j$$

is singular. Without loss of generality, we can assume that $a_{(1)(n-1)} = 0$ by subtracting a multiple of h.

Applying the Dunkl operator $D_{y_{n-1}-y_n}$, we compute $D_{y_{n-1}-y_n}g_0 =$
$a_{(n-1)(n-1)}(x_1+\cdots+x_{n-2}) + \sum_{k\neq n-1} a_{kk}(x_1+\cdots+x_{n-2}-x_{n-1}) + \sum_{k\neq n-1} a_{(k)(n-1)}(-x_k-x_{n-1})$.

Analyzing the coefficients of terms, we notice that

- The coefficient of x_{n-1} is
$$-\sum_{k\neq n-1} a_{kk} - \sum_{k\neq n-1} a_{(k)(n-1)}.$$
- The coefficient of all other x_j for $1 \le j < n-1$ is
$$\sum_k a_{kk} - a_{(j)(n-1)}.$$

Because g_0 is assumed to be singular, the action of any Dunkl operator on g_0 must give 0 by definition, so each coefficient must be 0. First consider the coefficient of x_i for $i < n-1$. Since we assumed that $a_{(1)(n-1)} = 0$ and we know the x_1 coefficient is 0, we have $\sum_k a_{kk} = 0$. Then for all other x_i for $i < n-1$, we must have $a_{(i)(n-1)} = 0$ also. Thus we now have

- The coefficient of x_{n-1} is
$$-\sum_{k\neq n-1} a_{kk}.$$
- The coefficient of all other x_j for $1 \le j < n-1$ can be written as $\sum_k a_{kk}$, or
$$a_{(n-1)(n-1)} + \sum_{k\neq n-1} a_{kk}.$$

Since both coefficients must be 0, their sum $a_{(n-1)(n-1)}$ must also be 0. This implies that
$$a_{(i)(n-1)} = 0$$
for all i.

This means that the polynomial we assume to be singular can be written as

$$g_1 := \sum_{\substack{i\le j\\ i,j<n-1}} a_{ij}x_ix_j.$$

Now consider $D_{y_{n-2}-y_n}g_0$ which is also 0 by assumption. By the analysis of $D_{y_{n-1}-y_n}g_0$, we have $\sum a_{kk} = 0$ and $a_{(n-1)(n-1)} = 0$, which gives $\sum_{k<n-1} a_{kk} = 0$ as well. So if we look at the coefficients of x_i for $i < n-2$ in $D_{y_{n-2}-y_n}g_0$, which must be 0, we conclude that $a_{(i)(n-2)} = 0$ for all $i < n-2$. Then looking at the coefficient of x_2, we find that $a_{(n-2)(n-2)} = 0$ also. Continuing inductively and considering the $D_{y_j-y_n}$ action, we get $a_{ij} = 0$ for all i, j. Thus g_0 was in fact a multiple of h. □

Since any singular polynomial we find is a multiple of h, we see that the dimension of the kernel of $\mathcal{B}$ is indeed 1. Therefore the dimension of $\mathcal{L}_{0,c}[2]$ is $\frac{n^2-n}{2} - 1 = \frac{n^2-n-2}{2}$, which verifies the conjecture for degree 2. $\square$

5.3 Degree 3

Theorem 5.4. *The dimension of $\mathcal{L}_{0,c}[3]$ is $n^2 - 2n$.*

Proof. Consider $\mathcal{M}_{0,c}[3]$, which has a basis given by all $x_i^3, x_i^2 x_j, x_i x_j x_k$ for $i,j,k < n$. The dimension of this basis is $n - 1 + (n-1)(n-2) + \binom{n-1}{3} = \frac{n^3-n}{6}$. We claim that the kernel of $\mathcal{B}$ has dimension $\frac{n^3-6n^2+11n}{6}$.

Recall that Lemma 5.3 gives that

$$h = \sum_{\substack{i \le j \\ i,j<n}} x_i x_j$$

is in the kernel, which means that any multiple of h is also in the kernel of $\mathcal{B}$.

With some computations using general Dunkl operator $D_{y_i - y_j}$, we can show the following:

Lemma 5.5. *The polynomial*

$$r := \sum_{\substack{i \le j \le k \\ i,j,k<n}} x_i x_j x_k$$

lies in the kernel of $\mathcal{B}$.

As we look for more singular polynomials, we also have as follows:

Lemma 5.6. *For all pairwise distinct $i, j, k < n$, the polynomials*

$$q_{i,j,k} := \sum_{\substack{m_i+m_j+m_k=3 \\ 0 \le m_i, m_j, m_k \le 3}} x_i^{m_i} x_j^{m_j} x_k^{m_k}$$

are in the kernel of $\mathcal{B}$.

Proof. By Lemma 3.1, if all Dunkl operators D_{y_ℓ} send the $q_{i,j,k}$ to a multiple of h, we will have proven this lemma. To check this, we have two cases.

Case 1: $\ell \ne i, j, k$. Without loss of generality, let $\ell = 1, i = 2, j = 3, k = 4$. We want to find $D_{y_1}(q_{2,3,4})$. To split this case further, we find that

- $D_{y_1}(x_2^3) = x_1^2 + x_1 x_2 + x_2^2$,
- $D_{y_1}(x_3^3) = x_1^2 + x_1 x_3 + x_3^2$,
- $D_{y_1}(x_4^3) = x_1^2 + x_1 x_4 + x_4^2$,
- $D_{y_1}(x_2^2(x_3 + x_4)) = x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + 2x_2^2$,
- $D_{y_1}(x_2(x_3 x_4 + x_3^2 + x_4^2)) = x_3^2 + x_4^2 + 2x_1 x_2 + 2x_2 x_3 + 2x_2 x_4 + x_3 x_4$,
- $D_{y_1}(x_3^2 x_4) = x_3^2 + x_1 x_4 + x_3 x_4$,

- $D_{y_1}(x_3x_4^2) = x_4^2 + x_1x_3 + x_3x_4$.

In total, we count three of each monomial, which means $D_{y_1}(q_{2,3,4}) = 3h$. Because the characteristic is 3, this is just 0.

Case 2: ℓ is one of i, j, k. Without loss of generality, let $\ell = 1, i = 1, j = 2, k = 3$. We want to find $D_{y_1}(q_{1,2,3})$. Again, we split up this case and find that

- $D_{y_1}(x_1^3) = -\sum_{i \neq 1}(x_1^2 + x_1x_i + x_i^2)$,
- if $i = 2$ or 3, $D_{y_1}(x_1^2(x_2 + x_3)) = -x_1x_2 - x_1x_3 - x_2x_3$, so in total we have $2(-x_1x_2 - x_1x_3 - x_2x_3)$,
- if $i \neq 2, 3$, $D_{y_1}(x_1^2(x_2 + x_3)) = (-x_i - x_1)(x_2 + x_3)$, so in total we have $\sum_{i \neq 1,2,3}((-x_i - x_1)(x_2 + x_3))$,
- if $i = 2$, $D_{y_1}(x_1(x_2x_3 + x_2^2 + x_3^2)) = x_1x_2 - x_3^2$ and likewise for $i = 3$, so in total we have $x_1x_2 + x_1x_3 - x_2^2 - x_3^2$,
- if $i \neq 2, 3$, $D_{y_1}(x_1(x_2x_3 + x_2^2 + x_3^2)) = -x_2^2 - x_3^2 - x_2x_3$, so in total we have $\sum_{i \neq 1,2,3}(-x_2^2 - x_3^2 - x_2x_3)$.

Combining these with computations from Case 1 and replacing all $x_n = -x_1 - \cdots - x_{n-1}$ (and thus replacing $x_n^2 = \sum_{i<n} x_i^2 + \sum_{i<j<n} 2x_ix_j$) let us determine the coefficient for each monomial, listed as follows:

- $x_1^2 : -n + 3$,
- $x_2^2 : -n + 3$,
- $x_3^2 : -n + 3$,
- $x_i^2 : -2$ for all $i \neq 1, 2, 3$,
- $x_1x_2 : -n + 3$,
- $x_1x_3 : -n + 3$,
- $x_1x_i : -2$ for all $i \neq 1, 2, 3$,
- $x_2x_3 : -n + 3$,
- $x_2x_i : -2$ for all $i \neq 1, 2, 3$,
- $x_3x_i : -2$ for all $i \neq 1, 2, 3$,
- $x_ix_j : -2$ for all $i, j \neq 1, 2, 3$.

Since $p|n-2$, we have 1 of each term in characteristic $p=3$, giving $D_{y_1}(q_{1,2,3}) = h$. Note that this conclusion does not follow for $p > 3$, which matches Conjecture 1.

Thus the action of all Dunkl operators D_{y_ℓ} on $q_{i,j,k}$ gives a multiple of h, which implies that $q_{i,j,k} \in \ker \mathcal{B}$ for all pairwise distinct i, j, k. $\square$

Now we know that our kernel must contain r and all such $q_{i,j,k}$, so its minimum dimension is $1 + \binom{n-1}{3} = \frac{n^3-6n^2+11n}{6}$.

To prove that this is its exact dimension, we find a basis for $\mathcal{L}_{0,c}[3]$.

Proposition 5.7. *Consider the set $S = \{x_i^3,\, x_j^2 x_k\}$ for all $1 < i < n$ and for all $1 \leq j, k < n$ such that $j \neq k$. Then S is a basis for $\mathcal{L}_{0,c}[3]$.*

We work towards proving this proposition. From calculations in Cases 1 and 2 above, we can easily confirm that $D_{y_1 - y_2} s \neq 0$ for all $s \in S$, so no element of S is in $\ker \mathcal{B}$.

Then, we want to show that $D_{y_1-y_2}$ does not send any linear combination of S to a multiple of h. Define

$$\phi := \sum_{j,k<n} c_{jk} x_j^2 x_k - c_{11} x_1^3.$$

Assume that

$$D_{y_1-y_2}(\phi) = k_1 h$$

for constant k_1, so that we want to show that all c_{jk} are 0. After computing the left hand side, we have the following:

- The coefficient of x_1^2 is $\sum_{j\geq 3}(c_{jj} - c_{1j}) - c_{21}$.
- The coefficient of x_2^2 is $\sum_{j\geq 3}(c_{2j} - c_{jj}) - c_{22} + c_{12}$.
- The coefficient of x_i^2 for all $i \neq 1, 2$ is $-c_{22} + c_{1i} - c_{2i} - c_{i1} + c_{i2}$.
- The coefficient of $x_1 x_2$ is $\sum_{j\geq 3}(c_{j2} - c_{j1}) - c_{12} + c_{21}$.
- The coefficient of $x_1 x_i$ for all $i \neq 1, 2$ is $-c_{22} + c_{ii} - c_{1i} + c_{2i} + \sum_{j\neq 1,2,i} c_{ji}$.
- The coefficient of $x_2 x_i$ for all $i \neq 1, 2$ is $-c_{22} - c_{ii} - c_{1i} + c_{2i} - \sum_{j\neq 1,2,i} c_{ji}$.
- The coefficient of $x_i x_j$ for all $i, j \neq 1, 2$ is $-c_{22}$.

Because we can add any multiple of h, we can let $c_{22} = 0$ without loss of generality. Then since the coefficient of all terms are equal, all coefficients listed above are 0 in this case.

Consider the coefficient of x_1^2 and the coefficient of x_2^2 which appear very similar. Since both are 0, plugging in $c_{22} = 0$ and adding the two give

$$\sum_{j\geq 3}(c_{2j} - c_{1j}) + c_{12} - c_{21} = 0.$$

Then the coefficient of x_i^2 is 0, so replacing i with j gives and rearranging gives

$$c_{2j} - c_{1j} = c_{j2} - c_{j1}$$

for all $j \neq 1, 2$. This then implies that

$$\sum_{j \geq 3} (c_{j2} - c_{j1}) + c_{12} - c_{21} = 0.$$

Now we subtract this from the coefficient of $x_1 x_2$ which is also equal to 0 to get

$$c_{21} - c_{12} - c_{12} + c_{21} = 0,$$

or

$$c_{12} = c_{21}.$$

Because we find that $c_{12} = c_{21}$ by adding a multiple of h, this means the constants will be equal regardless.

Now considering the Dunkl operators $D_{y_1 - y_\ell}$ for $3 \leq \ell \leq n-1$, we see the same corresponding equation results, so we know that $c_{1i} = c_{i1}$ for all $3 \leq i \leq n-1$. Using the coefficient of x_i^2 and the original Dunkl operator $D_{y_1 - y_2}$, since $c_{2j} - c_{1j} = c_{j2} - c_{j1}$, we now also have $c_{2i} = c_{i2}$ for all $3 \leq i \leq n-1$. Similarly using Dunkl operators $D_{y_1 - y_\ell}$ for $3 \leq \ell \leq n-1$, we find the following:

Lemma 5.8. *The coefficients $c_{ij} = c_{ji}$ when $i \neq j$.*

Because we cannot deduce more from $D_{y_1 - y_2}$ at this point, we consider a different Dunkl operator. Assume that

$$D_{y_1 - y_n}(\phi) = k_2 h$$

for constant k_2. For this Dunkl operator, we have the following from the left hand side:

- The coefficient of x_1^2 is $\sum_{j \neq 1} (c_{j1} - c_{1j})$.
- The coefficient of x_i^2 for all $i \neq 1$ is $\sum_{j \neq 1, i} (c_{ji} - c_{jj}) - c_{1i} - c_{i1}$.
- The coefficient of $x_1 x_i$ for all $i \neq 1$ is $\sum_{j \neq 1, i} (c_{jj} - c_{ji}) + \sum_{j \neq 1} c_{j1}$.
- The coefficient of $x_i x_j$ for all $i, j \neq 1$ is $\sum_{k \neq 1, i, j} c_{kk} + \sum_{k \neq 1, i} c_{ki} + \sum_{k \neq 1, j} c_{kj} - c_{ii} - c_{jj} + c_{1i} + c_{1j}$.

By assumption, all of these coefficients are equal. From Lemma 5.8, we have $c_{ij} = c_{ji}$ when $i \neq j$, so the coefficient of x_1^2 is just 0. Thus the other coefficients must be 0 as well. Then adding the coefficients of x_i^2 and $x_1 x_i$ gives

$$\sum_{j \neq 1} c_{j1} - c_{1i} - c_{i1} = 0.$$

After substituting $c_{1i} = c_{i1}$ due to Lemma 5.8, we have

$$2c_{i1} = \sum_{j \neq 1} c_{j1}.$$

Summing across all $i \neq 1$ gives

$$2 \sum_{i \neq 1} c_{i1} = (n-1) \sum_{j \neq 1} c_{j1}.$$

Therefore $\sum_{i \neq 1} c_{i1} = 0$, and from above, it follows that $c_{i1} = 0$ (and thus $c_{1i} = 0$) for all $i \neq 1$. Then considering the Dunkl operators $D_{y_\ell - y_n}$ for $2 \leq \ell < n$, we again have corresponding results, which we put together to see the following lemma:

Lemma 5.9. *The coefficients* $c_{ij} = 0$ *for all* $i \neq j$.

All we have left is to show the following:

Lemma 5.10. *The coefficients* $c_{ii} = 0$ *for all* $1 < i < n$ *as well.*

Proof. When we go back to the coefficients of terms for $D_{y_1 - y_2}(\phi)$, the coefficient of $x_2 x_i$ in that case now becomes $-c_{22} + c_{ii}$ since the others are 0 due to Lemma 5.9. Because we could assume that $c_{22} = 0$ without loss of generality so that the coefficient of $x_2 x_i$ must be 0 as well, this implies that

$$-c_{22} + c_{ii} = 0$$

for all $i \neq 1, 2$. Therefore $c_{ii} = 0$ for all $i \neq 1$. The case when $i = 1$ does not matter because it cancels out in ϕ. $\square$

Thus we have shown that there is no nontrivial ϕ such that $D_{y_i - y_n}(\phi)$ lies in the kernel, so the span of the $D_{y_i - y_n} s$ for all $s \in S$ has trivial intersection with $\ker \mathcal{B}[2]$. Now by Lemma 3.2, we have that the span of S has trivial intersection with $\ker \mathcal{B}[3]$. Hence S is a basis for $\mathcal{L}_{0,c}[3]$, and its dimension is $n^2 - 2n$, as desired.

$\square$

6 Future Work

Our work covers degrees 0 through 3 of the $p|n-2$ case. Future research could look towards extending this to greater degrees. Although it is not clear that all polynomials that lie in $\ker \mathcal{B}$ can be explicitly found as the kernel becomes very large for higher degrees, it would be helpful to conjecture a basis for the kernel, then use Lemma 3.1 and Lemma 3.2 to prove that it is indeed a basis. Exploring polynomials similar to the $q_{i,j,k}$ for higher degrees can help explain how the kernel grows so fast. Finally, note that from Conjecture 1, the dimension of $\mathcal{L}_{0,c}[6]$ should be $n-1$, so our approach would follow nicely for that degree.

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