ARRANGEMENTS OF SIMPLICES IN FINE MIXED SUBDIVISIONS

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Abstract. A regular simplex of side length n can be subdivided into multiple polytopes, each of which is a Minkowski sum of some faces of a unit simplex. Ardila and Billey have shown that exactly n of these cells must be simplices, and their positions must be in a "spread-out" arrangement. In this paper, we consider their question of whether every spread-out arrangement of simplices can be extended into such a subdivision, especially in the three-dimension case. We prove that a specific class of these arrangements, namely those that project down to a two-dimensional spread-out arrangement, all extend to a subdivision.

1. Introduction

Take an equilateral triangle of side length n, and remove n equilateral triangles of side length 1 from it. We want to tile the remaining shape with unit rhombi with angles $60°$ and $120°$. See Figure 1 for an example of such a tiling.

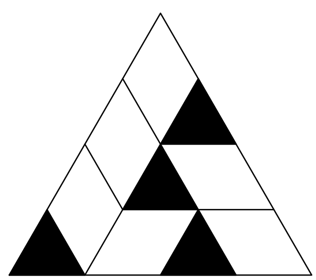

Figure 1. An example of the case $n = 4$.

This paper will examine the arrangements of the n removed triangles for which such a tiling (called a *fine mixed subdivision*) is possible, and the generalization of this problem to higher dimensions (with arrangements of n removed simplicies). We begin with some preliminary definitions and state the general problem in Section 2, then provide our own proofs of the two-dimensional case in Section 3.

The remainder of the paper focuses on potential approaches in the three-dimensional case. We introduce a combinatorial object known as *tropical pseudoplanes* in Section 4, based on [2], along with an alternative proof of the two-dimensional case. In Section 5, we show a method of inducting downwards for some specific arrangements of tetrahedra. In Section 6, we provide a construction of a subdivision for a specific class of arrangements which stay spread-out under a certain projection to two dimensions. Finally, in Section 7 we outline a few possible approaches to resolve the three-dimensional case in general.

2. Preliminaries

We begin by defining a few terms concerning polytopes.

Definition 2.1. A *(convex) polytope* P is the convex hull of a set of points $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in d-dimensional space $\mathbb{R}^d$,

$$P = \operatorname{conv}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i : \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \ \forall i \right\}.$$

The *dimension* of a polytope is the dimension of the smallest affine subspace containing it.

Polytopes can also be defined as the bounded intersection of closed half-spaces in $\mathbb{R}^d$.

Definition 2.2. A *face* of a polytope P is the intersection of P with the boundary of a closed half-space that contains P, *i.e.,* a hyperplane $A\mathbf{x} = \mathbf{z}$ such that all $\mathbf{x} \in P$ satisfy $A\mathbf{x} \leq \mathbf{z}$.

Both P and $\emptyset$ are faces of P, as seen by the inequalities $\mathbf{0x} \leq \mathbf{0}$ and $\mathbf{0x} \leq \mathbf{1}$. Here, $\mathbf{0}$ is the all-0s matrix. Observe that every face of a polytope is another polytope.

Definition 2.3. A *simplex* is the convex hull of d affinely independent vertices, i.e., d points that do not lie in a common $(d-2)$-dimensional hyperplane.

It is clear that every face of a simplex is also a simplex. We also denote the *standard regular simplex* in $\mathbb{R}^d$ by

$$\triangle_{d-1} := \operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d),$$

where $\mathbf{e}_i$ is the unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with a single 1 in the i-th coordinate. This embedding is useful for its symmetry, and in our case, the lattice points that it contains. Note that even though it is embedded in $\mathbb{R}^d$, it is still $(d-1)$-dimensional, hence the choice of subscript in the notation.

We will now define the type of polytopes that we consider in this paper.

Definition 2.4. The *Minkowski sum* of sets P and Q is defined as

$$P + Q := \{\mathbf{p} + \mathbf{q} : \mathbf{p} \in P, \mathbf{q} \in Q\}.$$

Note that this definition can easily be generalized to three or more summands.

Definition 2.5. A *fine mixed cell* is a polytope of the form

$$B_1 + B_2 + \cdots + B_n,$$

where the B_i's are faces of $\triangle_{d-1}$ that lie in independent affine subspaces such that

$$\sum_{i=1}^{n} \dim(B_i) = d - 1.$$

For example, the simplex $\triangle_{d-1}$ itself is always a fine mixed cell. See Figure 2 for some examples of fine mixed cells in two and three dimensions. Note that fine mixed cells cannot be rotated or reflected arbitrarily, and can only be translated by integer vectors.

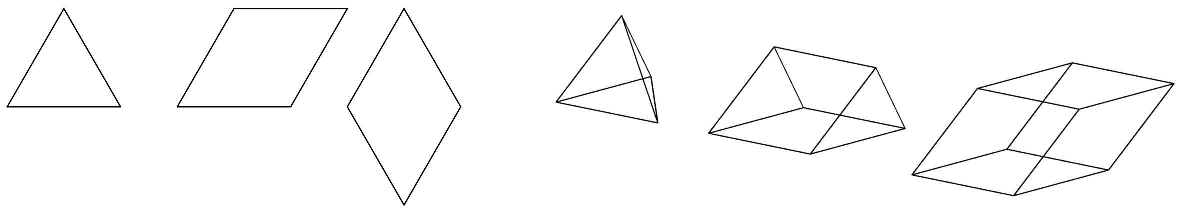

FIGURE 2. Some fine mixed cells in 2D and 3D.

For positive integers n and d, we use $n\triangle_{d-1}$ to denote $\mathrm{conv}(n\mathbf{e}_1, \ldots, n\mathbf{e}_d)$.

Definition 2.6. A *fine mixed subdivison* of $n\triangle_{d-1}$ is a decomposition of $n\triangle_{d-1}$ into fine mixed cells such that any two cells intersect at a face of both (possibly the empty face).

For example, a fine mixed subdivision when $d-1=1$ is simply dividing a length-n line segment into n unit segments, and when $d-1=2$ it is a tiling with unit equilateral triangles and unit rhombi, as shown in Figure 1.

Ardila and Billey ([1], Section 8) proved that a fine mixed subdivison of $n\triangle_{d-1}$ must have exactly n cells that are (unit) simplices. Moreover, in any smaller simplex of size k (*i.e.,* a translation of $k\triangle_{d-1}$), there must be at least k simplices. Let us call an arrangement of exactly n simplices that satisfies this condition *spread-out.*

Question 2.7 ([1], Conjecture 7.1)**.** Given a spread-out arrangement of n simplices inside $n\triangle_{d-1}$, does there always exist a fine mixed subdivison with simplices only at these positions?

In the same paper, Ardila and Billey ([1]) have shown that the answer to the above question is "yes" when $d-1 \leq 2$, but the question is still open for $d-1 \geq 3$. The rest of this paper will be dedicated to the two- and three-dimensional cases of this question.

3. Alternative Proofs of the Two-Dimensional Case

Section 6 of [1] gives an inductive proof that answers Question 2.7 when $d-1=2$. Here, we provide two of our own proofs.

Graph Theoretic Proof. Notice that we can subdivide $n\triangle_2$ into n^2 unit triangles, with $\frac{n(n+1)}{2}$ of them right-side up and the rest upside-down. Call the set of right-side up triangles A and the set of upside-down triangles B.

Let $S \subset A$ be a spread-out set of n triangles. Then, we can consider a bipartite graph between $A \setminus S$ and B, where two triangles in the graph are connected if and only if they are adjacent (share an edge). Every edge in this graph corresponds to a possible placement for a single rhombus, so it suffices to find a perfect matching on this graph.

We will use Hall's Marriage Lemma to show such a matching exists. First, we show the following lemma:

Lemma 3.1. *For any connected subset U of k triangles from B, say the smallest triangle containing all k of them is T, and T has size j. Then, those k triangles collectively have at least $j+k$ distinct neighbors in A.*

Proof. We proceed with strong induction. The base case is $k = 1$, in which case $j = 2$ and $j+k = 3$. Indeed, any triangle in B has 3 neighbors.

For the induction step, assume our lemma holds for any $k \leq i-1$, and consider some U with $|U| = i$. Notice U must have at least one triangle on the bottom row of T, otherwise T would be smaller. Consider removing the rightmost triangle of U on the bottom row, labeled x in Figure 3.

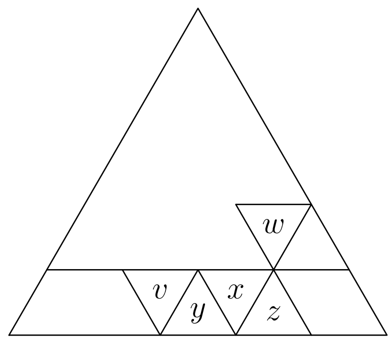

FIGURE 3. The leftmost triangle in the bottom row, and some of its neighbors.

We consider the following cases:

Case 1: Removing x does not disconnect U and does not make T smaller.

Then, $U \setminus \{x\}$ has at least $(i-1)+j$ neighbors by the induction hypothesis. Furthermore, no triangle from this set has z as a neighbor, as the only other triangle that has z as a neighbor, apart from x, is also on the bottom row, on the right of x. As $x \in U$ has z as a neighbor, U has at least $(i-1)+j+1 = i+j$ neighbors.

Case 2: Removing x does not disconnect U but does make T smaller.

As $i \geq 2$, x is adjacent to some other triangle in U, which must be in the second-to-bottom row. Hence, removing x can only move T's bottom edge by 1. Notice that even if x touches another edge of T, its neighbor in U must touch that edge as well, so removing x does not change the other edges of T. Hence, j decreases by 1, and by the induction hypothesis, $U \setminus \{x\}$ has at least $(i-1)+(j-1) = i+j-2$ neighbors.

Furthermore, no triangle in U, except for x, has y or z as a neighbor, as the only triangles with y or z as a neighbor are in the bottom row. Hence, U has at least $(i+j-2)+2 = i+j$ neighbors.

Case 3: Removing x disconnects U.

Notice that if the only neighbors of x in U were in the second-to-bottom row, removing x cannot disconnect U. Hence x is adjacent to some other triangle from U in the bottom row, which by definition must be on its left. Hence, $v \in U$. In order to split U by removing x, we must have $w \in U$ as well. In particular, removing x will not decrease the size of T.

Removing x splits U into two components; call them V and W such that $v \in V$ and $w \in W$. Say $|V| = k_1$ and the smallest bounding triangle of V has size j_1; define k_2 and j_2 similarly for W.

Then, by the induction hypothesis, V and W have at least $k_1 + j_1$ and $k_2 + j_2$ neighbors, respectively. None of these neighbors are shared, as any two triangles that share a neighbor in A are connected. Also, $k_1 + k_2 = i - 1$.

Notice that V must touch the left edge of T, and W must touch the right edge of T. Furthermore, as $w \in W$, the bounding triangle of W contains the top-right corner of v, so the bounding triangles of V and W touch. Hence, $j_1 + j_2 \geq j$.

Finally, z is still a neighbor to no triangle in U other than x. Hence, U has at least

$$(k_1 + j_1) + (k_2 + j_2) + 1 = (k_1 + k_2 + 1) + (j_1 + j_2) \geq i + j$$

neighbors, as desired.

This covers every case, completing the proof. □

Hence, if a subset $U \subseteq B$ with size k has smallest bounding triangle T with size j, then U has at least $j + k$ neighbors in A. Also, S has at most j elements of A in T by the spread-out condition. Thus, $A \setminus S$ has at least $(j + k) - j = k$ neighbors of U. As this holds for every subset U of B, by Hall's Marriage Theorem, there exists a perfect matching between $A \setminus S$ and B, as desired. □

Remark. While this proof offers a nice combinatorial representation of the problem in two dimensions, it does not generalize three or more dimensions, since there is no tiling of a simplex such that every fine mixed cell is the union of two tiles.

"Sliding Triangles" Proof. We say a triangle T is *saturated* by a set S of triangles if the number of elements of S contained in T equals the size of T.

Our goal is to first "slide" each triangle downwards to the bottom row; an example is shown in Figure 4.

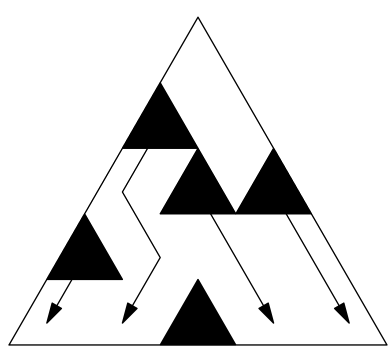

FIGURE 4. An example of the case $n = 5$.

We prove the following lemmas.

Lemma 3.2 ([1], Lemma 4.2)**.** *If T_1 and T_2 are both saturated by a set S_0, and their intersection is nonempty, both $T_1 \cap T_2$ and $T_1 \vee T_2$ (the smallest triangle containing both T_1 and T_2) are saturated by S_0 as well.*

Proof. Say T_1 is the set of lattice points in $n\triangle_2$ that majorize (a_1, a_2, a_3) and T_2 is the set of lattice points in $n\triangle_2$ majorize (b_1, b_2, b_3). For convenience, we call (a_1, a_2, a_3) the *apex* of T_1.

Then, their intersection is the set of lattice points that majorizes both triples, or the triangle with apex $(\max(a_1, b_1), \max(a_2, b_2), \max(a_3, b_3))$. Furthermore, a triangle T contains T_1 if and only if the apex of T_1 majorizes the apex of T, so $T_1 \vee T_2$ has apex $(\min(a_1, b_1), \min(a_2, b_2), \min(a_3, b_3))$.

The size of a triangle with apex (a_1, a_2, a_3) is $n - (a_1 + a_2 + a_3)$, so

$$\begin{aligned}
\text{size}(T_1) + \text{size}(T_2) &= 2n - (a_1 + a_2 + a_3 + b_1 + b_2 + b_3) \\
&= 2n - \sum_{i=1}^{3} (\max(a_i, b_i) + \min(a_i, b_i)) \\
&= \text{size}(T_1 \cap T_2) + \text{size}(T_1 \vee T_2).
\end{aligned}$$

In particular, the multiset $T_1 \cup T_2$ (counting the intersection twice) is contained in the multiset $(T_1 \cap T_2) \cup (T_1 \vee T_2)$. The former contains $\text{size}(T_1) + \text{size}(T_2)$ elements of S_0 (as both triangles are saturated), while the latter contains at most

$$\text{size}(T_1 \cap T_2) + \text{size}(T_1 \vee T_2) = \text{size}(T_1) + \text{size}(T_2).$$

Hence equality must hold, and both $T_1 \cap T_2$ and $T_1 \vee T_2$ are saturated by S_0. $\square$

Lemma 3.3. *Given a set S of spread-out triangles (of size n) and a triangle $T \in S$ not in the bottom row, Let T_1 and T_2 be the triangles on T's lower-left and lower-right, respectively. Then at least one of $(S \setminus \{T\}) \cup \{T_1\}$ and $(S \setminus \{T\}) \cup \{T_2\}$ is spread-out.*

In other words, we can "slide" T either down-and-left, or down-and-right.

Proof. Assume for sake of contradiction that neither set is spread-out. Let A_1 be a triangle that violates the spread-out condition for $(S \setminus \{T\}) \cup \{T_1\}$, and define A_2 similarly.

Then, both A_1 and A_2 are saturated by $S \setminus \{T\}$. By Lemma 3.2, $A_1 \vee A_2$ is also saturated by $S \setminus \{T\}$. However, T must be in $A_1 \vee A_2$, so $A_1 \vee A_2$ violates the spread-out condition for S, contradiction.

Hence the lemma holds. $\square$

To slide each triangle to the bottom row, we will first slide all the triangles in the top row into the second row. Then, we take all triangles in the second row and slide them into the third row (in any order), and so on. Note that Lemma 3.3 guarantees that we will always have a spread-out configuration and can always slide triangle down.

It is clear that the sliding paths are all disjoint, so we can fill each path with "horizontal" rhombi (each with a pair of horizontal edges), and the rest of the triangle can be filled with "vertical" rhombi, giving us our desired tiling.

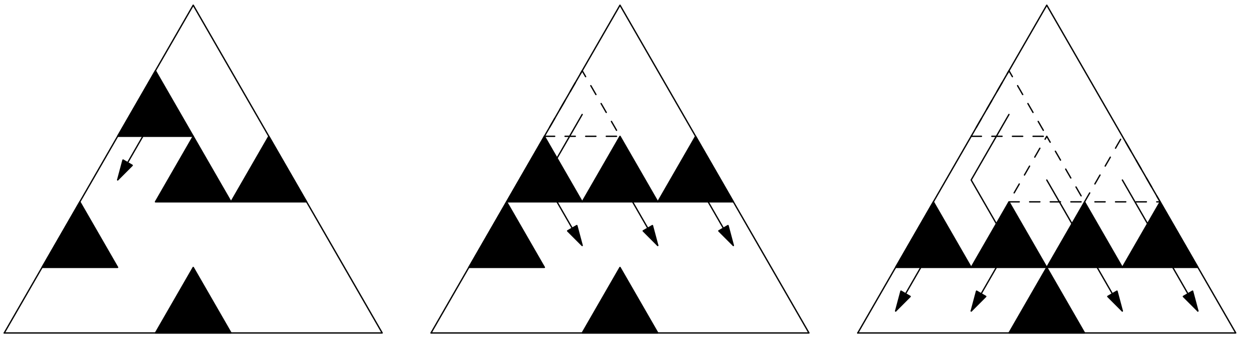

FIGURE 5. An example of the case $n = 5$.

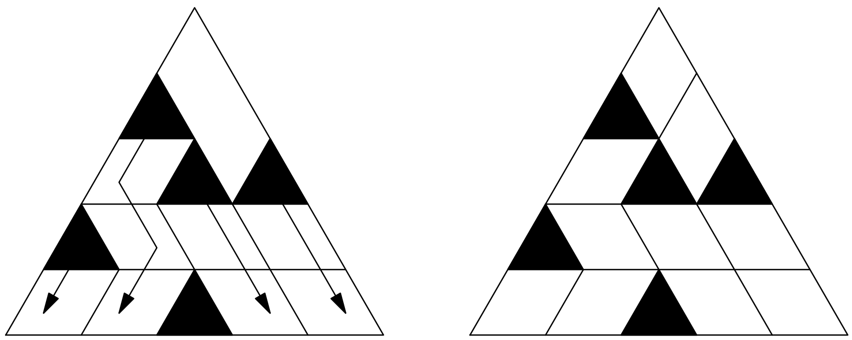

FIGURE 6. An example of the case $n = 5$.

□

In fact, this proof extends beyond the specific case in which the shape to be tiled is a triangle.

Theorem 3.4. *Let U be a connected set of upside-down equilateral triangles, and let V be the set of right-side-up triangles adjacent to (sharing an edge with) the triangles of U. Let S be a spread-out subset of V with size $|V| - |U|$. Then, the shape formed by $(U \cup V) \setminus S$ can be tiled with rhombi. (Such a set S may not exist.)*

Figure 7 shows an example of a set U and the resulting shape.

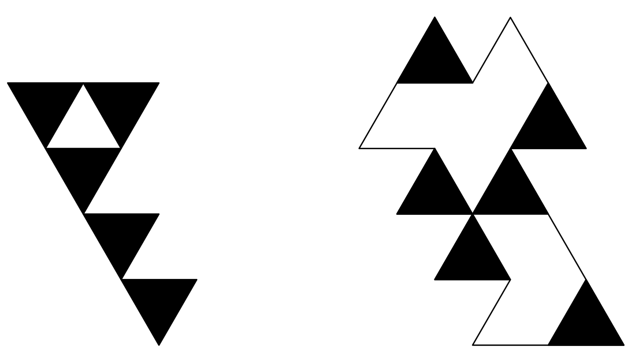

FIGURE 7. An example of U and the resulting shape.

Note that it is not guaranteed such an S even exists, in which case the theorem is vacuously true.

Proof. The previous "Sliding Triangles" proof applies almost verbatim here. We slide each triangle downwards until its bottom edge is on the boundary of $U \cup V$, and then tile the path with horizontal rhombi and the rest with vertical rhombi. □

Motivated by Theorem 3.4, we pose a similar question in three dimensions.

Question 3.5. In a tetrahedral-octahedral honeycomb, let U be a set of upside-down tetrahedra, let V be the set of octahedra adjacent to U, and let W be the set of right-side-up tetrahedra adjacent to V (where adjacent means shares a face). Let S be a spread-out subset of W with size $|W| + |U| - 2|V|$. Does there always exist a fine mixed subdivision of $(U \cup V \cup W)$ with tetrahedra at S?

4. Tropical Pseudolines and Pseudoplanes

In this section, we introduce the concept of *tropical pseudoplanes* (*tropical pseudolines* when $d - 1 = 2$), as defined in [2]. We then provide another proof of the two-dimensional case using pseudolines. For brevity, we will omit the word "tropical" for the remainder of this paper.

First, we define pseudolines when $d - 1 = 2$. Take a fine mixed subdivision of $n\triangle_2$. For each unit triangle in the subdivision, draw segments from its centroid to its three edges. Then, for each rhombus, draw segments connecting each opposite pair of midpoints of edges. The result is an arrangement of n pseudolines ("bent tropical lines"), as color-coded in Figure 8.

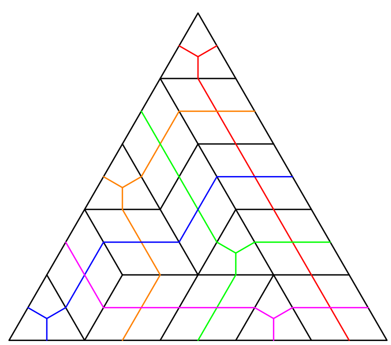

Figure 8. An example of pseudolines when $n = 5$. Each pseudoline is shown in a different color.

Pseudolines have some interesting properties. Each one consists of three "branches" emanating from the center of a triangle meeting a different edge of the triangle. Any two pseudolines intersect exactly once, and any cell can be defined as the intersection of two pseudolines uniquely (where a triangle can be thought of as intersecting its pseudoline with itself).

We can similarly define pseudoplanes when $d - 1 = 3$. Take a fine mixed subdivision of $n\triangle_3$. For each unit tetrahedron, first draw the segments from its centroid to each of the four faces, then fill in the planes between these segments to connect the centroid to each of the six edges. For each triangular prism, first draw the segment connecting the centroids of its two triangular faces, then draw planes connecting that segment to the midlines of each of the three rhombus faces. Finally, for parallelepipeds, draw in the three midplanes.

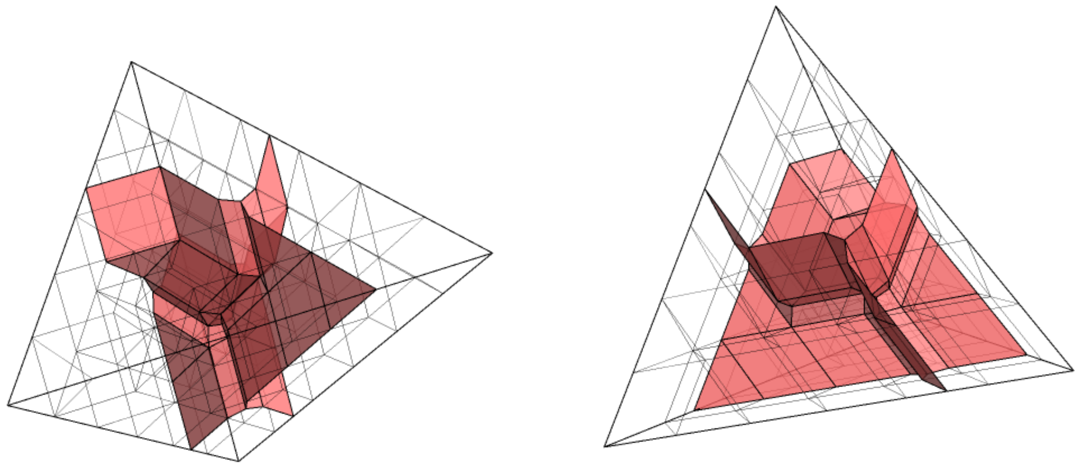

FIGURE 9. An example of a pseudoplane colored in red, from two different angles.

Pseudoplanes share very similar properties with pseudolines. Each one consists of four "branches" and six "sheets." Each branch intersects a face of $n\triangle_3$, and each component intersects an edge. Any three pseudoplanes intersect exactly once, and any cell can be defined as the intersection of three pseudoplanes uniquely (where a tetrahedron is the intersection of its pseudoplane with itself three times, and a triangular prism is the intersection of the same pseudoplane twice with a different one).

These pseudoplanes and pseudolines give us a combinatorial perspective towards understanding tilings, and attempting to construct them opens up various methods of creating fine mixed subdivisions, as we will see in the sections below.

Now we provide an alternative proof for Question 2.7 in the $d-1=2$ case.

Pseudoline Proof. For any set of n spread-out triangles, using Lemma 3.3, we can generate a pseudoline through one of the triangles with the property that no matter where we move the triangle along the psuedoline, the resulting configuration is still spread-out. Then, we can delete that pseudoline and "collapse" the remaining shape, as shown in Figure 10.

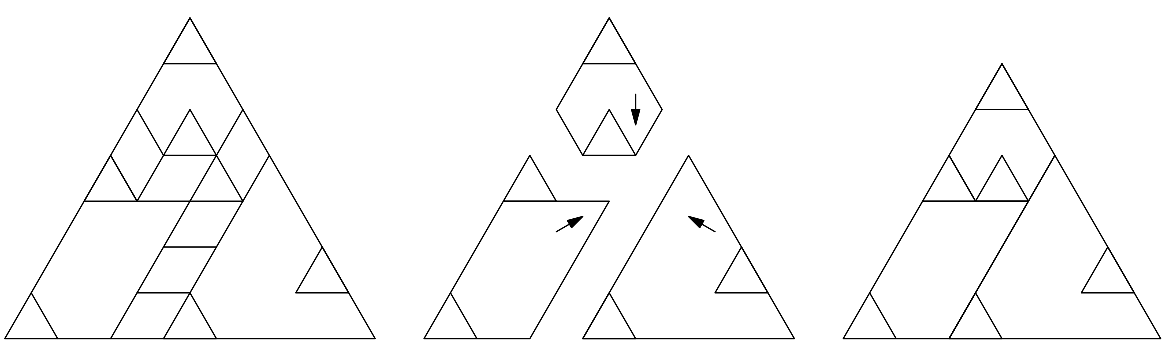

FIGURE 10. Creating and collapsing a pseudoline.

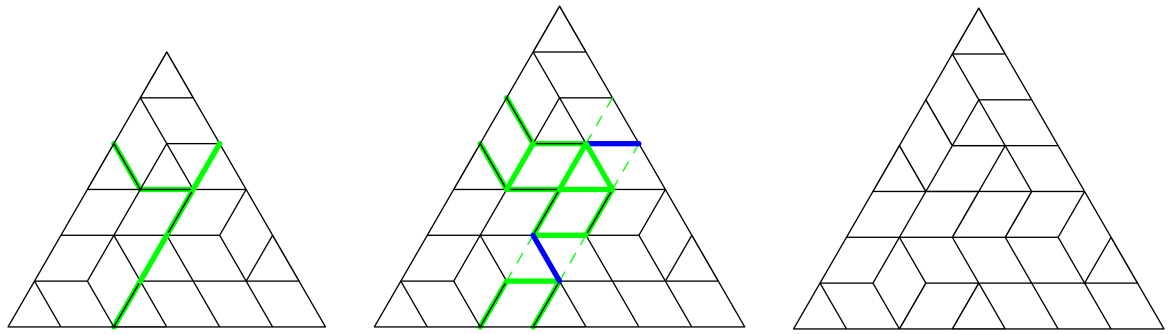

FIGURE 11. Adding back a pseudoline, drawn in green. Blue lines show "corrections".

Clearly, any triangle that is contained in one of the three "sections" still satisfies the spread-out condition.

After the collapse, consider a triangle T that overlaps with the pseudoline. Its pre-image from before the collapse is contained in a triangle T' of size one greater than that of T. Consider moving the unit triangle we removed along the pseudoline into T'. Then, T' must still satisfy the spread-out condition; removing that triangle, we see that T is also spread-out, because T has size one less than that of T', and has at least one less triangle now.

Hence, our new configuration is still spread-out. We can find a tiling of it, then attempt to add back the pseudoline. Whenever the pseudoline cuts "through" a rhombus, we can replace it with two rhombi, as shown in the dashed portions of Figure 11, corrected in blue. The rest of the pseudoline is shown in green.

Hence, we can reduce the size-n case to the size-$(n-1)$ case, so we can conclude via induction. □

Unfortunately, this does not fully generalize to $d-1=3$. The main problem is that when adding back the pseudoplane, if it cuts through any cells, it is not as apparent how to "fix" these cells, whereas in two dimensions it was easy to replace them with two rhombi. However, it may be possible to try to pick a specific pseudoplane, as opposed to arbitrarily choosing one, and attempting to find a tiling for the resulting size-$(n-1)$ configuration such that the pseudoplane will not intersect with any cells.

5. One Tetrahedron in the Bottom Layer

In this section, we will show that in a size-n case with exactly one tetrahedron in the last layer (along a triangular face), we can always derive a fine mixed subdivision by induction from the size-$(n-1)$ case.

First, we tile the top size-$(n-1)$ tetrahedron, which has $n-1$ spread-out tetrahedra in it, by the induction hypothesis.

Consider the second-bottom face's cross section (the bottom of the size-$(n-1)$ tiling). It must be a fine mixed subdivision of a size-$(n-1)$ triangle, where the given tetrahedron in the bottom layer intersects it at a point (as shown by the dot in Figure 12).

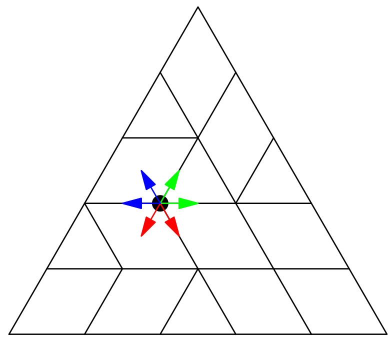

FIGURE 12. An example of the cross-section when $n = 6$. Each color represents the two arrows that point towards the same edge.

Notice the six directions around the dot labelled red, green, and blue. It is impossible for the dot to have a 180-degree angle around it, so at least one of each color edge must be present in the subdivision. (In this case, the lower blue edge, the rightwards red edge, and both green edges are present.) This property holds for every point. Hence, we can find a path from the dot to the lower edge of the triangle using only red directions, and likewise for the other two edges and colors. (This path may be non-unique.) The result resembles a pseudoline, as seen in Figure 13.

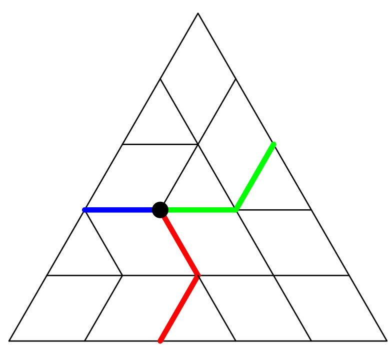

FIGURE 13. A pseudoline is drawn, each branch in a different color.

In each face we will draw an arrow to denote the Minkowski sum of the face with an edge in the arrow's direction. (While this is an aerial view, each arrow has a downwards component so that the cells occupy the bottom layer.) Notice how the pseudoline divides the tiling into three sections; we draw arrows in each one pointing away from the dot, as seen in Figure 14. (Two example cells are drawn in near the top, where the dotted edges are beneath the dashed edges, which are beneath the solid edges.)

Each edge along the pseudoline now borders two faces with different arrows. Take the Minkowski sum of that edge and the triangular face formed by these arrows. Along with the tetrahedron itself, this is a complete tiling of the bottom layer. This is shown in Figure 15, with dotted edges being on

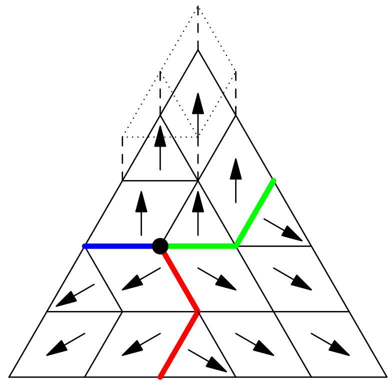

FIGURE 14. We start filling the bottom layer.

the bottom face, dashed in the bottom layer, and solid on top. Combining this with the size-$(n-1)$ subdivision earlier, we get a desired size-n subdivision.

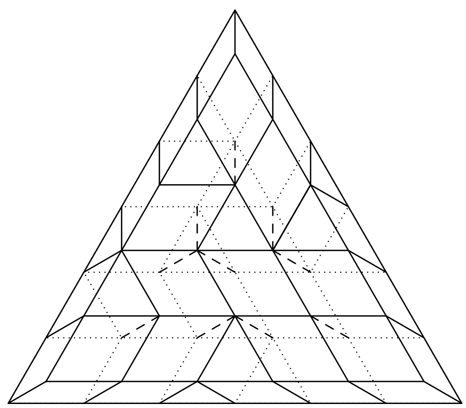

FIGURE 15. The bottom layer is completed.

In fact, we claim that the "converse" holds too:

Theorem 5.1. *If a fine mixed subdivision of $n\triangle_3$ has exactly one tetrahedron in the bottom layer, we can remove the entire bottom layer to obtain a fine mixed subdivision of $(n-1)\triangle_3$.*

Proof. Notice that each pseudoplane has one component that touches each edge, and one branch that touches each face. Number the tetrahedra 1 through n, with 1 being in the bottom layer. Each cell can be expressed as the intersection of three pseudoplanes of tetrahedra a, b, c, possibly repeated; we denote this cell by an *unordered* triple (a, b, c). (For example, tetrahedron 1 is denoted $(1, 1, 1)$.) A triangular prism must be of the form (a, a, b) for some $a \neq b$.

Assume for sake of contradiction that some triangular prism in the bottom layer is of the form (a, a, b) for $a \neq 1$ and $b \neq 1$. Then, it is the intersection of some branch of pseudoplane a with some component of pseudoplane b. It must be on the downwards branch of pseudoplane a, as that is the only branch that can extend downwards into the bottom layer. Likewise, the branch of pseudoplane b must be towards one of the three bottom edges, say towards edge E.

Now, the components of pseudoplanes a and b towards edge E cannot intersect. However, the downwards branch of pseudoplane a is part of its component towards E, which means that the intersection of the components of pseudoplanes a and b towards edge E must contain the triangular prism, contradiction.

Hence, any triangular prism in the bottom layer is part of pseudoplane 1. In particular, there are only $2n-2$ such prisms, one for each of the form $(1,1,x)$ and $(1,x,x)$ for $x \neq 1$, so there are at most $2n-2$ prisms in the bottom layer. Say there are $k \leq 2n-2$ prisms in the bottom layer.

Let us overlay a tetrahedral-octahedral honeycomb with our tiling. Notice that any cell that covers a right-side-up tetrahedron of this honeycomb in the bottom layer must be wholly contained in the bottom layer. There are $\frac{n(n+1)}{2}$ such tetrahedra; one of them is covered by tetrahedron 1, k of them are covered by prisms, and the remaining $\frac{n(n+1)}{2} - k - 1$ must be covered by parallelepipeds.

Each prism covers half an octahedron, while each parallelepiped covers one whole octahedron (or two halves) and one upside-down tetrahedron. There are $\frac{n(n-1)}{2}$ octahedra and $\frac{(n-1)(n-2)}{2}$ upside-down tetrahedra in the bottom layer; since each of these cells is completely contained in the bottom layer, we must have

$$\frac{1}{2}k + \left(\frac{n(n+1)}{2} - k - 1\right) \leq \frac{n(n-1)}{2} \quad \text{and} \quad \frac{n(n+1)}{2} - k - 1 \leq \frac{(n-1)(n-2)}{2}.$$

Both of these inequalities simplify to $k \geq 2n-2$. Since we also know $k \leq 2n-2$, we must have $k = 2n-2$, which means both inequalities above satisfy equality. In particular, this means that the cells covering the right-side-up bottom-layer tetrahedra in the honeycomb also cover all the octahedra and upside-down tetrahedra in the bottom layer, *i.e.,* they cover the whole bottom layer. We know these cells are all contained in the bottom layer as well, so they comprise exactly the bottom layer, and we can remove this layer. □

6. Main Theorem

We prove the following theorem:

Theorem 6.1. *Given an arrangement of n tetrahedrons in $n\triangle_3$, consider the triangle formed by an edge of the $n\triangle_3$ and the midpoint of the opposite edge. Suppose that the projection of the tetrahedra onto this triangle creates a spread-out configuration of triangles, so that it can be tiled by a two-dimensional tiling T. Then, there exists a fine mixed subdivision with those tetrahedra where each cell in the tiling projects to a cell of T.*

Proof. We construct such a tiling in three steps. Let E be the edge perpendicular to the plane of T, and let D be the edge opposite E. We define directions of "top"/"up" and "bottom"/"down" along edge E. Given a parallelogram in T, call it *Type-A* if it has an edge parallel to D, and *Type-B* otherwise. Define the *column* of a cell in T to be the portion of the large tetrahedron that projects to it. Illustrated in Figure 16 is an example of an arrangement of the tetrahedra, with the Type A and Type-B parallelograms labelled.

Step 1: Constructing Columns

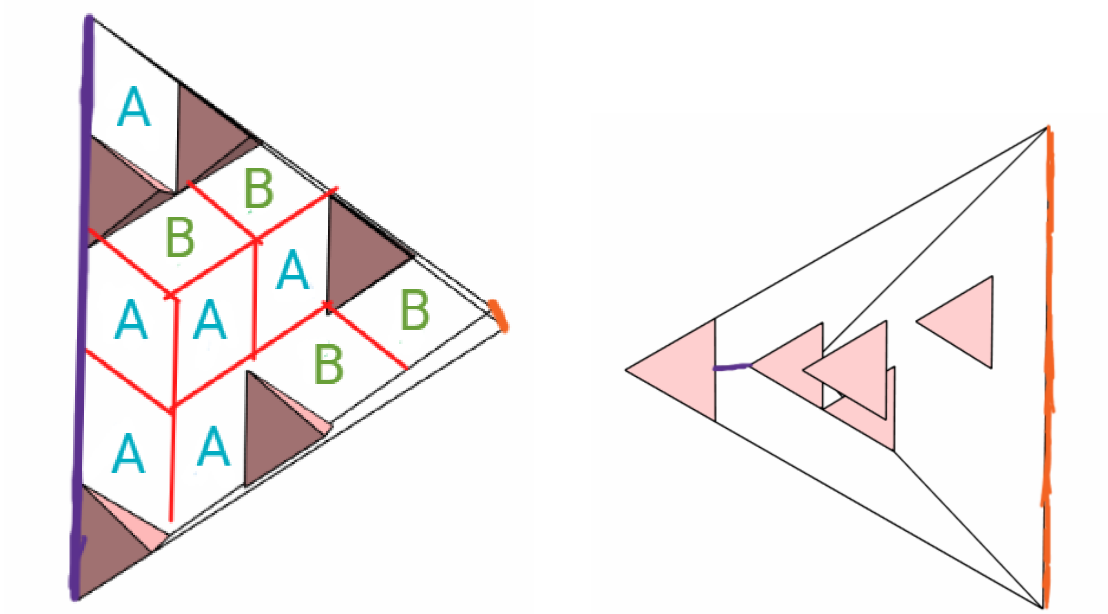

FIGURE 16. Left: Top view, with cell type labeled. Right: Side view. Edges D and E are drawn in purple and orange, respectively.

For each triangular cell in T, we fill the rest of its column (which already has a tetrahedron in it) by extending triangular prisms parallel to E, as shown in Figure 17.

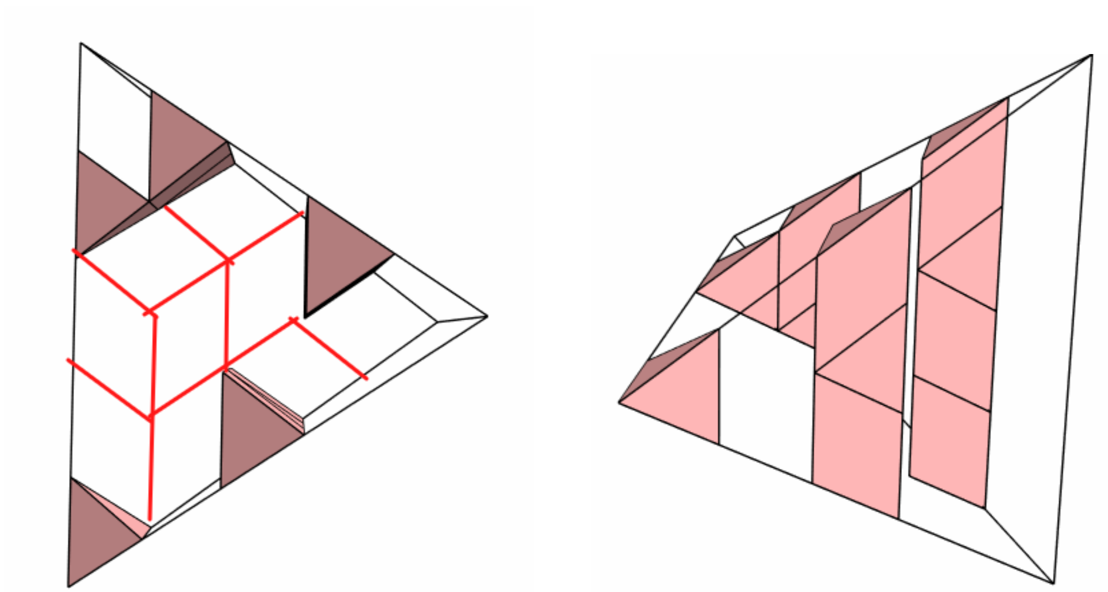

FIGURE 17. Constructing columns on each tetrahedron (Step 1).

Step 2: Extending Pseudolines

Consider the pseudolines in T, as shown in Figure 18. For each pseudoline, ignore the branch that points towards D, and construct triangular prisms from its tetrahedron along the remaining two branches.

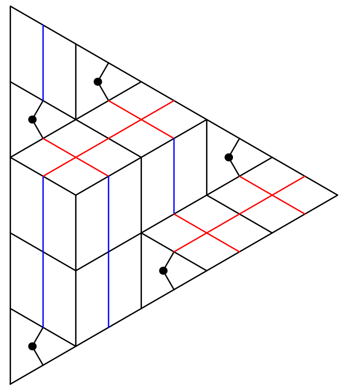

FIGURE 18. The pseudolines of T, blue in Type-A cells and red in Type-B cells.

Each Type-A cell in T will only have one branch through it, as we discarded the branch pointing towards D. Hence, any prism we place that projects to a Type-A cell extends parallel to D.

On the other hand, each Type-B cell is the intersection of two branches. Neither of them are parallel to D, so any prism we place along these branches can either extend upwards or downwards (in addition to along the pseudoline). To decide which direction we extend in, the pseudoline which is currently higher extends upwards, and the lower one extends downwards (so they always extend away from each other). If they are at the same height, then we arbitrarily pick one to extend upwards and one to extend downwards.

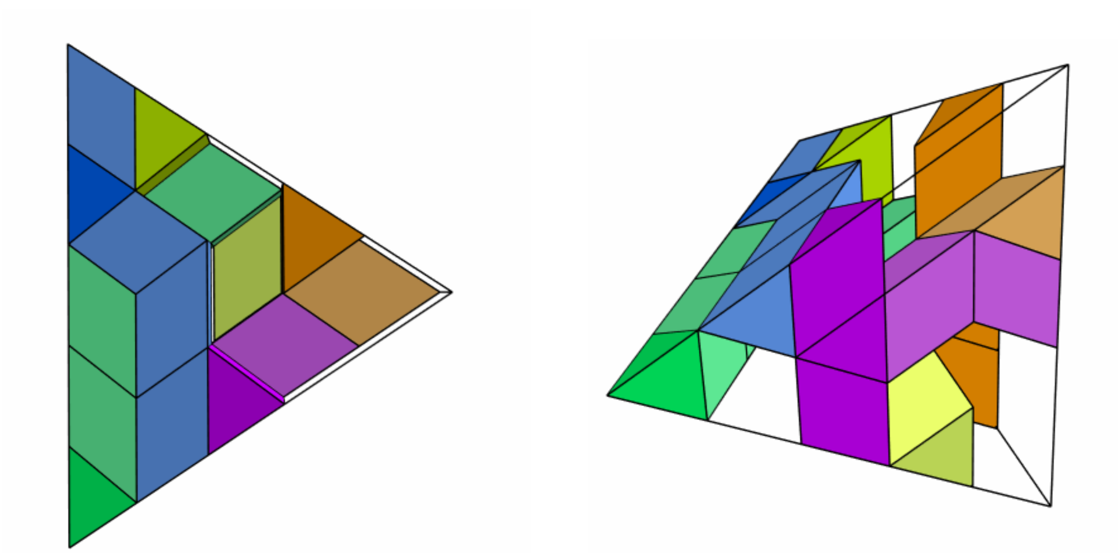

FIGURE 19. Extending pseudolines (Step 2). The pseudolines are color-coded based on their tetrahedron (which are in darker shades of the same color).

We decide when pseudolines extend upwards or downwards starting along D, going by edge layer until we reach E. (This is because whether a pseudoline goes up or down depends on its current height, which depends on how it extended in the previous edge layers.)

Step 3: Filling Remaining Columns

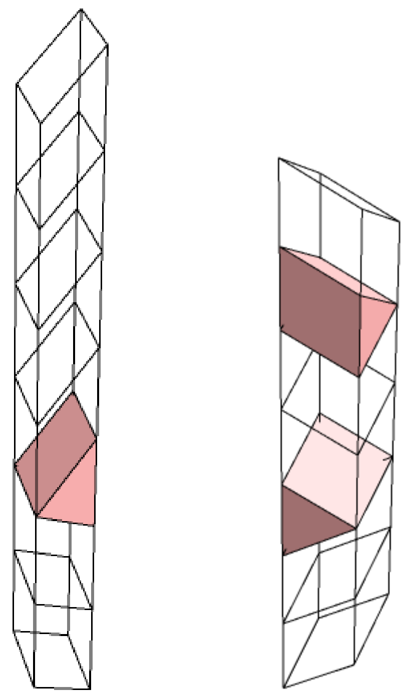

Figure 20. Left: A type-A column. Right: A type-B column. The triangular prisms in the columns are solid while the parallelepipeds are transparent.

At this point, we claim that we can fill the remaining columns with parallelepipeds.

Each column stemming from a Type-A cell has only one prism in it which runs parallel to D. This prism's top face is parallel to the top face of the large tetrahedron, and their bottom faces are also parallel, so the remaining spaces can be filled with parallelepipeds.

Each column stemming from a Type-B cell has two prisms in it. The lower prism must extend downwards by our construction, so its bottom face is parallel to the bottom face of the large tetrahedron, so we can fill the space between them (if there is one) with parallelepipeds. Its top face and the bottom face of the higher prism are also parallel (both are squares and may coincide), so we can fill the space between them with parallelepipeds as well. Finally, the top face of the higher prism is parallel to the top face of the large tetrahedron, so we can fill the space between them with parallelepipeds as well.

Hence, the whole tetrahedron is tiled. It remains to show that all the cells intersect properly.

Clearly, the cells within a single column all intersect properly with each other. Whenever two columns meet, they either intersect at a rectangular face or a trapezoidal face. If they intersect at a rectangular face, it must be subdivided into unit squares on both columns, so they must intersect properly.

If they intersect at a trapezoidal face, it must be subdivided into one triangle and unit parallelograms on both columns. Furthermore, the triangular face must be at the same position on both columns,

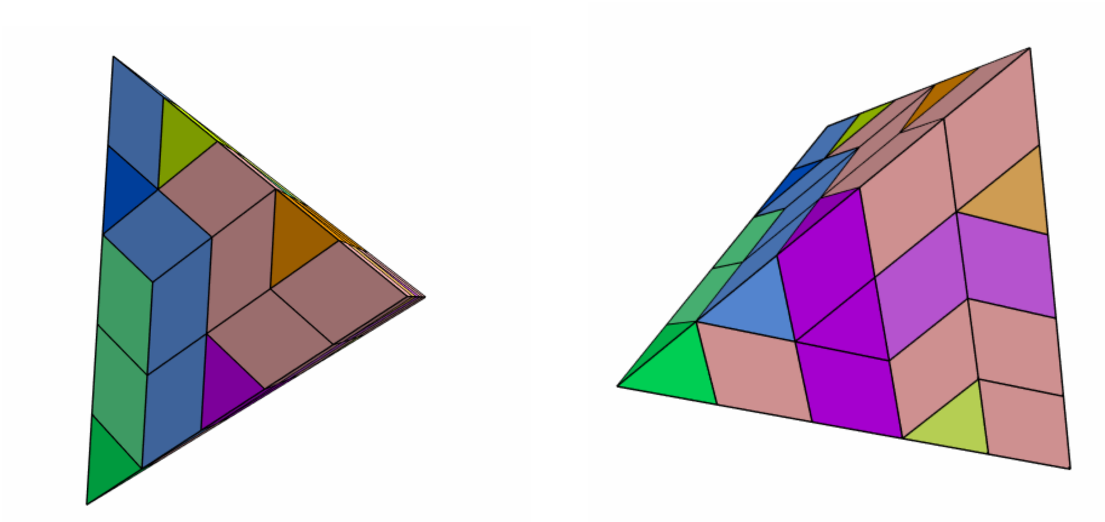

FIGURE 21. Completing the tiling (Step 3).

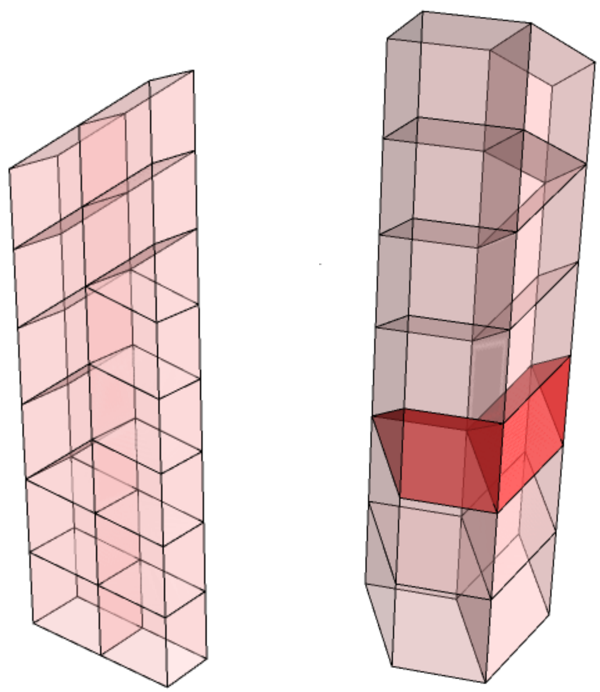

FIGURE 22. Two types of column intersections. Left: rectangular intersection. Right: trapezoidal intersection, where the two prisms with shared triangular face are highlighted.

because they arose from the pseudolines we constructed in Step 2. Once the triangle is placed, the remaining unit parallelograms are fixed in the same way on both columns, so they intersect properly.

Hence, this construction always produces a valid subdivision. □

Remark. This theorem covers some special cases for the tetrahedra's arrangements:

(1) There is one tetrahedron in each edge layer (layers starting from an edge towards the opposite edge).
(2) There is one tetrahedron in each face layer (layers starting from a vertex towards the opposite face).
(3) All the tetrahedra are touching the same face.

The first two cases hold because in the two-dimensional case, one triangle per row automatically implies a spread-out arrangement. The last case holds because the tetrahedra must form a two-dimensional spread-out arrangement along that face, which the projection will preserve.

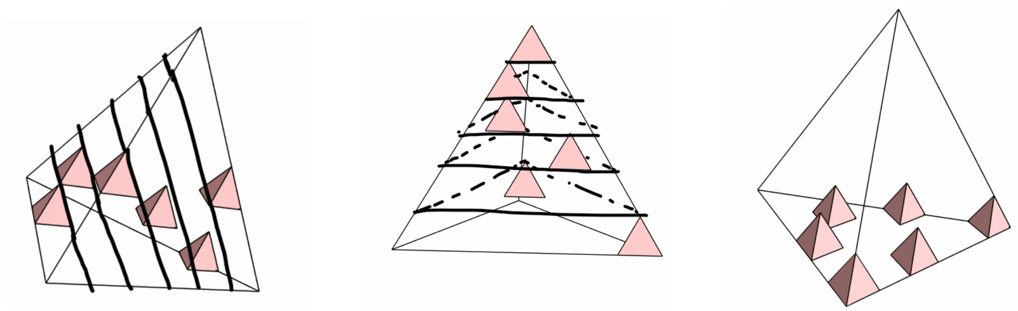

FIGURE 23. Three special cases of the main theorem. Left: One in each edge layer. Center: One in each face layer. Right: All on one face.

7. Possible Approaches to Resolve the General $d-1=3$ Case

Here, we outline possible approaches to the problem in three dimensions.

One idea is to try to generalize our main theorem to "projections" along "bent lines," instead of straight edges. This way, the "columns" are not straight columns, but pseudolines, or segments of pseudoplanes. This could allow our theorem to generalize to more arrangements of tetrahedra.

Another possible approach is to "slide" the tetrahedra (as we did in two dimensions) to a more favorable arrangement, such as one our main theorem covers. We can find a subdivision with those tetrahedra, then attempt to "slide" them back to their original positions. However, this is much more difficult to do in three dimensions than 2 because the cells have to intersect properly – attempting to slide tetrahedra will disrupt how they intersect with their neighboring cells.

A third approach is similar to the second proof of Section 3. In three dimensions, there are 12 possible "sliding" directions. For each face, three of these directions are towards that face, three are away, and six are neither.

Select an edge E of the large tetrahedron which borders faces A and B. Of the three sliding directions towards A, only one of them is away from B, while the other two remain the same distance from B. Similar to Lemma 3.3, we know that we can pick one of these two directions to slide a unit tetrahedron in, and we can "by layer" slide all the tetrahedra onto face A this way. They must still be spread-out. Let us call each path we created a "branch," which is part of the pseudoplane we will construct.

Then, we can, as above, slide all the tetrahedra onto edge E. This time, though, we do not only slide triangles; we will slide the whole branch in a similar manner. This gives us a component of a pseudoplane for each tetrahedron that touches edge E.

This approach has flaws, most notably that one of the pseudoplanes we create may not intersect properly with a tetrahedra adjacent to it. However, it is a direct and plausible approach to the problem.

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