MODULI SPACES OF MORPHISMS OF CONE STACKS

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ABSTRACT. We study morphisms between *cone stacks*, objects defined by Cavelieri, Chan, Ulirsch, and Wise as a framework for moduli problems in tropical geometry. We construct a cone stack $[\Sigma, \Gamma]$ parameterizing morphisms between fixed cone stacks Σ and Γ. We also briefly discuss applications to logarithmic geometry.

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1. INTRODUCTION

1.1. **Tropicalization and cone stacks.** Let X be a variety and D a divisor on X. When X is a toric variety and D is the complement of the dense open torus in X, we can associate a *lattice fan* (N_X, Σ_X) to X encoding the combinatorics of D. Remarkably, this extends to an equivalence between the category of normal toric varieties with torus-equivariant maps and the category of lattice fans—in other words, the geometry of X is *entirely* described by the combinatorial data of its lattice fan. In the absence of a torus action, we can still produce a *cone complex* $\Sigma_{(X,D)}$ encoding the combinatorics of the pair (X, D). However, it comes without a canonical embedding $\Sigma_X \hookrightarrow N$ into a lattice, and we cannot hope to recover X in its entirety from $\Sigma_{(X,D)}$. This represents an example of *tropicalization*, a term used in many contexts to indicate replacement of geometric objects by combinatorial or piecewise-linear models.

Moduli spaces provide an important example of this situation. In particular, we can consider the complement D of the smooth locus $\mathcal{M}_{g,n}$ within the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. In [4], Cavalieri, Chan, Ulirsch, and Wise introduced *cone stacks* as a general foundation for moduli spaces in tropical geometry, an important development [2] that forms the basis of the present work. As the main example they constructed a moduli stack for *tropical curves*, a cone stack parameterizing families of certain decorated graphs that models the moduli space of algebraic curves $\mathcal{M}_{g,n}$. Since then, the framework of cone stacks has been applied to various moduli problems in tropical geometry: see [7, 9, 11, 12, 15].

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Kato's theory of *logarithmic structures*, laid out in [8], has played an important role in formalizing this process. In essence, a log structure M_X on X distinguishes certain functions on each étale neighborhood of X as "monomials," which makes X look étale-locally like a toric variety. For example, a pair (X, D) can be represented as a logarithmic structure by letting M_D be the sheaf of functions on X that are invertible away from D. Conversely, one can recover the divisor D from the *logarithmic scheme* (X, M_D), though in general not every logarithmic structure comes from a divisor.

With mild hypotheses, one can associate to a logarithmic scheme X a logarithmic algebraic stack $\mathcal{A}_X$ called the *Artin fan* of X [3, Prop. 3.2.1]. This turns out to be a combinatorial object generalizing the fan of a toric variety. Most important for our purposes is the equivalence of 2-categories between Artin fans and cone stacks [4, Theorem 3], allowing us to speak of the *cone stack* $\Sigma(\mathcal{X})$ *associated to a logarithmic algebraic stack* $\mathcal{X}$. Figure 1 illustrates the relationships between all of these categories.

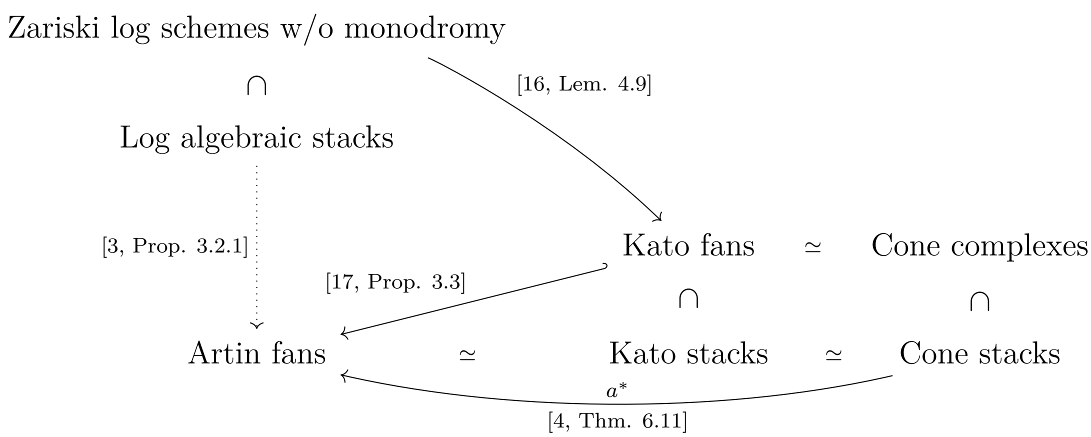


FIGURE 1. A commutative of functors from logarithmic catgories to combinatorial categories. The dotted arrow indicates that the association of Artin fans to logarithmic algebraic stacks generally fails to be functorial. For more information on the unlabeled equivalences involving Kato fans and Kato stacks, see [4, Remark 5.6].

1.2. **Mapping stacks.** Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks over a site (C, τ). The *mapping stack* of $\mathcal{X}$ and $\mathcal{Y}$ is characterized by its functor of points

$$\mathrm{Map}(\mathcal{X}, \mathcal{Y}) \colon \mathsf{C}^o \to \mathbf{Gpd}$$
$$c \mapsto \mathrm{Hom}_{\mathbf{St}_{\mathsf{C}}}(\mathcal{X} \times c, \mathcal{Y}).$$

This is the "right definition" in the sense that it yields an adjunction

$$\mathrm{Hom}_{\mathbf{St}_{\mathsf{C}}}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathrm{Hom}_{\mathbf{St}_{\mathsf{C}}}(\mathcal{X}, \mathrm{Map}(\mathcal{Y}, \mathcal{Z})),$$

making the category of stacks on C into a cartesian-closed category. This is what is meant by "moduli space of morphisms" of either cone stacks or logarithmic algebraic stacks.

The notion of mapping stack extends the familiar notions of "mapping object" in other areas of mathematics. A simple example is the vector space $\mathrm{Hom}_{k-\mathsf{Vect}}(V, W)$ of linear maps between vector spaces V and W. Other examples include the compact-open topology on the set $\mathrm{Hom}_{\mathsf{Top}}(X, Y)$ of continuous maps between topological spaces X and Y, as well as the category of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$, whose morphisms are natural transformations.

Mapping stacks have been studied in various contexts, with the typical goal being to prove representability of $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ by a *geometric stack* in the case that $\mathcal{X}$ and $\mathcal{Y}$ are sufficiently nice geometric stacks (in the terminology of [4, Section 1]). For example, Noohi has proven that when $\mathcal{X}$ and $\mathcal{Y}$ are topological stacks and $\mathcal{Y} = [Y_1 \rightrightarrows Y_0]$ admits a groupoid presentation by compact topological spaces, $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ is a topological stack [10]. In the algebraic context, Hall and Rydh showed that $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ is an algebraic stack assuming properness of $\mathcal{X}$ and other mild conditions [6]. Most relevant for our purposes, Wise proves an analagous result for logarithmic algebraic stacks in [16].

The main result of this paper is the following.

Theorem 1.1. *If $\mathcal{X}$ and $\mathcal{Y}$ are cone stacks and $\mathcal{X}$ has finitely many faces, then the mapping stack* $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ *is representable by a cone stack.*

Of particular interest is the case where $\mathcal{X}$ and $\mathcal{Y}$ correspond to the tropicalizations of logarithmic schemes. In that case we can ask about the relationship between their mapping stack and that of their tropicalizations.

2. Mapping stacks of cone stacks

We briefly recall the definitions of [4]. Let N be a finitely generated free abelian group; let M be the dual lattice $\mathrm{Hom}(N, \mathbb{Z})$; denote $N \otimes \mathbb{R}$ by $N_{\mathbb{R}}$. A *rational polyhedral cone* is a pair (N, σ) consisting of a lattice N and a subset $\sigma \subset N_{\mathbb{R}}$ equal to the intersection of finitely many half-spaces $H_i = \{u \in N_{\mathbb{R}} \mid \langle u, v_i \rangle \geq 0\}$, where $v_i \in M$ are chosen such that σ contains no nontrivial linear subspaces. A morphism of cones $(N, \sigma) \to (N', \sigma')$ is a group homomorphism $N \to N'$ under which the image of σ lies in σ'. The resulting category is denoted RPC. A *face* τ of a cone σ is a subset of the form $u^{-1}(0) \cap \sigma$ for $u \in M$.

A *cone complex* Σ is a poset diagram of face inclusions in RPC such that all faces of cones in Σ are also in Σ. A morphism of cone complexes $f\colon \Sigma \to \Gamma$ is a collection of morphisms $f_i\colon \sigma_{i_1} \to \gamma_{i_2}$, one for each cone σ_{i_1} of Σ, none of which factor through a proper face map into γ_{i_2}, commuting with face inclusions, such that f forms an order-preserving map between posets. The resulting category is denoted RPCC. Letting the covering families be the sets of face inclusions $\{\sigma_i \to \Sigma\}$ defines a Grothendieck topology τ_{face} on RPCC called the *face topology*. A morphism of cone complexes $f\colon \Sigma \to \Gamma$ is called *strict* if each component $f_i\colon \sigma_{i_1} \to \gamma_{i_2}$ is an isomorphism, and the class of such morphisms is denoted $\mathbb{S}$.

The triple $(\mathsf{RPCC}, \tau_{\text{face}}, \mathbb{S})$ defines a geometric context [4, Prop. 2.6]. The geometric spaces and geometric stacks in this context are respectively called *cone spaces* and *cone stacks*. We use the notation $[X, Y]$ in this section to denote the mapping stack.

Our proof of Theorem 1.1 will proceed in stages of successive generalization, beginning with the simplest case of maps from a cone to a cone. We first state a foundational lemma.

Lemma 2.1. *Let σ, θ, and γ be cones. There is a natural bijection*

$$\mathrm{Hom}(\sigma \times \theta, \gamma) \cong \mathrm{Hom}(\sigma, \gamma) \times \mathrm{Hom}(\theta, \gamma).$$

In other words, RPC *has biproducts.*

Proof. Define the forward direction of the correspondence by sending the map $f\colon \sigma \times \theta \to \gamma$ to the pair $(f_\sigma\colon s \mapsto f(s, 0), f_\theta\colon t \mapsto f(0, t))$. These can be obtained by composing f with the face morphisms $\sigma \times 0$ and $0 \times \gamma$. Since RPC is a category, f_σ and f_γ are cone morphisms.

Define the reverse direction by sending the pair (f_σ, f_θ) to the map $f\colon \sigma\times\theta\to\gamma$ defined by $f(s,t)=f_\sigma(s)+f_\gamma(t)$. The image of f lies within γ because γ is closed under addition, and f is $\mathbb{Z}$-linear because f_σ and f_γ are. We observe that the two directions are mutual inverses, establishing a bijection. □

Proposition 2.2. *Let σ and γ be cones. Then $[\sigma,\gamma]\cong\bigsqcup_{\mathrm{Hom}(\sigma,\gamma)}\gamma$.*

Proof. By [4, Prop. 2.3], it suffices to check this on cones. That is, we need to exhibit a natural isomorphism between $[\sigma,\gamma](\theta)=\mathrm{Hom}(\sigma\times\theta,\gamma)$ and $\mathrm{Hom}(\theta,\bigsqcup_{\mathrm{Hom}(\sigma,\gamma)}\gamma)$ for all cones θ. We observe that a choosing a morphism from θ to $\bigsqcup_{\mathrm{Hom}(\sigma,\gamma)}\gamma$ is equivalent to choosing an index in $\mathrm{Hom}(\sigma,\gamma)$, then choosing a cone morphism in $\mathrm{Hom}(\theta,\gamma)$. But by Lemma 2.1, this data is equivalent to an element of $\mathrm{Hom}(\sigma\times\theta,\gamma)$. □

By [4, Remark 2.2], cone complexes are equivalently described in [1] as follows. The category of rational polyhedral cone complexes is the full subcategory of $\mathsf{PSh}(\mathsf{RPC})$ consisting of presheaves of the form $\Sigma:=\varinjlim_I \mathrm{Hom}_{\mathsf{RPC}}(-,\sigma_i)$, where I is a poset diagram in RPC consisting of proper face morphisms and identities. We abuse notation and write $\Sigma=\varinjlim\sigma_i$.

Proposition 2.3. *Let σ be a cone and let $\Gamma=\varinjlim\gamma_i$ be a cone complex. Then the presheaf colimit $\varinjlim\,[\sigma,\gamma_i]$ of the induced diagram of cone complexes is a cone complex representing $[\sigma,\Gamma]$.*

Proof. First we show that the colimit of the diagram of cone complexes $[\sigma,\gamma_i]$ is a cone complex. By Proposition 2.2, the objects of this diagram are just disjoint unions of cones γ_i. Likewise, face inclusions inside $[\sigma,-]$ map to coproducts of face inclusions outside $[\sigma,-]$. Hence, we may view this diagram as a poset diagram of cones and face morphisms, giving rise to a cone complex.

Now we show that $\varinjlim\,[\sigma,\gamma_i]$ actually represents the mapping stack $[\sigma,\Gamma]$. Let θ be a cone. We have natural isomorphisms

$$\begin{aligned}[\sigma,\Gamma](\theta)&\cong\mathrm{Hom}(\sigma\times\theta,\Gamma)\\&\cong\Gamma(\sigma\times\theta)\\&\cong\Big(\varinjlim\gamma_i\Big)(\sigma\times\theta)\\&\cong\varinjlim\mathrm{Hom}(\sigma\times\theta,\gamma_i)\\&\cong\varinjlim[\sigma,\gamma_i](\theta).\end{aligned}$$
□

Groupoid presentations will be an important tool for us. Recall that if $(R\rightrightarrows U)$ is a geometric groupoid, the quotient stack $[U/R]$ is the stackification of the category fibred in groupoids

$$[U/R]^{pre}(T):=(\mathrm{Hom}(T,R)\rightrightarrows\mathrm{Hom}(T,U)).$$

We show that in the specific context of RPCC, we can sidestep stackification.

Lemma 2.4. *Let $\mathcal{X}$ be a category fibred in groupoids over RPCC. Then $\mathcal{X}$ restricts to the same category fibred in groupoids over RPC as its stackification.*

Proof. This is follows from the construction of stackification. Every statement involving a covering $\{U_i\to U\}$ trivializes when U is a cone, so the fiber over each cone U is left unchanged by stackfication. □

In light of [4, Proposition 2.3], we will think of stacks (resp. sheaves) on RPCC as categories fibred in groupoids (resp. sets) over RPC. Combined with Lemma 2.4, this will yield simple descriptions of maps to cone stacks (resp. spaces).

Proposition 2.5. *Let σ be a cone, and let $(R \rightrightarrows U)$ be a groupoid of cone spaces. The functor $[\sigma, -]$ commutes with the quotient operation. In other words, we have a natural equivalence of categories fibred in groupoids over* RPC

$$[\sigma, [U/R]] \simeq [[\sigma, U]/[\sigma, R]].$$

Proof. The quotient stack $[\sigma, [U/R]]$ is defined as the stackification of the category fibred in groupoids over RPCC whose fiber over θ is the groupoid $(R(\theta) \rightrightarrows U(\theta))$. So if θ is a cone, then Lemma 2.4 justifies the first isomorphism in the following.

$$[\sigma, [U/R]](\theta) \cong (\mathrm{Hom}(\sigma \times \theta, R) \rightrightarrows \mathrm{Hom}(\sigma \times \theta, U))$$
$$\cong (\mathrm{Hom}(\theta, [\sigma, R]) \rightrightarrows \mathrm{Hom}(\theta, [\sigma, U])) \cong [[\sigma, U]/[\sigma, R]](\theta).$$

We conclude that $[\sigma, [U/R]]$ is isomorphic to $[[\sigma, U]/[\sigma, R]]$. ☐

Corollary 2.6. *Let σ be a cone, and let Y be a cone space. The mapping stack $[\sigma, Y]$ is representable by a cone space.*

Proof. Any cone space Y can be presented as the quotient $[\Gamma_0/\Gamma_1]$ of a groupoid of cone complexes. The mapping stacks $[\sigma, \Gamma_0]$ and $[\sigma, \Gamma_1]$ are cone complexes by Proposition 2.3, so the proposition above yields a presentation of $[\sigma, Y]$ by a groupoid of cone complexes. The quotient is in fact a cone *space* because $[\sigma, Y]$ is fibered in sets. ☐

Corollary 2.7. *Let σ be a cone, and let $\mathcal{Y}$ be a cone stack. The mapping stack $[\sigma, \mathcal{Y}]$ is representable by a cone stack.*

Proof. Any cone stack $\mathcal{Y}$ can be presented as the quotient $[Y_0/Y_1]$ of a groupoid of cone spaces. The mapping stacks $[\sigma, Y_0]$ and $[\sigma, Y_1]$ are cone spaces by Corollary 2.5, so the proposition above yields a presentation of $[\sigma, \mathcal{Y}]$ by a groupoid of cone spaces. ☐

To prove Theorem 1.1, we need the machinery of *2-colimits*, which is summarized in the appendix. The combinatorial description of cone spaces given in [4, Definition 2.12] can be rephrased as follows.

Lemma 2.8. *Let $\mathcal{X}$ be a cone stack. Let $D\colon \mathsf{C} \to \mathbf{GpdFib}_{\mathsf{RPC}}$ be the forgetful functor from the category $\mathsf{C} \subset \mathsf{RPC}/\mathcal{X}$ of strict morphisms and strict 2-commutative triangles. The tautological cocone over D with apex $\mathcal{X}$ induces an equivalence of categories fibered in groupoids*

$$T\colon \text{2-colim}\, D \to \mathcal{X}.$$

Proof. It suffices to show that T restricts to an equivalence on fibers over each cone θ. Similar to colimits of presheaves, 2-colimits of categories fibered in groupoids are computed fiberwise in the following sense.

Let $\mathrm{fib}_\theta\colon \mathbf{GpdFib}_{\mathsf{RPC}} \to \mathbf{Gpd}$ be the strict 2-functor sending a category fibered in groupoids to its fiber over θ. According to [5, Theorem 3.2.25], the fiber of 2-colim D over θ is the 2-colimit of the diagram of (discrete) groupoids $D_\theta := \mathrm{fib}_\theta \circ D$.

In turn, we can compute the 2-colimit of D_θ by applying the Grothendieck construction to $D_\theta\colon (\mathsf{C}^o)^o \to \mathbf{Gpd}$ to obtain the category fibered in groupoids $\int D_\theta \to \mathcal{C}^o$, then formally inverting all morphisms to obtain a groupoid $K(\int D_\theta)$ [5, Theorem 3.2.9]. We shall show that

$$T_\theta\colon K\left(\int D_\theta\right) \to \mathcal{X}(\theta)$$

is essentially surjective and fully faithful.

T_θ IS ESSENTIALLY SURJECTIVE. On objects, T_θ acts by the composition map

$$\bigsqcup_{(s\colon \sigma\to\mathcal{X})\in\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\theta,\sigma) \xrightarrow{\bigsqcup s_*} \mathrm{Hom}(\theta,\mathcal{X}).$$

We observe that if k is a θ-point of $\mathcal{X}$, the θ-points of 2-colim D mapping to k under T_θ are precisely the factorizations of ξ through strict maps $\sigma\to\mathcal{X}$. Such a factorization always exists by [4, Lemma 2.18], so T_θ is surjective.

T_θ IS FULLY FAITHFUL. If

$$\xi = \left(\theta \xrightarrow{\xi^\sharp} \sigma \xrightarrow{s} \mathcal{X}\right) \qquad \text{and} \qquad \zeta = \left(\theta \xrightarrow{\zeta^\sharp} \tau \xrightarrow{t} \mathcal{X}\right)$$

are objects of $\int D_\theta$, then the morphisms from ξ to ζ can be thought of as restrictions of the codomain of $\xi^\sharp$ that yield $\zeta^\sharp$. Specifically, we have

$$\mathrm{Hom}_{\int D_\theta}(\xi,\zeta) = \bigsqcup_{f\in\mathrm{Hom}_{\mathsf{C}}(t,s)} \mathrm{Hom}_{\sigma(\theta)}(\xi^\sharp, f^\sharp\circ\zeta^\sharp),$$

and since the cone σ is fibered in sets, this can be simplified to

$$\mathrm{Hom}_{\int D_\theta}(\xi,\zeta) = \{(f^\sharp,\varphi_f)\in\mathrm{Hom}_{\mathsf{C}}(t,s)\mid \xi^\sharp = f^\sharp\circ\zeta^\sharp\}.$$

We will show that the collection of all morphisms in $\int D$ form a left multiplicative system in the sense of [13, Definition 04VC]. The first condition holds trivially. The second condition, that every solid diagram

$$\begin{array}{ccc} \xi & \longrightarrow & \zeta \\ \big\downarrow & & \big\downarrow \\ \eta & \dashrightarrow & \gamma \end{array}$$

can be completed to a dotted diagram, is satisfied by letting γ be the initial factorization of the map $T_\theta(\xi)\colon\theta\to\mathcal{X}$ guaranteed by [4, Lemma 2.18]. Indeed, that γ factors uniquely through any factorization of $T_\theta(\xi)$ implies that ζ and η restrict to γ, and that the square commutes. The third condition, that any solid diagram

$$\eta \xrightarrow{a} \xi \underset{c}{\overset{b}{\rightrightarrows}} \zeta \overset{d}{\dashrightarrow} \gamma$$

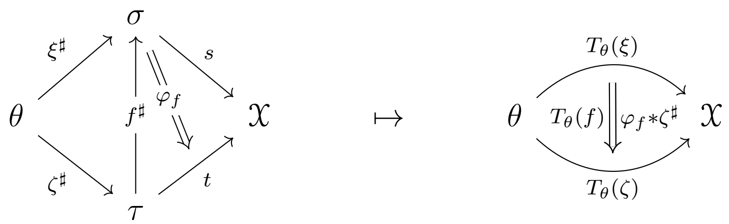
with $b\circ a = c\circ a$ can be completed to a dotted diagram with $d\circ b = d\circ c$, holds by the same argument. Having shown that the collection of all morphisms in $\int D_\theta$ is a left multiplicative system, we can finally describe the morphisms of the localization $K(\int D_\theta)$ [13, Equation 05Q1]:

$$\mathrm{Hom}_{K(\int D_\theta)}(\xi,\zeta) = \operatorname*{colim}_{(g\colon\zeta\to\eta)\in\zeta/\int D_\theta} \mathrm{Hom}_{\int D_\theta}(\xi,\eta).$$

From this it is clear how T_θ acts on morphisms:

$$T_\theta\colon \Big([(f^\sharp,\varphi_f)] : (\xi^\sharp,s)\to(\zeta^\sharp,t)\Big)\mapsto \varphi_f * \zeta^\sharp.$$

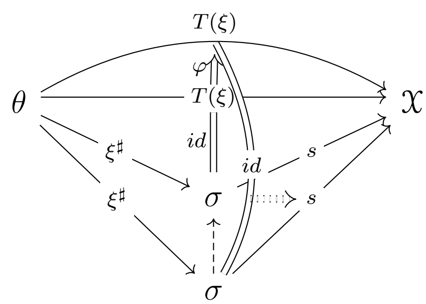
Diagramatically,



To show that a functor between groupoids is fully faithful, it suffices to check that it induces a bijection on automorphism groups. Furthermore, since [4, Lemma 2.18] asserts that every connected component of $K(\int D_\theta)$ contains an object ξ realizing the initial factorization of its image $T_\theta(\xi)$, it suffices to check that *those* automorphism groups are mapped bijectively by T_θ. In other words, we restrict attention to $\xi = (\xi^\sharp, s)$ for which the image of $\xi^\sharp$ lies outside any proper face of its codomain σ. But the colimit describing the automorphisms of such ξ is taken over the category of restrictions of ξ, and no nontrivial restrictions of ξ exist. It follows that

$$\mathrm{Aut}_{K(\int D_\theta)}(\xi) \cong \mathrm{Aut}_{\int D_\theta}(\xi).$$

Let $\varphi\colon T_\theta(\xi) \Rightarrow T_\theta(\xi)$ be an automorphism in $\mathcal{X}(\theta)$. Composing φ with the identity 2-morphism realizing the commutativity of the triangle formed by ξ yields a new factorization of $T(\xi)$ through a strict map from a cone to $\mathcal{X}$. By hypothesis, ξ is initial among such factorizations, so there exists a unique 2-commutative diagram



in which the dotted 2-cell must yield φ when prepasted with $\xi^\sharp$. In other words, there is a unique automorphism of ξ mapping to φ under T_θ. Therefore T_θ is fully faithful. □

Proof of Theorem 1.1. By Lemma 2.8, we have $[\mathcal{X}, \mathcal{Y}] = [\text{2-colim}\, \sigma_i, \mathcal{Y}]$. By Proposition A.3, we have $[\text{2-colim}\, \sigma_i, \mathcal{Y}] = \text{2-lim}[\sigma_i, \mathcal{Y}]$. This is a 2-limit of cone stacks by Corollary 2.7. Whenever the category of strict maps to $\mathcal{X}$ has a finite equivalent subcategory, this 2-limit becomes finite, in which case the theorem then follows from Proposition A.7 that finite 2-limits of cone stacks are cone stacks. □

Appendix A. 2-limits and 2-colimits of stacks

This appendix collects general results about 2-categories of stacks that the author was unable to locate in the literature. However, we make no claims as to the originality of these results. Our main reference is [5, Chapter 2]. We work over an arbitrary base category C.

The literature contains several notions of 2-categorical limit. For our purposes, we seek a definition that generalizes the familiar 2-fiber product of stacks in the simplest way.

Stacks over C form a strict (2,1)-category $\mathbf{St}_{\mathsf{C}}$. We could therefore view $\mathbf{St}_{\mathsf{C}}$ as a (1-)category enriched in groupoids. Enriched category theory then furnishes a definition of *weighted limit*, which some authors call a 2-limit in the groupoid-enriched context [14]. Intuitively, the F-weighted limit of a diagram D is the universal "F-shaped" cone over D. We have no need for this much control, and would prefer to say something like "the 2-limit of a diagram is the universal 2-commutative cone over that diagram." In [14], this is called a *pseudo-limit*. We will use the following definition, which is a special case of [5, Definition 2.5.1].

Definition A.1. Let I be a small category, which we view as a 2-category with trivial 2-morphisms. Let $D\colon I \to \mathbf{St}_{\mathsf{C}}$ be a strict 2-functor. In this situation we call D a *diagram of stacks*. Denote by $\Delta_{\mathcal{Z}}\colon I \to \mathbf{St}_{\mathsf{C}}$ the constant 2-functor at $\mathcal{Z} \in \mathrm{Ob}(\mathbf{St}_{\mathsf{C}})$. A *pseudo-cone* over the diagram D is a pseudo-natural transformation $\eta\colon \Delta_{\mathcal{Z}} \Rightarrow D$. We write $\mathsf{PsNat}(\Delta_{\mathcal{Z}}, D)$ for the category whose objects are pseudo-cones and whose morphisms are modifications.

Definition A.2. Let $D\colon I \to \mathbf{St}_{\mathsf{C}}$ be a diagram of stacks. The *2-limit* of D is a stack $L \in \mathrm{Ob}(\mathbf{St}_{\mathsf{C}})$ with a pseudo-cone $\eta\colon \Delta_L \Rightarrow D$ such that

$$\eta \circ -\colon\ \mathrm{Hom}(Z, L) \cong \mathsf{PsNat}(\Delta_Z, \Delta_L) \to \mathsf{PsNat}(\Delta_Z, D)$$

is an equivalence of categories, natural in Z.

Pseudo-cocones and *2-colimits* are defined dually.

Proposition A.3. *The contravariant mapping stack functor converts 2-colimits in the first argument to 2-limits.*

Proof. Let $D\colon I \to \mathbf{St}_{\mathsf{C}}$ be a diagram of stacks. By [5, Remark 2.5.2(vi)], the 2-colimit of D corresponds exactly to the 2-limit of the opposite functor $D^o\colon I^o \to \mathbf{St}_{\mathsf{C}}^o$. The mapping stack functor realizes a 2-adjunction

$$[-, \mathcal{X}]^o : \mathrm{Stacks} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathrm{Stacks}^o : [-^o, \mathcal{X}]\ .$$

See [5, Definition 2.4.15] for details. Right 2-adjoints preserve 2-limits, by [5, Proposition 2.5.9]. Hence, $[-^o, \mathcal{X}]$ maps the (opposite of) the 2-colimit of D to the 2-limit of the composite $[-^o, \mathcal{X}] \circ D^o\colon I^o \to \mathbf{St}_{\mathsf{C}}$. ☐

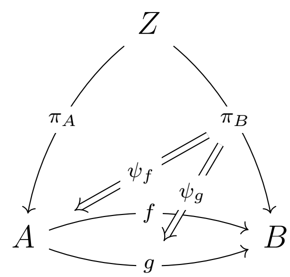
Fact A.4. *In any 2-category, the category of pseudo-cones over a parallel pair of arrows $f, g : A \rightrightarrows B$ is equivalent to the category of pseudo-cones over the following diagram.*

$$B \xrightarrow{\ \Delta\ } B \times B \xleftarrow{(f,g)} A$$

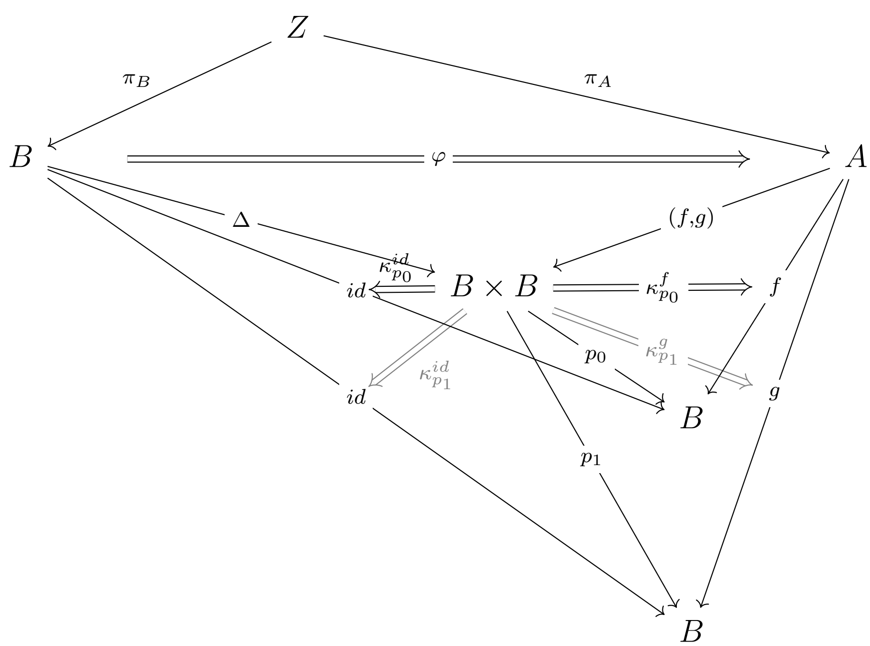
Proof. Suppose we are given a pseudo-cone over the cospan diagram. Such a thing is determined up to isomorphism by a 2-commutative square

$$\begin{array}{ccc} Z & \xrightarrow{\ \pi_A\ } & A \\ {\scriptstyle \pi_B}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle (f,g)} \\ B & \xrightarrow{\ \Delta\ } & B \times B \end{array} .$$

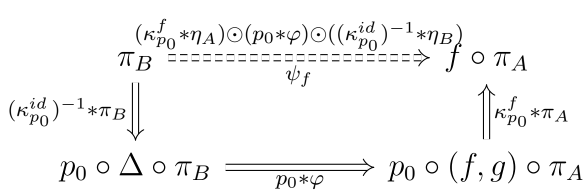
In order to obtain a pseudo-cone over the parallel pair $f, g\colon A \rightrightarrows B$, we must provide 2-cells ψ_f and ψ_g as in the diagram below.

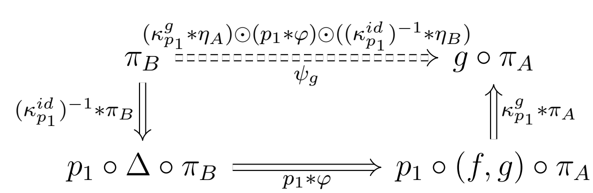


To do this, we will first augment the square above with the pseudo-cones corresponding to the maps Δ and (f, g).

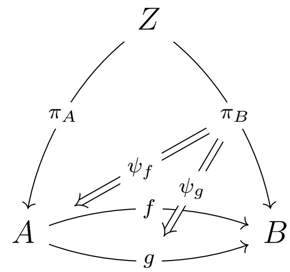


The 2-cells labeled κ are needed because the universal property of the product $B \times B$ does not guarantee that Δ and (f, g) actually map to their respective defining cones upon composing with the projections p_0 and p_1, but only that they map to *isomorphic* cones. Depending on the 2-category at hand, there may be a construction of $B \times B$ which renders the κ cells superfluous. In any case, we can now define ψ_f and ψ_g as the following composites.





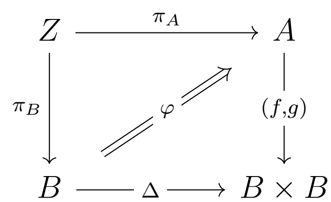
We have shown how to obtain a pseudo-cone over the diagram $f, g : A \rightrightarrows B$. We can carry out this process in reverse: given a pseudo-cone



we can define 2-cells φ_0 and φ_1 as the composites below.

$$\begin{array}{ccc} p_0 \circ \Delta \circ \pi_B & \overset{\varphi_0}{\dashrightarrow} & p_0 \circ (f, g) \circ \pi_A \\ {\scriptstyle \kappa^{id}_{p_0} * \pi_B} \Downarrow & & \Uparrow {\scriptstyle (\kappa^{id}_{p_0})^{-1} * \pi_A} \\ \pi_B & \underset{\psi_f}{\Longrightarrow} & f \circ \pi_A \end{array} \qquad \begin{array}{ccc} p_1 \circ \Delta \circ \pi_B & \overset{\varphi_1}{\dashrightarrow} & p_1 \circ (f, g) \circ \pi_A \\ {\scriptstyle \kappa^{id}_{p_1} * \pi_B} \Downarrow & & \Uparrow {\scriptstyle (\kappa^{id}_{p_1})^{-1} * \pi_A} \\ \pi_B & \underset{\psi_g}{\Longrightarrow} & g \circ \pi_A \end{array}$$

These 2-cells form a modification from the pseudo-cone induced by $\Delta \circ \pi_B$ to the pseudo-cone induced by $(f, g) \circ \pi_A$. By the universal property of the product, this modification induces a 2-cell $\varphi_0 \times \varphi_1 \colon \Delta \circ \pi_B \Rightarrow (f, g) \circ \pi_A$, yielding the pseudo-cone



as desired. We omit the verification that these maneuvers induce bijections on modifications of pseudo-cones, but the argument is entirely formal. $\square$

Fact A.5. *Let* $\mathbf{C}$ *be a 2-category, and let* $F \colon I \to \mathbf{C}$ *be a diagram. The category of pseudo-cones over* F *is equivalent to the category of pseudo-cones over the following diagram.*

$$\prod_{\mathrm{Ob}(I)} Fi \; \underset{\prod Ff \circ p_{s(f)}}{\overset{\prod p_{t(f)}}{\rightrightarrows}} \; \prod_{\mathrm{Mor}(I)} Ft(f)$$

Proof. The data of a pseudo-cone over the equalizer diagram with apex Z is equivalent to the data of a map $\pi \colon Z \to \prod_{\mathrm{Ob}(I)} Fi$ and a 2-cell

$$\varphi \colon \left(\prod_{f \in \mathrm{Mor}(I)} p_{t(f)} \right) \circ \pi \Rightarrow \left(\prod_{f \in \mathrm{Mor}(I)} Ff \circ p_{s(f)} \right) \circ \pi.$$

We can obtain a pseudo-cone η over F from this data by defining $\eta_i = p_i \circ \pi$ and setting η_f as the composite

$$\begin{array}{ccc} p_{t(f)} \circ \pi & \overset{\eta_f}{\dashrightarrow} & Ff \circ p_{s(f)} \circ \pi \\ \Downarrow & & \Uparrow \\ p_f \circ (\prod_{\mathrm{Mor}(I)} p_{t(f)}) \circ \pi & \underset{p_f * \varphi}{\Longrightarrow} & p_f \circ (\prod_{\mathrm{Mor}(I)} Ff \circ p_{s(f)}) \circ \pi \end{array}$$

Here we have left implicit the vertical 2-cells associated to the product; the situation is similar to that of Fact A.4.

Conversely, suppose we have a pseudo-cone $\eta : \Delta_Z \Rightarrow F$. Restricting η to the discrete category $\mathrm{Ob}(I) \subset I$ gives a collection of maps $\eta_i \colon Z \to Fi$, and taking their product yields a map

$$\pi := \left(\prod_{Ob(I)} \eta_i \right) \colon Z \to \prod_{\mathrm{Ob}(I)} Fi.$$

We define φ_f as the composite

$$\begin{array}{ccc}
p_f \circ (\prod_{\mathrm{Mor}(I)} p_{t(f)}) \circ \pi & \overset{\varphi_f}{\Longrightarrow} & p_f \circ (\prod_{\mathrm{Mor}(I)} Ff \circ p_{s(f)}) \circ \pi \\
\Downarrow & & \Uparrow \\
p_{t(f)} \circ \pi & & Ff \circ p_{s(f)} \circ \pi \\
\Downarrow & & \Uparrow \\
\eta_{t(f)} & \overset{\eta_f}{\Longrightarrow} & Ff \circ \eta_{s(f)}
\end{array}$$

again leaving the vertical 2-cells implicit. By the universal property of the product, the 2-cells φ_f induce a 2-cell

$$\varphi \colon \left(\prod_{f \in \mathrm{Mor}(I)} p_{t(f)} \right) \circ \pi \Rightarrow \left(\prod_{f \in \mathrm{Mor}(I)} Ff \circ p_{s(f)} \right) \circ \pi,$$

which together with π forms a pseudo-cone over the equalizer diagram. Again, we omit the proof that this process induces a bijection on modifications of pseudo-cones over the two diagrams. □

Corollary A.6. *The 2-limit of any finite diagram of stacks can be written as a 2-fiber product of products of stacks in the diagram.*

Proposition A.7. *Let* $(\mathsf{C}, \tau, \mathbb{P})$ *be a geometric context* [4, Section 1]*. Let* $F \colon I \to \mathbf{St}_{\mathsf{C}}$ *be a finite diagram of geometric stacks. The 2-limit of* F *is also a geometric stack.*

Proof. It is known that 2-fiber products and finite products of geometric stacks are geometric (see [13, Tag 04TD] for a proof in the algebraic case). Then use Corollary A.6. □

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References

[1] Dan Abramovich, Lucia Caporaso, and Sam Payne, *The tropicalization of the moduli space of curves* (February 15, 2014), available at `1212.0373[math]`.

[2] Dan Abramovich, Qile Chen, Steffen Marcus, Martin Ulirsch, and Jonathan Wise, *Skeletons and fans of logarithmic structures*, Nonarchimedean and tropical geometry, 2016, pp. 287–336. Series Title: Simons Symposia.

[3] Dan Abramovich, Qile Chen, Steffen Marcus, and Jonathan Wise, *Boundedness of the space of stable logarithmic maps*, Journal of the European Mathematical Society **19** (2017), no. 9, 2783–2809, available at `1408.0869[math]`.

[4] Renzo Cavalieri, Melody Chan, Martin Ulirsch, and Jonathan Wise, *A moduli stack of tropical curves*, Forum of Mathematics, Sigma **8** (2020), e23, available at `1704.03806[math]`.

[5] Ofer Gabber and Lorenzo Ramero, *Foundations for almost ring theory – Release 7.5* (September 2018), available at `0409584[math]`.

[6] Jack Hall and David Rydh, *Coherent tannaka duality and algebraicity of hom-stacks*, Algebra & Number Theory **13** (September 21, 2019), no. 7, 1633–1675, available at `1405.7680[math]`.

[7] Alana Huszar, Steffen Marcus, and Martin Ulirsch, *Clutching and gluing in tropical and logarithmic geometry* (August 17, 2018), available at `1706.07554[math]`.

[8] Kazuya Kato, *Logarithmic structures of fontaine-illusie*, Algebraic Analysis, Geometry, and Number Theory (1989), 191–224.

[9] Samouil Molcho and Jonathan Wise, *The logarithmic picard group and its tropicalization* (March 16, 2022), available at `1807.11364[math]`.

[10] Behrang Noohi, *Mapping stacks of topological stacks*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2010** (January 2010), no. 646.

[11] Dhruv Ranganathan, Keli Santos-Parker, and Jonathan Wise, *Moduli of stable maps in genus one and logarithmic geometry I*, Geometry & Topology **23** (December 30, 2019), no. 7, 3315–3366, available at `1708.02359[math]`.

[12] ———, *Moduli of stable maps in genus one and logarithmic geometry II*, Algebra & Number Theory **13** (October 9, 2019), no. 8, 1765–1805, available at `1709.00490[math]`.

[13] The Stacks project authors, *The stacks project*, 2022.

[14] Ross Street, *Limits indexed by category-valued 2-functors*, Journal of Pure and Applied Algebra **8** (June 1976), no. 2, 149–181 (en).

[15] Martin Ulirsch, *Non-archimedean geometry of artin fans* (December 22, 2018), available at `1603.07589[math]`.

[16] Jonathan Wise, *Moduli of morphisms of logarithmic schemes*, Algebra & Number Theory **10** (June 20, 2016), no. 4, 695–735, available at `1408.0037[math]`.