THE GEOMETRY AND LIMITS OF YOUNG PARTITION FLOW POLYTOPES

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Abstract. In 2017, Mészáros, Simpson, and Wellner demonstrated that certain flow polytopes resulting from Young tableaux are easily decomposed into simplices, and others have a natural relation to the well-known Tesler and CRY polytopes. Within a family of polytopes determined by a single tableaux shape, they introduced the limiting polytope. The limiting polytope is a useful notion since it is easy to decompose into a product of simplices. In this work, we use geometric decomposition to further examine the limiting process within each family of polytopes. Our main results analyze the family of hooks, and we demonstrate an algorithm to get geometric decompositions.

Keywords: flow polytopes, inequalities, Catalan numbers, young tableaux

Contents

1. Introduction 1
2. Some definitions and the connection between partitions and polytopes 2
3. Geometric decomposition from a polytope 6
3.1. An algebraic interpretation of the digraph 6
3.2. Turning the equations into solvable inequalities 6
3.3. An algorithm for the volume computation of any flow polytope 7
4. Results for specific Young shapes 8
5. Future directions 12
6. Acknowledgements 12
References 13

1. Introduction

Flow polytopes are extremely important for combinatorial optimization due to their relation to maximum matching and minimum cost problems (e.g. see Chapter 13 of [9]). By better understanding the characteristics of these structures, such as their geometry and volume, we gain more insight in solving these optimizations. Additionally, in recent works, flow polytopes have also been shown to have connections to representation theory [1], diagonal harmonics [7], geometric and algebraic combinatorics due to connections with Schubert polynomials [5], toric geometry [4], and more, making polytope theory an extremely rich area of study.

In 2008, Baldoni and Vergne [1] proved the following general formula for volumes of flow polytopes in terms of the Kostant Partition Function:

Theorem 1.1 ([1], Theorem 38)**.** *Let $a = (a_1, a_2, \ldots a_n, -\sum_{i=1}^n a_i)$, satisfying $a_i \geq 0$ for $i \in [n]$. Then, for a graph G on vertex set $[n+1]$ with N edges, we have the following:*

$$\operatorname{vol} \mathcal{F}_G(\boldsymbol{a}) = \sum_i (N-n)! \cdot \prod_{j=1}^{n} \frac{a_j^{i_j}}{(i_j)!} K_{G'}(i_1 - t_1^G, i_2 - t_2^G, \ldots, i_n - t_n^G)$$

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and

$$K_G(\boldsymbol{a}) = \sum_{i} \binom{a_1 + t_1^G}{i_1}\binom{a_2 + t_2^G}{i_2} \cdots \binom{a_n + t_n^G}{i_n} \cdot K_{G'}(i_1 - t_1^G, i_2 - t_2^G, \ldots, i_n - t_n^G)$$

where both sums are over all weak compositions $i = (i_1, i_2, \ldots, i_n)$ *of* $N - n$ *with* n *parts.* G' *is the restriction of* G *onto* $[n]$*, and* t_i^G *for* $i \in [n]$*, stands for the outdegree of vertex* i *in* G *minus* 1*.*

The factorials and binomials suggest a very combinatorial interpretation of the above formula — it seems as if the volume of a flow polytope counts some other object. However, no such interpretation has been found, which is why utilizing this formula is extremely cumbersome. Moreover, past methods for calculating volumes have included constant term identities and multivariate Laurent series; neither of them preserves the hypothesized combinatorial background of the formula. In addition, the Baldoni-Vergne formula does not reflect the actual structure of a flow polytope as the scaled sum of products of simplices.

Furthermore, in 2000, Chan, Robbins, and Yuen [2] discovered the polytope named for them (the CRY Polytope) and conjectured that the volume could be found by a product of consecutive Catalan numbers. Shortly after, Zeilberger [11] proved the CRY conjecture using the Morris constant term identity.

Theorem 1.2 ([11])**.** *The Morris Constant Term can be written as*

$$M_n(a, b, c) = \prod_{j=0}^{n-1} \frac{\Gamma(a - 1 + b + (n - 1 + j)\frac{c}{2})\Gamma(\frac{c}{2} + 1)}{\Gamma(a + j\frac{c}{2})\Gamma(b + j\frac{c}{2})\Gamma(\frac{c}{2}(j + 1) + 1)}.$$

Corollary 1.3. *By setting* $a = b = c = 1$*, the volume of* CRY_{n+1} *is given by* $M_n(1, 1, 1) = \prod_{i=1}^{n-1} C_i$ *where* C_i *is the* i*th Catalan number.*

However, with how prominent the Catalan numbers are in combinatorics with various counting identities (see [10, Ex. 6.19] for examples), finding a more combinatorial proof of the CRY volume has been an area of study since then. Polytope theory has only gotten richer and more studied in the years since with works such as from Mészáros and Morales [6], who made significant progress in examining these polytopes combinatorially.

In 2017, Mészáros, Simpson, and Wellner [8] focused on the polytopes related to partitions and introduced the notion of a limiting polytope. The families of polytopes resulting from partitions show an interesting connection between some of the more elusive polytopes, like the CRY polytope, and the easier-to-compute ones, like the limiting polytopes. In this paper, we expand on these ideas by examining volume formulas for various different partition shapes. In particular, we wish to outline a way to gain information as to how the polytopes that fall in between the limiting polytope and the harder-to-understand polytopes work and what the limiting process means combinatorially.

By studying the inequalities representing a flow polytope, we develop an algorithm (see Section 3.3) to compute a volume formula for any flow polytope in a way that preserves the simplex structure of the shapes [3]. Given any polytope, we represent the volume as a sum of products of simplices, which yields information about the geometric structure of the polytope and points to potential improvements in our combinatorial understanding as well. We then apply this algorithm to specific classes of polytopes, like those defined by a hook-shaped partition, to develop explicit volume formulas and study how the simplices making up the polytope change as it gets closer to the limiting polytope.

2. Some definitions and the connection between partitions and polytopes

Let G be a loopless directed acyclic connected graph on the vertex set $[n + 1]$ with m edges. To each edge (i, j) with $i < j$, of G, associate the positive type A_n root $\alpha(i, j) = \mathbf{e}_i - \mathbf{e}_j$ where $\mathbf{e}_i$ and $\mathbf{e}_j$ are members of the standard basis of $\mathbb{R}^n$. Let $S_G := \{\{\alpha(e)\}\}_{e \in E(G)}$ be the multiset of roots corresponding to the multiset of edges of G. Define M_G as the $(n + 1) \times m$ matrix whose columns are the vectors in S_G.

Definition 2.1 (Netflow vector)**.** The **netflow** is an integer vector $\mathbf{a} = (a_1, a_2, \ldots, a_n, -\sum_{i=1}^{n} a_i) \in \mathbb{Z}^{n+1}$ such that each $a_i \geq 0$. An **a-flow** $\mathbf{f}_G$ on G is a vector $\mathbf{f}_G = (b_k)_{k\in[m]}, b_k \in \mathbb{R}_{\geq 0}$ such that $M_G\mathbf{f}_G = a$. In other words, for all $1 \leq i \leq n$, we have the following equation:

$$\sum_{e=(g,i)\in E(G)} b(e) + a_i = \sum_{e=(i,j)\in E(G)} b(e).$$

Definition 2.2 (Flow polytope)**.** The **flow polytope** $\mathcal{F}_G(\mathbf{a})$ associated to a graph G on the vertex set $[n+1]$ and the netflow $\mathbf{a} = (a_1, a_2, \ldots a_n, -\sum_{i=1}^{n} a_i)$ is the set of all **a**-flows $\mathbf{f}_G$ on G. Thus, $\mathcal{F}_G(\mathbf{a}) = \{\mathbf{f}_G \in \mathbb{R}^m_{\geq 0} \mid M_G\mathbf{f}_G = \mathbf{a}\}$. So, the flow polytope of G exists in $\mathbb{R}^m$ space.

Definition 2.3 (Kostant partition fucntion)**.** The **Kostant Partition Function** K_G evaluated at the vector $\mathbf{b} \in \mathbb{Z}^{n+1}$ gives the number of representations of a vector into nonnegative integer linear combinations of positive roots. In context to the flow polytope $\mathcal{F}_G(\mathbf{b})$, it provides the number of lattice points and is defined as follows:

$$K_G(\mathbf{b}) = \#\left\{(f(e))_{e\in E(G)} | \sum_{e\in E(G)} f(e)\alpha(e) = \mathbf{a} \text{ and } f(e) \in \mathbb{Z}_{\geq 0}\right\}.$$

It also has generating function

$$\sum_{b\in\mathbb{Z}^{n+1}} K_G(\mathbf{b}) x_1^{a_1} x_2^{a_2} \ldots x_{n+1}^{-\sum_i a_i} = \prod_{(i,j)\in E(G)} \left(1 - x_i x_j^{-1}\right)^{-1}.$$

Definition 2.4 (Partition)**.** A **partition** λ of an integer $n > 0$ is a monotonic increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers $\lambda_i > 0$ satisfying $\sum_{1\leq i\leq n} \lambda_i = n$. Each λ_i is called a **part** of λ and its **length** (denoted $\ell(\lambda)$) is equal to the number of parts.

Definition 2.5 (Family of polytopes)**.** For each λ, we define the family of flow polytopes $\mathcal{F}_{(\lambda,\mathbf{a})}$. Given a partition λ, let Y be its corresponding left-justified Young Diagram. Pick an integer n satisfying $n - i \geq \lambda_i$ for all $i \in [\ell(\lambda)]$. Then, we can place Y in the upper triangle of an $n \times n$ matrix M, with the top edges of Y flush with the top edges of M and likewise for the right edges. Let Y' be the set of entries (i, j) of M that lie inside of Y, and define $G(\lambda, n)$ as the directed graph:

$$G(\lambda, n) := \big([n+1], \{(i, n+1) : i \in [n]\} \cup Y'\big).$$

Definition 2.6 (Limiting polytope)**.** For a partition λ and netflow $\mathbf{a} \in \mathbb{Z}^n_{>0}$, the **limiting polytope of the family** $\mathcal{F}_{(\lambda,\mathbf{a})}$, denoted as $\mathcal{F}^{\lim}_{(\lambda,\mathbf{a})}$, is the polytope $\mathcal{F}_{G(\lambda,\ell(\lambda)+\lambda_1)}$.

We highlight the geometric difference between limiting and non-limiting polytopes in Figure 1 by focusing on the family $\mathcal{F}_{((3,2,1),\mathbf{a}}$. Note that the family $\mathcal{F}_{[n,n-1,\ldots,1],(1,\ldots,1,-n)}$ contains the Tesler polytope and the family $\mathcal{F}_{[n,n-1,\ldots,1],(1,0,0,\ldots,-1)}$ contains the CRY polytope. Thus these families generalize the well-known polytopes that are a huge focus of study.

From the 2017 Mészáros, Simpson and Wellner [8], we get a volume formula of the following.

Theorem 2.7. *The normalized volume of a limiting polytope is*

$$\operatorname{vol} \mathcal{F}^{\lim}_{(\lambda,\boldsymbol{a})} = \left(\sum_{i\in[\ell(\lambda)]} \lambda_i\right)! \prod_{i\in[\ell(\lambda)]} \frac{a_i^{\lambda_i}}{\lambda_i!}.$$

In more recent work, Mészáros and Morales show in [6] that with compounded reduction trees (CRTs), a general combinatorial breakdown for all flow polytopes exists.

Theorem 2.8 ([6], Lemma 3.4)**.** *Let G be a graph on the vertex set $[n+1]$ with positive integer netflow $\boldsymbol{a} = (a_1, \ldots, a_n, -\sum_{i=1}^{n} a_i)$ and an integer $i \in [n+1]$ such that vertex i has both incoming and outgoing edges in G. Then,*

$$\mathcal{F}_G(\boldsymbol{a}) = \bigcup_{\mathrm{T}\in\mathcal{T}_{\mathcal{I}_i\cup\{i\},\mathcal{O}_i}} \mathcal{F}_{G_T^{(i)}}(\boldsymbol{a}).$$

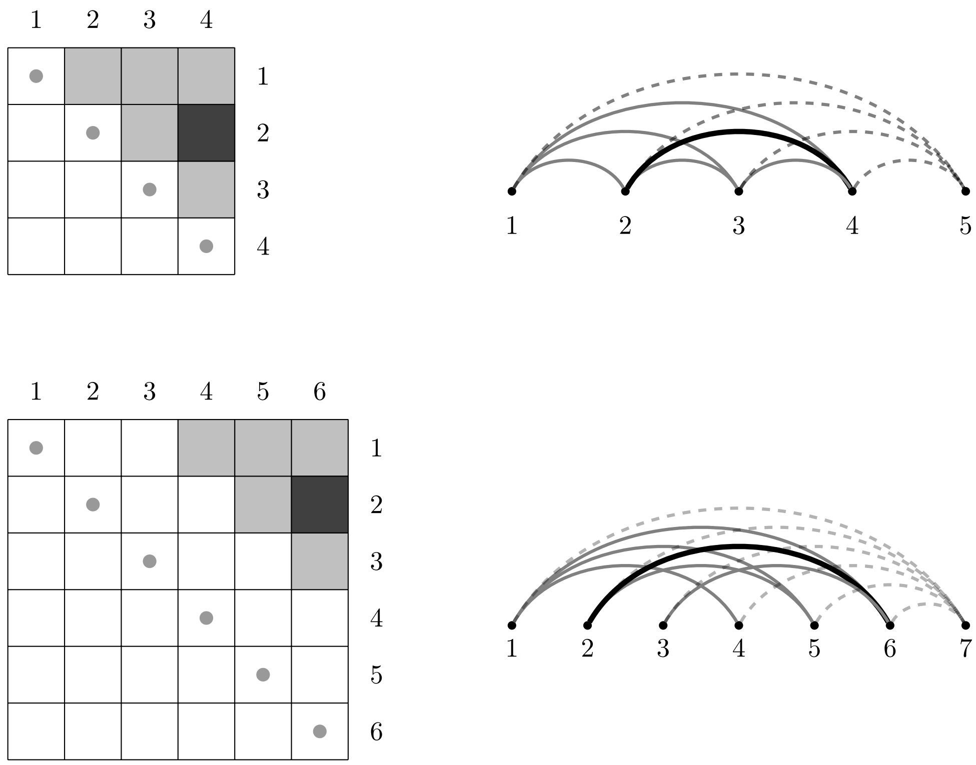

FIGURE 1. The diagram provides two examples of how to convert a Young diagram into a digraph. Each filled-in block in the Young diagram corresponds with the edge on the digraph sharing the same color. The top row features the left-justified Young diagram of $\lambda = (3, 2, 1)$, the diagram in a 4×4 matrix and the corresponding graph on five vertices. We call vertex 5 the **sink** because every vertex connects to it. The bottom row has the left-justified Young diagram of $\lambda = (3, 2, 1)$ in a 6×6 matrix and its corresponding graph on seven vertices. This is the limiting polytope of the family $\mathcal{F}_{((3,2,1),\mathbf{a})}$. Geometrically, the Young diagram fits entirely in the top-right quadrant. In the digraph, this means no edge has both an incoming and outgoing edge (excluding edges going to the sink, which are the dashed edges).

Theorem 2.8 implies that a flow polytope can be expressed as the union of the leaves of its CRT. To build upon this idea further, we characterize the graphs that appear as leaves.

Definition 2.9. Let $\mathbf{m} = (m_1, \dots m_n)$ be a tuple of positive integers. Then, $G(\mathbf{m})$ is the graph on $[n+1]$ with m_i edges $(i, n+1)$.

Theorem 2.10 (Mészáros-Morales [6])**.** *Given $G(\boldsymbol{m})$ on the vertex set $[n+1]$ with $\boldsymbol{m} = (m_1, \dots m_n)$ a tuple of positive integers, $\boldsymbol{a} = (a_1, \dots a_n) \in \mathbb{Z}^n_{\geq 0}$, the normalized volume of $\mathcal{F}_{G(\boldsymbol{m})}(\boldsymbol{a})$ is*

$$\mathrm{vol}(\mathcal{F}_{G(\boldsymbol{m})}(\boldsymbol{a})) = (\#E(G(\boldsymbol{m})) - n)! \cdot \prod_{i=1}^{n} \frac{a_i^{m_i - 1}}{(m_i - 1)!}.$$

We can see that the volume formulas found in these two papers agree.

Theorem 2.11. *The process of compounded subdivision above is consistent with the partition-based volume formula when applied to limiting polytopes.*

The proof of Theorem 2.11 will appear after the following related lemmas.

Theorem 2.12 (Mészáros-Morales [6])**.** *Given the flow polytope $\mathcal{F}_G(\boldsymbol{a})$ with G a graph on the vertex set $[n+1]$ and $a_i \geq 0$ for $i \in [n]$, the leaves of any compounded reduction tree R_G rooted at G are of the form $G(\boldsymbol{m})$ with $m_i = 1$ if and only if $a_i = 0$ and $\sum_{i=1}^{n} m_i = \#\mathrm{E(G)}$.*

Lemma 2.13. *After completing compounded subdivision on the digraph, G, of any limiting digraph, there is just 1 leaf.*

Proof. By definition of the limiting polytope, if we consider G' to be the restriction of G onto $[n]$, then no vertex in G' has both an incoming and outgoing edge.

In G, since each vertex has an edge connecting it to the sink, a vertex with at least one incoming edge has exactly 1 outgoing edge. So, each vertex in G can be classified as having either only outgoing edges (in which case we cannot do compounded subdivision on it) or some number of incoming edges and exactly 1 outgoing edge.

Let v be a vertex in G with $n > 0$ incoming edges and 1 outgoing edge. When doing compounded subdivision on it, the number of branches in the CRT is given by $\binom{\ell + r - 2}{\ell - 1}$ where there are ℓ elements in the left ordered set and r elements in the right ordered set because that is how many bipartite non-crossing trees can be formed. By definition of the compounded subdivision process, r is simply the set of outgoing edges, which we know to be 1. Thus, v has exactly 1 branch. Since, at each step, there is exactly 1 branch, the entire CRT has only 1 leaf. □

Lemma 2.14. *For $1 \leq i \leq n$, whenever $a_i > 0$,*

$$\lambda'_i = m_i - 1$$

where $G(\boldsymbol{m})$ is the singular leaf of G's CRT. Moreover, in general, $\sum m_i = \sum \lambda'_i + n$.

Proof. The second part of the claim is immediate. If compounded subdivision is done on G, then by Theorem 2.12, $\sum m_i = \#E(G)$. However, the number of edges in G also equals $\sum_{i=1}^{n} \lambda'_i + n$. The sum of the parts of the tuple corresponds to edges not involving the sink, and there is 1 edge going to the sink for each $i \in [n]$.

For the first part of the claim, note that in G, the outdegree of vertex i is simply $\lambda'_i + 1$ as we have λ'_i edges that do not go to the sink and 1 edge that goes to the sink. We will show that the outdegree remains constant throughout the CRT process for $a_i > 0$. Suppose there is edge $e = (i, j)$.

- If subdivision happens on i, then $j = n + 1$. If $a_i > 0$, then e is preserved, so the outdegree of i does not change.
- If subdivision happens on j, then there exists edge $e' = (j, n + 1)$. By subdivision process, e and e' will be replaced by $(i, n + 1)$. Even then, the outdegree of i stays the same.

So, whenever $a_i > 0$, the outdegree of the ith vertex, $\lambda_i + 1$, remains constant. Moreover, we showed that any edge e will end up taking the form $(i, n + 1)$. However, m_i is defined by the number of edges of the form $(i, n + 1)$. Since all outgoing edges of i take this form, $m_i = \text{outdeg}(i) = \lambda'_i + 1$.

□

We now prove Theorem 2.11.

Proof. Let G be the digraph associated to the limiting polytope of a partition λ, and let $n = \ell(\lambda) + \lambda_1$. Make λ' the tuple formed by adding $0's$ to λ until the length of λ' is n. By Lemma 2.13, since G has only 1 leaf, Theorem 2.8 implies that $\mathcal{F}_G(\mathbf{a}) = \mathcal{F}_{G_T}(\mathbf{a})$ where G_T is the leaf of the CRT.
By Theorem 2.10,

$$\operatorname{vol} \mathcal{F}_{G_T}(\mathbf{a}) = (\#E(G_T) - n) \cdot \prod_{i=1}^{n} \frac{a_i^{m_i - 1}}{(m_i - 1)!}.$$

From Lemma 2.14, $\#E(G_T) - n = \sum_{i=1}^{n} m_i - n = \sum_{i=1}^{n} \lambda'_i$, and for all i with $a_i > 0$, we have $m_i - 1 = \lambda'_i - 1$. Thus, we obtain the following:

$$\operatorname{vol} \mathcal{F}_G(\mathbf{a}) = \operatorname{vol} \mathcal{F}_{G_T}(\mathbf{a}) = (\#E(G_T) - n) \cdot \prod_{i=1}^{n} \frac{a_i^{m_i - 1}}{(m_i - 1)!} = \left(\sum_{i \in \ell(\lambda)} \lambda_i \right)! \cdot \prod_{i \in \ell(\lambda)} \frac{a_i^{\lambda_i}}{(\lambda_i)!}.$$

□

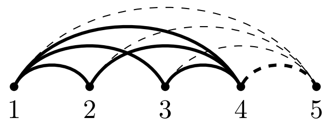

FIGURE 2. The Digraph $G((3, 1, 1), 4)$ with solid edges from the Young shape and dashed edges for the sink edges.

Note that $a_i \geq 0$ by definition, and when $a_i = 0$, the entire volume becomes 0 (according to both formulas), so they match there as well. We can switch from λ' to λ because for each additional 0 we added to λ to form λ', m_i must equal 1 at which point $\frac{a_i^{m_i-1}}{(m_i-1)!} = 1$, so it becomes irrelevant to the volume.

3. Geometric decomposition from a polytope

In this section, we explain how to convert any flow polytope into a set of inequalities. Then, we break down each volume computation into a generalized algorithm dependent on the entries of the netflow vector.

3.1. An algebraic interpretation of the digraph.

Consider the digraph of a flow polytope as first defined in Definition 2.5. As an example, we will work with $G((3, 1, 1), 4)$, which is shown in Figure 2. For notation purposes, we will call the edge going from vertex i to vertex j, x_{ij}. By definition of the flow polytope, the total flow coming into a vertex must equal the flow leaving. For example, the only flow coming into vertex 1 comes from the netflow vector itself, which has value a_1. The edges $x_{12}, x_{13}, x_{14}, x_{15}$ carry flow out of the vertex, meaning we must have

$$x_{12} + x_{13} + x_{14} + x_{15} = a_1.$$

For vertex 2, there is incoming flow from both the netflow, a_2, as well as the edge x_{12}. Flow leaves the vertex through the edges x_{24} and x_{25}, meaning

$$x_{24} + x_{25} = a_2 + x_{12}.$$

Similarly, we get the following equations for vertex 3 and vertex 4:

$$x_{34} + x_{35} = a_3 + x_{13}$$

$$x_{45} = a_4 + x_{14}.$$

In general, we form these equations in a similar manner: for each vertex, the total flow leaving must equal the total flow coming in.

3.2. Turning the equations into solvable inequalities.

These equations fully define the flow polytope, so by finding all possible solutions to the system, we get the flow polytope. To solve these equations, first notice that we can turn each equation into an inequality, using the fact that every edge must have nonnegative flow. For example, in

$$x_{12} + x_{13} + x_{14} + x_{15} = a_1,$$

as long as $x_{12} + x_{13} + x_{14} \leq a_1$, we get a working solution as x_{15} is automatically defined as $a_1 - (x_{12} + x_{13} + x_{14})$. Using similar logic, we get the following system of inequalities to solve:

$$x_{12} + x_{13} + x_{14} \leq a_1$$

$$x_{24} \leq a_2 + x_{12}$$

$$x_{34} \leq a_3 + x_{13}.$$

Note that we do not have to worry about $x_{45} = a_4 + x_{14}$ as the equation implies that x_{45} is fixed by a_4 and x_{14}, and since a_4 is given and we already solve for x_{14} in the first inequality, that solution determines x_{45}.

Remark 3.1. We refer to inequalities of the form $x_{12} + x_{13} + x_{14} \leq a_1$, where there is just one term on the right side of the inequality as **type A.** We refer to inequalities of the form $x_{24} \leq a_2 + x_{12}$, where there are 2 or more terms on the right side of the inequality as **type B.**

In the past two sections, we have worked through some illuminating examples of how this process works in some specific cases. In the next section, we discuss how the process works in general and present the method to get the volume formula.

3.3. **An algorithm for the volume computation of any flow polytope.** Here, we provide pseudocode for a general algorithm that can be used to solve for the volume of a flow polytope. Then, we go into the details of how the algorithm works.

Algorithm 3.1. Input: Flow Polytope $\mathcal{F}$.

(1) Create the set of n inequalities as shown in Section 3.2.
(2) Take inequality n and create a list of all m cases that it has.
(3) For i going from 1 to m, consider case j of inequality n.
    (a) If case j is a Type A inequality: add the simplex it represents to the value.
    (b) Else:
        (i) If case j is a Type B base case: update the volume accordingly.
        (ii) See what bijections case j creates and how it affects the other inequalities.
    (c) Create a new flow polytope f' that is f with inequality n removed and other inequalities changed based on the bijections from case j.
    (d) Print output.

Output: Current volume and function with input f'.

Theorem 3.2. *The algorithm 3.1 will properly decompose the inequalities to output the correct geometric volume of the relevant flow polytope.*

Proof. We start with the inequality:

$$\sum_{j=k+1}^{k+\ell} x_{ij} \leq a_i + \sum_{m=p+1}^{p+n} x_{mi}.$$

It can be split into $n+1$ cases, where case 1 is given by

$$\sum_{j=k+1}^{k+\ell} x_{ij} \leq a_i$$

and case $\alpha > 1$ is

$$a_i + \sum_{m=p+1}^{p+\alpha-2} x_{mi} \leq \sum_{j=k+1}^{k+\ell} x_{ij} \leq a_i + \sum_{m=p+1}^{p+\alpha-1} x_{mi}.$$

Case 1 is a Type A inequality and requires no further breakdown. Assuming no cases from other Type B inequalities in the polytope affect $\sum_{j=k+1}^{k+\ell} x_{ij}$, the solution set is simply an ℓ-simplex scaled by a_i: it has volume $\frac{1}{\ell!} a_i^{\ell}$.

The other cases require further breakdown through recursion. In general, assume we have the inequality

$$a_i + \sum_{m=q+1}^{q+\beta} x_{mi} \leq \sum_{j=k+1}^{k+\ell} x_{ij} \leq a_i + \sum_{m=q+1}^{q+\beta+1} x_{mi}.$$

It is solved using the following method:

(1) Break the inequality into ℓ more cases. For case $\gamma \in [\ell]$, consider

$$\sum_{j=k+1}^{k+\ell-\gamma} x_{ij} \leq a_i + \sum_{m=q+1}^{p+\beta} x_{mi} \leq \sum_{j=k+1}^{k+\ell-\gamma+1} x_{ij} \leq \sum_{j=k+1}^{k+\ell} x_{ij} \leq a_i + \sum_{m=q+1}^{q+\beta+1} x_{mi}.$$

By using these cases, we consider all possibilities for what range $a_i + \sum_{m=q+1}^{p+\beta} x_{mi}$ lies in. This method makes every case disjoint, ensuring no possibilities are missed nor double counted.

(2) Split each inequality

$$\sum_{j=k+1}^{k+\ell-\gamma} x_{ij} \leq a_i + \sum_{m=q+1}^{p+\beta} x_{mi}$$

into further cases. It is the same form of inequality that we originally started with. So, we must again break this inequality down into its several cases using the process outlined above, creating a recursive cycle.

Eventually, the recursion reaches a base case, which is when $\ell = 1$ (or when $\beta = 0$). Once this happens, there is no more recursion left to do since the LHS is 0, implying the following term vanishes:

$$\sum_{j=k+1}^{k+\ell-\gamma} x_{ij} \leq a_i + \sum_{m=q+1}^{p+\beta} x_{mi}.$$

This is the Type B base case. □

This process can be formally written in code, and we have a proof of concept written in Java, although ideally, this would be formatted to be compatible with other volume computation software. We have included the information on where to find this code [3].

4. Results for specific Young shapes

To begin our specific detailed looks at geometric breakdowns, we start with rectangles, polytopes whose partition has a Young Tableaux in the shape of a rectangle. In other words, we have $\lambda = (a, a, a, \ldots, a)$ for some a. For example, Figure 3 displays the Young Tableaux and digraph for the rectangle $\lambda = (3, 3)$ with $n = 5$.

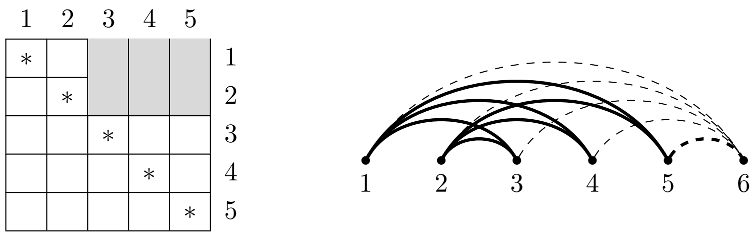

Figure 3. From left to right: the left-justified Young diagram of $\lambda = (3, 3)$, the diagram in a 5×5 matrix, and the corresponding graph on six vertices.

The case of rectangles, though, is not as telling about the limiting process due to the following lemma.

Lemma 4.1. *A flow polytope with a corresponding rectangle partition is always limiting.*

Proof. Let $\lambda = (a, a, \cdots, a)$ where there are b a's. From [8], a flow polytope $\mathcal{F}_{G(\lambda,n)}(\mathbf{a})$ is limiting for all $n \geq \lambda_1 + \ell(\lambda)$. However, for $\mathcal{F}_{G(\lambda,n)}(\mathbf{a})$ to be a valid flow polytope, we need n to be large enough such that the Young Tableaux of λ fits in the top right half of the $n \times n$ square. Since $\lambda = (a, a, \ldots, a)$, this implies that the bottom left corner of λ's Young Tableaux, which is an $a \times b$ rectangle, is in the top right half of the $n \times n$ square. In order for this to happen, we must have $n \geq a + b$.
However, $\mathcal{F}_{G(\lambda,n)}(\mathbf{a})$ is limiting when $n \geq \lambda_1 + \ell(\lambda)$. In the case of a rectangle, $\lambda_1 = a$ and $\ell(\lambda) = b$, which means the flow polytope is limiting for all $n \geq a + b$. □

At that point, since rectangles are always limiting, their volumes are given by Theorem 2.7:

$$\operatorname{vol}\mathcal{F}^{\lim}_{(\lambda,\mathbf{a})} = \left(\sum_{i\in[\ell(\lambda)]}\lambda_i\right)!\prod_{i\in[\ell(\lambda)]}\frac{a_i^{\lambda_i}}{\lambda_i!} = (a\cdot b)!\prod_{i\in[\ell(\lambda)]}\frac{a_i^a}{a!}.$$

However, we can also derive this formula using geometric decomposition, as shown below.

Theorem 4.2. *Let $\lambda = (a, a, \cdots, a)$ where $\ell(\lambda) = b$. Then,*

$$\operatorname{vol}\mathcal{F}_{G(\lambda,n)}(\boldsymbol{a}) = (a\cdot b)!\prod_{i\in[\ell(\lambda)]}\frac{a_i^a}{a!}$$

for all $n \geq a + b$.

Proof. From Lemma 3.1 of [8], $\operatorname{vol}\mathcal{F}_{G(\lambda,n)}(\mathbf{a})$ is constant for $n \geq a+b$, since $\mathcal{F}$ is limiting at $n = a+b$. So, showing the equation holds for $n = a + b$ suffices.

We proceed with geometric decomposition. Vertex 1 has incoming flow a_1 and outgoing flow through its a edges (excluding the one going to the sink), $x_{1(b+i)}$ for $i \in [1, a]$. So, we have:

$$\sum_{i=1}^{a} x_{1(b+i)} \leq a_1.$$

Similarly, for $j = 2$ to $j = b$, vertex j only has incoming flow from the netflow but has a outgoing edges, $x_{j(b+i)}$ for $i \in [1, a]$. So, we have

$$\sum_{i=1}^{a} x_{j(b+i)} \leq a_j$$

for j from 1 to b. For vertices x_k for $k = b+1$ to $a+b$, note that x_k only has one outgoing edge: to the sink. So, all incoming flow simply leaves through the sink-bound edge, meaning there is no inequality to solve. So, by solving the inequalities of the form

$$\sum_{i=1}^{a} x_{j(b+i)} \leq a_j$$

for j from 1 to b, we solve the decomposition. Note that all b inequalities are independent of each other as there are no common variables. Moreover, the solution set for each inequality is simply an a-dimensional simplex with side length a_j. The volume of such simplices are $\frac{a_j^a}{a!}$ for $j \in [1, b]$. Thus, since the overall volume is given by the product of all these simplices, we have:

$$\prod_{j=1}^{b}\frac{a_j^a}{a!}.$$

The final step is normalizing the volume. Since $\lambda_1 = a$ and $\ell(\lambda) = b$, $\dim\mathcal{F}_{(G,n)}(\mathbf{a}) = a + b$. Thus,

$$\operatorname{vol}\mathcal{F}_{(G,n)}(\mathbf{a}) = (a+b)!\prod_{j=1}^{b}\frac{a_j^a}{a!}.$$

□

Remark 4.3. The above argument can also be generalized to prove Theorem 2.7 as a whole.

In the case of hook Young shapes, we see the following:

Definition 4.4 (Hook)**.** A flow polytope $\mathcal{F}_{G(\lambda,n)}(\mathbf{a})$ is a hook if its partition λ is of the form $(n, 1, 1, \cdots 1)$.

We can use the inequalities to solve for the volume of all hooks. Before we do, though, we must define the elementary symmetric sums.

Definition 4.5 (bth elementary symmetric sum)**.** Given variables $x_1, x_2, \cdots x_n$, the bth elementary symmetric sum $e_b(x_1, x_2, \cdots, x_n)$ $(0 \le b \le n)$ is given by $\sum_{x_1, x_2, \cdots, x_n} \prod_{i=1}^{n} x_{a_i}$, or in other words, the sum of the products of all sets of b elements in $\{x_1, x_2, \cdots, x_n\}$. We define $e_0(x_1, x_2, \cdots, x_n) = 1$.

Theorem 4.6. *Let $\lambda = (a, 1, 1, \cdots 1)$ where there are b 1's and $a > b$. Then,*

$$\operatorname{vol} \mathcal{F}_{(G,a+1)}(\boldsymbol{a}) = \sum_{j=0}^{b} \frac{1}{(a+j)!} a_1^{a+j} e_{b-j}(a_2, a_3, \ldots, a_{b+1})$$

and

$$\operatorname{vol} \mathcal{F}_{(G,a+1+x)}(\boldsymbol{a}) = \sum_{j=0}^{b-x} \frac{1}{(a+j)!} a_1^{a+j} \prod_{i=2}^{x+1} a_i e_{b-x-j}(a_{x+2}, a_{x+3}, \ldots, a_{b+1})$$

for $x > 0$.

Proof. First, we will consider the case where $n = a + 1$ and then show how the volume changes as n grows.

When $n = a + 1$, the corresponding digraph has $n + 1 = a + 2$ vertices. Vertex 1 is connected by an edge to every other vertex, vertices 2 through $b + 1$ each have one edge going to vertex n and another going to the sink, and vertices $b + 2$ through n only have an edge going to the sink. Thus, for the inequalities, we must only consider vertices 1 through $b + 1$. Vertex 1 has incoming flow from a_1 while vertices 2 through $b + 1$ have incoming flow from the netflow as well as the edge x_{1i} where $i \in [2, b + 1]$. This leads to the inequalities:

$$\sum_{j=2}^{n} x_{1j} \le a_1$$

$$x_{in} \le a_i + x_{1i}$$

where $i \in [2, b + 1]$.

Notice that each inequality of the form $x_{in} \le a_i + x_{1i}$ is type B. We can consider them in two cases:

(1) $x_{in} \le a_i$
(2) $a_i \le x_{in} \le a_i + x_{1i}$

In case 1, the solution set is simply a scaled 1-dimensional simplex with volume a_i. Moreover, the constraint $\sum_{j=2}^{n} x_{1j} \le a_1$ is left untouched.

For case 2, let $x'_{in} = x_{in} - a_i > 0$. Then, we simply need $x'_{in} < x_{1i}$. For that, consider the following mapping:

- $x'_{in} \to y_0$
- $x_{1i} \to y_0 + y_1$

where $y_0, y_1 > 0$. Note that this mapping satisfies $a_i \le x_{in} \le a_i + x_{1i}$, but it also changes the constraint $\sum_{j=2}^{n} x_{1j} \le a_1$. We now have

$$x_{12} + x_{13} + \cdots + x_{1(i-1)} + y_0 + y_1 + x_{1(i+1)} + \ldots + x_{1n} \le a_1.$$

Essentially, we add one more variable to the inequality, making it represent a scaled $(n+1)$-dimensional simplex instead of n-dimensional.

There are a total of b inequalities, and each of them can either follow case 1 or 2. If they all follow case 1, then the volume is simply $\frac{1}{(n-1)!} a_1^{n-1} a_2 a_3 \cdots a_{b+1}$. This can be written in terms of elementary symmetric sums as

$$\frac{1}{(n-1)!} a_1^{n-1} e_b(a_2, a_3, \ldots, a_{b+1}).$$

If one of them follows case 2, then we need to sum over all possibilities:

$$\sum_{i=1}^{b} \frac{1}{n!} a_1^n \prod_{j=2}^{i-1} a_j \cdot \prod_{j=i+1}^{b+1} a_j.$$

This sum iterates through all $n-1$ possibilities for which inequality follows case 2 and then that corresponding a_i value is dropped from the volume. So, we have

$$\frac{1}{n!} a_1^n (a_1 a_2 \cdots a_b + a_1 a_2 \ldots a_{b-1} a_{b+1} + \cdots + a_2 a_3 \cdots a_{b+1}).$$

However, that sum is simply the elementary symmetric sum $e_{b-1}(a_2, a_2 \cdots a_{b+1})$. So, the volume for this case is

$$\frac{1}{n!} a_1^n e_{b-1}(a_2, a_3 \cdots a_{b+1}).$$

Similarly, if we have j inequalities satisfying case 2, then the volume changes in a similar manner:

$$\frac{1}{(n-1+j)!} a_1^{n-1+j} e_{b-j}(a_2, a_3, \ldots a_{b+1}).$$

To get the total volume, we must sum through all possible values of j, ranging from 0 to b, yielding

$$\operatorname{vol} \mathcal{F}_{(G,a+1)}(\mathbf{a}) = \sum_{j=0}^{b} \frac{1}{(n-1+j)!} a_1^{n-1+j} e_{b-j}(a_2, a_3, \cdots a_{b+1}).$$

Substituting in $n = a+1$, we get the desired claim.

Now, we examine what happens when n increases by 1. We start off with investigating vertex 1 and its a edges. Assume that originally, vertex 1 connects with vertices $(n-a+1), (n-a+2), \ldots n$. Then, when we increase n by 1, all of the vertices get shifted, meaning vertex 1 now connects with vertices $(n-a+2), (n-a+3), \ldots, (n+1)$. The key to notice is that there is no longer an edge connecting vertex 1 with vertex $(n-a+1)$.

Also recall that originally, vertex $(n-a+i)$ had two cases:

(1) $x_{(n-a+1)n} \leq a_{(n-a+1)}$
(2) $a_{(n-a+1)} \leq x_{(n-a+1)n} \leq a_{(n-a+1)} + x_{1(n-a+1)}$

Now, because edge $x_{1(n-a+1)}$ no longer exists, the second case is no longer valid. Thus, only 1 case exists: $x_{in} \leq a_i$. So, in order to go from $\operatorname{vol} \mathcal{F}_{(G,n)}$ to $\operatorname{vol} \mathcal{F}_{(G,n+1)}$, we must remove all instances where vertex $n-a+1$ follows case 2. Hence, we must completely remove the term representing the scenario when each vertex $n-a+1$ through n satisfies case 2. In all other cases, since vertex i automatically assumes case 1, the degree of the symmetric polynomial sum essentially decreases by 1, as there is one less Case 2 inequality, and we multiply the term by a_{n-a+1} as that corresponds to the solution set of $x_{in} \leq a_i$.

When this process of increasing n by 1 continues x times, we can only have a maximum of $n-x$ Case 2 type B inequalities, as x of the type B inequalities are now forced to be Case 1. Thus, we get the formula:

$$\operatorname{vol} \mathcal{F}_{(G,a+1+x)}(\mathbf{a}) = \sum_{j=0}^{b-x} \frac{1}{(a+j)!} a_1^{a+j} \prod_{i=2}^{x+1} a_i e_{b-x-j}(a_{x+2}, a_{x+3}, \ldots, a_{b+1})$$

as claimed. □

Theorem 4.7. *Let $\lambda = (a, 1, \ldots, 1)$ where there are b 1's and $b = a+c-1$, $c > 0$. Then,*

$$\operatorname{vol} \mathcal{F}_{(G,a+1)}(\boldsymbol{a}) = \sum_{j=0}^{b} \frac{1}{(a+j)!} a_1^{a+j} \prod_{i=2}^{c+1} a_i e_{b-j}(a_{c+2}, a_{c+3}, \ldots, a_{c+b+1})$$

and

$$\operatorname{vol} \mathcal{F}_{(G,a+1+x)}(\boldsymbol{a}) = \sum_{j=0}^{b-x} \frac{1}{(a+j)!} a_1^{a+j} \prod_{i=2}^{c+x+1} a_i e_{b-x-j}(a_{x+c+2}, a_{x+c+3}, \ldots, a_{c+b+1})$$

for $x > 0$.

Proof. The formed polytope is essentially the same as in Theorem 4.6. Vertex 1 is connected to a vertices and b vertices have 2 edges: one to vertex n and the other to the $n+1$. However, when $b = a+c-1 > a$, there are c vertices with 2 edges that do not also have an incoming edge from vertex 1. Thus, their only incoming flow is from the netflow, meaning they must satisfy $x_{in} \leq a_i$ where i is one of the c vertices. Thus, we must include the $\prod_{i=2}^{c=1} a_i$ in the volume, as that represents these c vertices. Otherwise, the setup is the exact same.

The only exception is that the potential vertices, which are Case 2 type B must be shifted c indices, as the first c vertices must be type A. Thus, the volume formula has $e_{b-j}(a_{c+2}, a_{c+3}, \dots a_{c+b+1})$ instead of $e_{b-j}(a_2, a_3, \dots, a_{b+1})$. □

5. Future directions

When considering the volume of polytopes of a partition family with Young diagram size n vs. $(n-1)$, we see that some of the terms collapse. It is not clear at what rate these terms vanish or exactly how this corresponds. The reductions of these terms have some hope of providing the insights needed to transform the volumes of the limiting polytopes, which have a more clear combinatorial meaning, to the more complex polytopes such as the CRY and Tesler polytopes. It would be interesting to find a combinatorial expression of the limiting process and how it affects the family of polytopes geometrically. Perhaps considering a series of repeated gifts of candy, where the limit steps would have clear rules for how the students give out candy, could be a setting in which the limit steps are analyzable. For example, for the CRY polytope, one could do something like the following.

The nth child first gives all of their candy to the other $(n-1)$ children. Then the $(n-1)$st child gives all of their candy to the $(n-2)$ remaining children. The $(n-2)$nd child gives all of their candy to the $(n-3)$ remaining children. We continue in this fashion, and then the 3rd child gives all their candy to the 2 remaining children. The 2nd child gives all of their candy to the 1 remaining child. At each step, we can try to associate this to the geometric decomposition formulas and, thus, hopefully, get a clear look at how things are changed when we take a limit step.

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