PRIMES Math Problem Set: Solutions

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Solution to General Math Problems

Problem G1

A polynomial $f(x)$ has complex coefficients. It turns out that $f(x) \cdot f'(x)$ is a degree five polynomial whose x^5, x^4, x^1, x^0 coefficients are respectively 3, 10, 25, 12. Determine the polynomial f.

Solution

The answers are

$$\begin{aligned} f(x) &= x^3 + 2x^2 + 3x + 4, \\ f(x) &= x^3 + 2x^2 + \frac{-3+\sqrt{73}}{2}x + \frac{9-3\sqrt{73}}{8}, \\ f(x) &= x^3 + 2x^2 + \frac{-3-\sqrt{73}}{2}x + \frac{9+3\sqrt{73}}{8} \end{aligned}$$

and the negations of these (for a total of six possible answers).

Notice that we have

$$2f(x)f'(x) = 6x^5 + 20x^4 + \bigstar x^3 + \bigstar x^2 + 50x^1 + 24$$

where $\bigstar$ represents coefficients that are not known. The left-hand side is the derivative of the polynomial $f(x)^2$, so it follows (by integrating both sides) that

$$f(x)^2 = x^6 + 4x^5 + \bigstar x^4 + \bigstar x^3 + 25x^2 + 24x + \bigstar.$$

Apparently, f is a cubic polynomial. By replacing f with $-f$ if necessary, we may as well assume f is monic. So we are seeking constants a, b, c such that

$$(x^3 + ax^2 + bx + c)^2 = x^6 + 4x^5 + \bigstar x^4 + \bigstar x^3 + 25x^2 + 12x + \bigstar.$$

We therefore get the system of equations

$$\begin{aligned} 2a &= 4, \\ 2ac + b^2 &= 25, \\ 2bc &= 24. \end{aligned}$$

Evidently $a = 2$, and now solving the resulting cubic equation gives $(b, c) = (3, 4)$ as well as the two solutions $(b, c) = \left(\frac{-3\pm\sqrt{73}}{2}, \frac{9\pm3\sqrt{73}}{8}\right)$. This gives the solutions claimed.

Problem G2

Scientists have found a vaccine that produces undesirable side effects with probability p. Initially, the number p is distributed uniformly across the interval $[0, 0.1]$. To test the vaccine, the scientists test the vaccine on 148374 volunteers and find that no one experiences adverse side effects.

Find the smallest real number λ such that the scientists can assert $p < \lambda$ with probability at least 95%. Round your answer to four significant figures.

Solution

Let $n = 148374$ for brevity. By Bayes' theorem, we require λ to satisfy

$$\begin{aligned} 0.95 &\le \mathbb{P}(p < \lambda) \\ &= \frac{\int_0^\lambda (1-p)^n \, dp}{\int_0^{0.1} (1-p)^n \, dp} \\ &= \frac{\frac{1}{n+1}(1-(1-\lambda)^{n+1})}{\frac{1}{n+1}(1-(1-0.1)^{n+1})} \\ &= \frac{1-(1-\lambda)^{n+1}}{1-0.9^{n+1}}. \end{aligned}$$

Using exact methods will give $\lambda = 2.019 \cdot 10^{-5}$ as the optimal choice. Actually, this can be approximated very closely by hand by simply commenting the denominator is very nearly 1 for large n; so essentially we want $\lambda \approx 1 - 0.05^{1/(n+1)}$ which gives the same approximation above.

Problem G3

Let p be an odd prime number. Calculate the number of triples $(a, b, c) \in \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p$ for which $a + b + c = a^3 + b^3 + c^3 = 1$.

Solution

If $p = 3$, then we are simply counting the number of triples which satisfy $a + b + c \equiv 1 \pmod 3$, which is 9.

Otherwise, notice that

$$0 = (a + b + c)^3 - (a^3 + b^3 + c^3) = 3(a + b)(b + c)(c + a).$$

Hence the solution set is exactly the number of triples (a, b, c) such that $a + b + c = 1$ and some two are negatives of each other. In other words, (a, b, c) is a permutation of $(1, t, -t)$.

This is now a fairly routine counting problem. There are 3 ways to pick which entry to be 1, and p choices for the other two entries. This overcounts $(1, 1, -1)$ and so on twice each. So the answer is $3p - 3$.

Problem G4

We roll a fair six-sided die and let s_1 be the result of the roll. Then, we roll s_1 fair six-sided dice and let s_2 be the sum of the rolls. Then, we roll s_2 fair six-sided dice and let s_3 be the sum of the rolls. The process continues to generate an infinite sequence $(s_1, s_2, \dots)$.

(a) Find the probability that 3 appears in the sequence.

(b) Find the expected value of s_n, for each integer n.

(c) We say the sequence grows exponentially if there exists a constant $c > 1$ such that $s_n > c^n$ for all sufficiently large integers n. Does the sequence grow exponentially almost surely?

Solution

(a): The sequence is nondecreasing, so we can solve this by considering just a finite number of states. Let a be the probability of achieving 3 from s_n if $s_n = 1$, and let b be the probability of achieving 3 from $s_n = 2$. Then it follows that

$$\begin{aligned} a &= \frac{1}{6} \cdot a + \frac{1}{6} \cdot b + \frac{1}{6} \cdot 1 + \frac{3}{6} \cdot 0, \\ b &= \frac{1}{36} \cdot b + \frac{2}{36} \cdot 1 + \frac{33}{36} \cdot 0. \end{aligned}$$

Solving gives $b = 2/35$ and $a = 37/175$, so the answer is $37/175$.

(b): $s_n = (7/2)^n$ by induction. Indeed, s_{n+1} in general is equal to the sum of s_n independent dice rolls, and each dice roll contributes $7/2$. To make this rigorous we may write

$$s_{n+1} = \sum_{k \geq 1} X_k \cdot \mathbf{1}_{s_n > k}$$

where X_k is a dice roll, and $\mathbf{1}_{s_n > k}$ is the relevant indicator variable. Then by taking linearity of expectation we get

$$\mathbb{E}[s_{n+1}] = \sum_{k \geq 1} \frac{7}{2} \cdot \mathbf{P}(s_n > k) = \frac{7}{2} \sum_{k \geq 1} \mathbf{P}(s_n > k) = \frac{7}{2} \mathbb{E}[s_n]$$

proving the claim.

(c): Yes (and the bounds are quite weak).

Notice that $s_{n+1} \geq \frac{3}{2} s_n$ holds as long as at least half the dice roll greater than 1, so it holds with probability at least $\frac{1}{2}$, say.

Now by the central limit theorem, if we take $c = \sqrt[3]{3/2}$, then for sufficiently large N, the probability that $s_n < c^n = (3/2)^{n/3}$ decays exponentially in n. Taking a union bound across large enough N will give a total probability less than 1, as desired.

Problem G5

For each positive integer $n \geq 4$, find all positive real numbers a_1, a_2, $\ldots$, a_n such that

$$a_i^2 = 19a_{i+1} + 20a_{i+2} + 21a_{i+3}$$

holds for all $i = 1, \ldots, n$ with indices taken modulo n.

Solution

The answer is that all numbers must be equal to 60 (which works).

The largest number is at most 60, since if M is maximal, then $M^2 \leq 60M$.

The smallest number is at least 60, since if m is minimal, then $m^2 \geq 60m$.

So all the numbers are equal to 60.

Problem G6

If s is a finite binary string, then we denote by $f(s)$ the sum of the squares of the lengths of the *consecutive runs* of f. For example, $f(10110001111100) = 1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 2^2 = 44$.

Suppose that a binary string s of length n is specified by letting the ith bit be 1 with probability p_i and 0 with probability $1 - p_i$, all independent. We wish to calculate the expected value of $f(s)$ given the values of n, p_1, p_2, …, p_n.

(a) Exhibit an algorithm with the best runtime you can find, in terms of n.

(b) Give the best lower bounds you can on the runtime of such an algorithm.

Solution

The answer is that $O(n)$ time is possible (and this is asymptotically best possible since it takes time to read the input). This is essentially the same as CodeForces 235B *Let's Play Osu!*, which in turn appears to be inspired by the rhythm game *Osu* (in which hitting k consecutive notes correctly earns $300(1 + 2 + \cdots + k)$ points).

In what follows, we abbreviate "consecutive run" to "block".

We first show how to calculate the contribution from blocks of 1's; the analogous calculation from blocks of 0's then gives the solution.

The idea is that the score due to a block of length k may be written as $n^2 = 1 + 3 + 5 + \cdots + (2k - 1)$, which lets us simply sum the scores from left to right. For example, the input string 10110001111100 can be decomposed as the following gains:

Bit	1	0	1	1	0	0	0	1	1	1	1	1	0	0
Score	+1		+1	+3				+1	+3	+5	+7	+9		

So let X_i be the number of additional points scored by the i'th bit, which means that

$$X_i = \begin{cases} X_{i-1} + 2 & i\text{th bit is 1 and } (i-1)\text{st bit is 1} \\ 1 & i\text{th bit is 1 and } (i-1)\text{st bit is 0} \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$X_i = \mathbf{1}_{i\text{th bit is 1}} \cdot \left(X_{i-1} + 2 - \mathbf{1}_{(i-1)\text{st bit is 0}}\right)$$

with $\mathbf{1}_E$ be the indicator variable for event E. Taking expectation of both sides,

$$\mathbb{E}[X_i] = p_i(\mathbb{E}[X_{i-1}] + 2 - (1 - p_{i-1})).$$

So $\mathbb{E}[X_i]$ can be computed recursively for $i = 1, 2, \ldots, n$ in linear time.

Similarly the contribution of analogously defined variable for the runs of 0's can be computed. This completes the solution.

Solution to Advanced Math Problems

Problem M1

For positive integers n, find a closed form for

$$\sum_{\substack{a+b+c+d=n \\ a,b,c,d\geq 0}} 2^{a+2b+3c+4d}$$

in terms of n.

Possible hint: use generating functions.

Solution

We use generating functions. Because $\frac{1}{1-2X} = \sum_k 2^k X^k$, it follows the desired answer is equivalent to the coefficient of x^n in

$$F(X) = \frac{1}{1-2X} \cdot \frac{1}{1-4X} \cdot \frac{1}{1-8X} \cdot \frac{1}{1-16X}.$$

We may express $F(X)$ using partial fractions as

$$\frac{-1/21}{1-2X} + \frac{2/3}{1-4X} - \frac{8/3}{1-8X} + \frac{64/21}{1-16X}.$$

This gives the answer of

$$\frac{-1}{21} \cdot 2^n + \frac{2}{3} \cdot 4^n - \frac{8}{3} \cdot 8^n + \frac{64}{21} \cdot 16^n.$$

Problem M2

We say a real number α is *good* if there exist nonzero integers m and n such that $e^{\alpha m}$ is an integer divisor of 2020^n.

(a) Let V denote the set of real numbers which are the sum of two good numbers. Show that V is a $\mathbb{Q}$-vector space under addition.

(b) Calculate $\dim V$ and give an example of a basis of V.

Solution

In general, if θ^m is an integer divisor of 2020^n for some n, then $\theta = 2^x 5^y 101^z$ for some rational numbers x, y, z, with the same sign. This gives a characterization of good numbers.

Hence V consists of those β such that $e^\beta = 2^x 5^y 101^z$ for rational numbers x, y, z. This immediately implies that V is closed under addition and rational multiplication, and so it is indeed a vector space, spanned by $\log 2$, $\log 5$, $\log 101$. By the fundamental theorem of arithmetic it follows readily that these numbers are $\mathbb{Q}$-linearly independent, so they actually form a basis, and in particular $\dim V = 3$.

Problem M3

Let T be a finite tournament. For any vertex v, the indegree and outdegree of v is denoted by $\operatorname{indeg} v$ and $\operatorname{outdeg} v$, respectively. For each positive integer d we then define

$$A_d = \sum_v (\operatorname{indeg} v)^d \qquad B_d = \sum_v (\operatorname{outdeg} v)^d.$$

(a) Find all d such that $A_d = B_d$ holds for any tournament T.

(b) Prove or disprove: if $A_3 \geq B_3$ then $A_4 \geq B_4$.

(c) Prove or disprove: if $A_4 \geq B_4$ then $A_5 \geq B_5$.

Solution

For $d = 1$ the statement is true since $A_1 = B_1 = \binom{n}{2}$; we claim also $A_2 = B_2$. Indeed,

$$\begin{aligned} A_2 - B_2 &= \sum_v (\operatorname{indeg} v)^2 - \sum_v (\operatorname{outdeg} v)^2 \\ &= \sum_v (\operatorname{indeg} v + \operatorname{outdeg} v)(\operatorname{indeg} v - \operatorname{outdeg} v) \\ &= \sum_v (n-1)(\operatorname{indeg} v - \operatorname{outdeg} v) \\ &= 0. \end{aligned}$$

However, for $d \geq 3$ one can take a tournament with outdegree sequence $(3, 1, 1, 1)$, (and hence indegree sequence $(0, 2, 2, 2)$), so $A_d = 3^d + 3$ while $B_d = 3 \cdot 2^d$, which forces $d \leq 3$.

Part (b) is true, and the inequalities are equivalent actually. In what follows, we let $k = n - 1$ and set

$$\begin{aligned} \operatorname{indeg} v &= \frac{k}{2} + x_v \\ \operatorname{outdeg} v &= \frac{k}{2} - x_v \end{aligned}$$

for some (integer or half-integer) x_v. Since $\sum_v \operatorname{indeg} v = \sum_v \operatorname{outdeg} v = \binom{n}{2}$ it follows that $\sum_v x_v = 0$. Now

$$\begin{aligned} A_3 - B_3 &= \sum_v \left(\frac{k}{2} + x_v\right)^3 - \sum_v \left(\frac{k}{2} - x_v\right)^3 \\ &= \sum_v \left[2x_v^3 + 6\left(\frac{k}{2}\right)^2 x_v\right] \\ &= 2\sum_v x_v^3 \end{aligned}$$

while

$$\begin{aligned} A_4 - B_4 &= \sum_v \left(\frac{k}{2} + x_v\right)^4 - \sum_v \left(\frac{k}{2} - x_v\right)^4 \\ &= \sum_v \left[4x_v^3\left(\frac{k}{2}\right) + 4x_v\left(\frac{k}{2}\right)^3\right] \\ &= 2k\sum_v x_v^3 \end{aligned}$$

and so one is nonnegative if and only if the other is.

Part (c) is false. One counterexample is to consider a tournament on 16 vertices which has 1 vertex of indegree 5, 10 vertices of indegree 7, and 5 vertices of indegree 9. (One can verify, say by Landau's theorem, that a tournament with this degree sequence actually exists). In that case we have

$$\begin{aligned} A_4 &= B_4 = 57440 \\ A_5 &= 466440 \\ B_5 &= 466560 \end{aligned}$$

so this exhibits the desired counterexample.

Problem M4

A particle is initially on the number line at a position of 0. Every second, if it is at position x, it chooses a real number $t \in [-1, 1]$ uniformly and at random, and moves from x to $x + t$.

Find the expected value of the number of seconds it takes for the particle to exit the interval $(-1, 1)$.

Possible hint: for each $0 < x < 1$, let $E(x)$ denote the expected value of the amount of time until the particle exits the interval. You may assume without proof that $E(x)$ is a well-defined and analytic function on the interval $(0, 1)$.

Solution

We let $E(x)$ be the expected value of the time until exiting if the particle starting from a position of x, where $0 < x < 1$. For $x \geq 0$, obviously $E(x) = E(-x)$ by symmetry.

Apparently,

$$E(x) = 1 + \frac{1}{2} \int_{x-1}^{x+1} E(y)\, dy.$$

Let's define the constant $C = 2 + \int_0^1 E(y)\, dy$. Then for $0 \leq x < 1$, we recover the statement

$$2E(x) = C + \int_0^{1-x} E(y)\, dy.$$

Actually, note that by setting $x = 0$ we get $2E(0) = 2C - 2$ or $E(0) = C - 1$.

Differentiate once:

$$2E'(x) = -E(1 - x) \tag{1}$$

Differentiate again:

$$2E''(x) = E'(1 - x) = -\frac{1}{2} E(x) \tag{2}$$

Consequently, from $E'' = -E/4$ (by (2)) we conclude

$$E(x) = a \sin \frac{x}{2} + b \cos \frac{x}{2}$$

for some constants a and b, valid for $0 < x < 1$.

Returning to (1) we should have, for all $0 < x < 1$, the identity

$$\begin{aligned} a \cos \frac{x}{2} - b \sin \frac{x}{2} &= -a \sin \frac{1-x}{2} - b \cos \frac{1-x}{2} \\ &= -a \left[\sin \frac{1}{2} \cos \frac{x}{2} - \cos \frac{1}{2} \sin \frac{x}{2} \right] - b \left[\cos \frac{1}{2} \cos \frac{x}{2} + \sin \frac{1}{2} \sin \frac{x}{2} \right] \\ &= \left(-a \sin \frac{1}{2} - b \cos \frac{1}{2} \right) \cos \frac{x}{2} + \left(a \cos \frac{1}{2} - b \sin \frac{1}{2} \right) \sin \frac{x}{2} \end{aligned}$$

This can only hold for all $0 < x < 1$ if we have

$$\begin{aligned} a &= -a \sin \frac{1}{2} - b \cos \frac{1}{2} \\ -b &= a \cos \frac{1}{2} - b \sin \frac{1}{2} \end{aligned}$$

which both imply

$$\frac{a}{b} = -\frac{\cos \frac{1}{2}}{1 + \sin \frac{1}{2}} = -\frac{1 - \sin \frac{1}{2}}{\cos \frac{1}{2}}$$

So, let us write

$$E(x) = \lambda\left[-\cos\frac{1}{2}\sin\frac{x}{2} + \left(1+\sin\frac{1}{2}\right)\cos\frac{x}{2}\right].$$

for some constant λ.

$$\begin{aligned}
C = E(0) + 1 = 2 + \int_0^1 E(y)\,dy \\
\implies \lambda\left[1+\sin\frac{1}{2}\right] &= 1 - \lambda\cos\frac{1}{2}\int_0^1 \sin(y/2)\,dy + \lambda\left(1+\sin\frac{1}{2}\right)\cdot\int_0^1\cos(y/2)\,dy \\
&= 1 - \lambda\cos\frac{1}{2}\left(2 - 2\cos\frac{1}{2}\right) + \lambda\left(1+\sin\frac{1}{2}\right)\cdot 2\sin\frac{1}{2} \\
\lambda &= \frac{1}{\cos\frac{1}{2}\cdot\left(2-2\cos\frac{1}{2}\right) + \left(1-2\sin\frac{1}{2}\right)\left(1+\sin\frac{1}{2}\right)} \\
&= \frac{1}{2\cos\frac{1}{2} - \sin\frac{1}{2} - 1}
\end{aligned}$$

Finally, we seek $E(0)$:

$$E(0) = \lambda\cdot\left(1+\sin\frac{1}{2}\right) = \frac{1+\sin\frac{1}{2}}{2\cos\frac{1}{2}-\sin\frac{1}{2}-1} \approx 5.3653.$$

Problem M5

Suppose G is a finite group and $\varphi\colon G \to G$ a homomorphism. Denote by $0 \le k \le 1$ the fraction of elements $g \in G$ which satisfy

$$\varphi(g) = g^2.$$

(a) Give an example where $k = 0.03$.

(b) If $k \neq 1$, how large can you get k to be?

Solution

This problem is based off an infamous exercise in Herstein's *Topics in Algebra* in which the condition $\varphi(g) = g^2$ is instead $\varphi(g) = g^{-1}$. The solution is analogous.

For (b), the answer is $k = 3/4$. An example achieving the equality case is to choose $G = Q_8$ the quaternion group, and let $\varphi(i) = \varphi(j) = -1$ and $\varphi(k) = +1$.

To show $k > 3/4$ can't work, we define the set

$$S = \left\{ \varphi(g) = g^2 \right\}.$$

Fix any $s \in S$. Note that if $g \in S$ is an element for which $gs \in S$, then

$$gsgs = \varphi(gs) = \varphi(g)\varphi(s) = ggss \implies gs = sg$$

and so g is in $C_G(s)$, the centralizer of s. By principle of inclusion-exclusion, the number of $g \in S$ which have this property is greater than $\frac{3}{4}|G| + \frac{3}{4}|G| - |G| = \frac{1}{2}|G|$, so in other words, $|C_G(s)| > \frac{1}{2}|G|$. Since $C_G(s)$ is a subgroup of G though, we need $C_G(s) = G$. In other words, s lies in the center of G.

Thus the center of G contains all of S, but since $|S| \ge \frac{3}{4}|G| > \frac{1}{2}|G|$, the center coincides with G — that is, G is abelian. But now if $g \in G$ is any element, again we can find some $s \in S$ such that $gs \in S$, and now we have

$$gsgs = \varphi(gs) = \varphi(g)\varphi(s) = \varphi(g)ss \implies \varphi(g) = g^2$$

so φ is actually the map $g \mapsto g^2$ on the whole group. Hence $k = 1$, violating the assumption.

As for (a), one may take the product of the quaternion group $G = Q_8 \times \mathbb{Z}/25\mathbb{Z}$, with $\varphi\colon Q_8 \times \mathbb{Z}/25\mathbb{Z} \to Q_8 \times \mathbb{Z}/25\mathbb{Z}$ acting on the first component as in the previous example and trivially on the second component. This gives $\frac{3}{4} \cdot \frac{1}{25} = 0.03$.

Problem M6

A unit regular tetrahedron is a tetrahedron whose edge lengths are all equal to 1. Two unit regular tetrahedrons $ABCD$ and $WXYZ$ lie in Euclidean space. The labelings of $ABCD$ and $WXYZ$ are *oppositely* oriented.

(a) How small can $\max(AW, BX, CY, DZ)$ be?

(b) Generalize from 3 dimensions to n dimensions.

Solution

The answer to (b) is $\sqrt{(n+1)/\left\lfloor \frac{1}{2}(n+1)^2 \right\rfloor}$ and hence the answer to (a) is $\sqrt{2}/2$. This problem was suggested by Nikolai Beluhov, who previously proposed the special case $n = 2$ as problem 4 of grade 10 on the 2012 Autumn Mathematical Tournament in Bulgaria.

Let $A_1 \dots A_{n+1}$ and $B_1 \dots B_{n+1}$ be the two unit regular n-simplices with opposite orientations. We are going to use two lemmas and one well-known theorem.

Lemma M6.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be any isometry which flips orientation. Let P be any point, $Q = f(P)$, and let M denote the midpoint of PQ. As P varies, the locus of M is contained in some $(n-1)$-hyperplane.*

Proof. Suppose that f maps the orthonormal basis $Oe_1e_2 \dots e_n$ onto $O'e'_1e'_2 \dots e'_n$. Then f is the composition of some linear isometry h with transformation matrix H and the translation $\overrightarrow{OO'}$.

Since h is an isometry, $HH^\top = E$, and since h flips orientation, $\det H = -1$. We have that

$$H^\top(E+H) = H^\top E + H^\top H = H^\top + E = (E+H)^\top.$$

Therefore

$$\begin{aligned} -\det(E+H) &= \det H^\top \det(E+H) = \det\left[H^\top(E+H)\right] \\ &= \det(E+H)^\top = \det(E+H), \end{aligned}$$

implying that $\det(E+H) = 0$.

Now $\vec{OM}$ satisfies

$$\vec{OM} = \frac{1}{2}(\vec{OP} + \vec{OQ}) = \frac{1}{2}(E+H)\vec{OP} + \frac{1}{2}\vec{OO'}.$$

Since $\det(E+H) = 0$, the rank of the transformation matrix $E+H$ is at most $n-1$, and so all points M lie in some $(n-1)$-hyperplane when P varies, as needed. □

Lemma M6.2. *We say that two faces of an n-simplex are complementary when their vertex sets form a partitioning of the vertex set of the entire simplex. The shortest distance between two complementary faces of $A_1A_2 \dots A_{n+1}$ equals*

$$d = \sqrt{\frac{n+1}{\left\lfloor \frac{1}{2}(n+1)^2 \right\rfloor}}$$

and is attained exactly when one face is a $\lfloor (n+1)/2 \rfloor$-face and the other one is a $\lceil (n+1)/2 \rceil$-face.

Proof. Let C_i be the point whose ith coordinate is 1 and other coordinates are 0. The points C_1, C_2, ..., C_{n+1} then form a regular n-simplex in $\mathbb{R}^{n+1}$ whose side length is $\sqrt{2}$. The distance from its center $O = (1/(n+1), 1/(n+1), \dots, 1/(n+1))$ to its k-face $C_1C_2 \dots C_k$ equals the distance from O to the center of this k-face. That center is

$$O_k = (1/k, 1/k, \dots, 1/k, 0, 0, \dots, 0),$$

and so that distance is $\sqrt{\frac{\ell}{k(n+1)}}$, where $\ell = (n+1) - k$.

The distance between two complementary faces of dimensionalities k and ℓ, then, equals the sum of the distances from O to these faces, which is given exactly by

$$\sqrt{\frac{\ell}{k(n+1)}} + \sqrt{\frac{k}{\ell(n+1)}} = \frac{k+\ell}{\sqrt{k\ell(n+1)}} = \sqrt{\frac{n+1}{k\ell}}.$$

For a unit n-simplex, the corresponding distance will instead equal $\sqrt{\frac{n+1}{2k\ell}}$, and so it will be minimized exactly when the difference $|k - \ell|$ is as small as possible. $\square$

Lemma M6.3 (Radon's theorem)**.** *Let P_1, P_2, ..., P_{n+2} be $n+2$ points in n-dimensional Euclidean space. Then there exists at least one partitioning of the set $\{P_1, P_2, \dots, P_{n+2}\}$ into two subsets A and B such that their convex hulls $\mathcal{H}(A)$ and $\mathcal{H}(B)$ have a common point.*

We now solve the problem. By Lemma M6.1, the midpoints M_1, M_2, ..., M_{n+1} of segments A_1B_1, A_2B_2, ..., $A_{n+1}B_{n+1}$ lie in some $(n-1)$-hyperplane α.

Consider the $n+1$ balls S_1, S_2, ... S_{n+1} of centers A_1, A_2, ... A_{n+1} and radii $d/2$.

Claim. There is at least one ball S_i such that α does not intersect the interior of S_i.

Proof. Suppose, for the sake of contradiction, that α intersects all of these balls in interior points R_1, R_2, ..., R_{n+1}. By Radon's theorem, there exists at least one partitioning of the set $\{1, 2, \dots, n+1\}$ into two subsets U and V such that the convex hulls of subsets $R_U = \{R_i \mid i \in U\}$ and $R_V = \{R_i \mid i \in V\}$ have a common point X.

Consider, then, the complementary faces

$$F_U = \mathcal{H}(\{A_i \mid i \in U\}) \qquad \text{and} \qquad F_V = \mathcal{H}(\{A_i \mid i \in V\})$$

of $A_1A_2 \dots A_{n+1}$ together with the $(n-1)$-hyperplane β that is parallel to both of them and lies mid-way between them.

Since d is at most the distance between F_U and F_V, we have that β separates all balls with indices in U from all balls with indices in V. Consequently, β separates R_U and R_V, and hence also their convex hulls. We have arrived at a contradiction with $\mathcal{H}(R_U)$ and $\mathcal{H}(R_V)$ having a common point X. $\square$

Suppose α does not intersect the ball S_1. Then the distance between A_1 and B_1 is at least twice the radius of S_1, that is, at least d.

On the other hand, a longest segment of length exactly d is attained when $B_1B_2 \dots B_{n+1}$ is the reflection of $A_1A_2 \dots A_{n+1}$ across any $(n-1)$-hyperplane that lies mid-way between two complementary faces of dimensions $\lfloor \frac{n+1}{2} \rfloor$ and $\lceil \frac{n+1}{2} \rceil$. This completes the main part of the solution.

Problem M7

Let G be a finite simple graph with n vertices. Say that two Hamiltonian paths P_1 and P_2 of G are *neighbors* if they have exactly $n-2$ edges in common; also say a Hamiltonian path P and a Hamiltonian cycle C of G are *neighbors* if every edge of P is also an edge of C. Finally, we say that two Hamiltonian cycles C_1 and C_2 of G are *equivalent* if there exist some number of Hamiltonian paths $P_1, P_2, \ldots, P_k$ of G such that every pair of consecutive terms in the sequence $C_1, P_1, P_2, \ldots, P_k, C_2$ are neighbors.

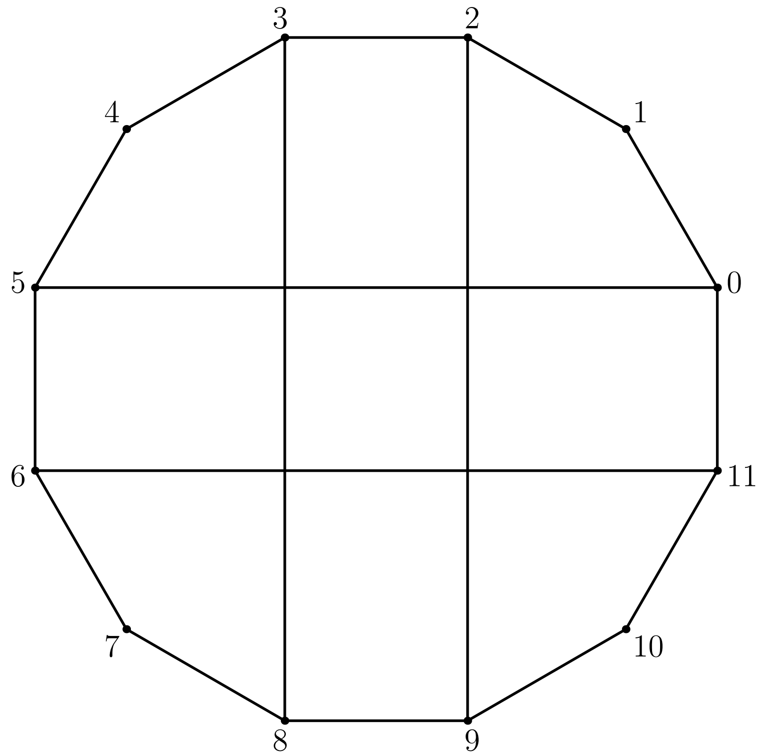
(a) Give an example of a graph G with at least two inequivalent Hamiltonian cycles.

(b) Give an example of a graph G with at least 2020 inequivalent Hamiltonian cycles or prove that no such graph exists.

Solution

This problem is exercise 79 in the current draft of section 7.2.2.4, Hamiltonian Paths and Cycles, of Donald Knuth's book *The Art of Computer Programming.* (See `https://cs.stanford.edu/~knuth/fasc8a.ps.gz` for the most recent version.)

We start with the solution to (a). Let H be the graph whose vertex set is $V(H) = \mathbb{Z}/12\mathbb{Z}$, the integers modulo 12. Draw an edge between x and $x+1$ for all $x \in V(H)$, as well as an edge between y and $y+5$ for $y \in \{0, 3, 6, 9\}$, for a total of 16 edges. This gives the graph H below.



This graph has exactly two Hamiltonian cycles, $C_1 = 0—1—\cdots—11—0$ and $C_2 = 0—1—2—9—10—11—6—7—8—3—4—5—0$. It is straightforward to check by hand that they are not equivalent. (We need to examine only a small number of cases because the graph is highly symmetric.)

We now proceed to part (b).

For the problem, we construct graph G as follows. Let k be any positive integer such that $2^k \geq 2020$.

- First we take k copies of H, denoted $H_0, H_1, \ldots, H_{k-1}$, where for all i the vertices of graph H_i are $v_{i,j}$ with $0 \leq j \leq 11$, so that vertex $v_{i,j}$ of H_i corresponds to vertex j of H. (Thus vertices $v_{i,j'}$ and $v_{i,j''}$ of H_i are joined by an edge if and only if vertices j' and j'' of H are joined by an edge.)
- Create k additional vertices $u_0, u_1, \ldots, u_{k-1}$.
- For all i we join vertex u_i to vertices $v_{i,0}$ and $v_{(i+1 \bmod k),1}$.

This completes our description of graph G. It has a total of $13k$ vertices and $18k$ edges.

We claim that G has exactly 2^k Hamiltonian cycles, and describe them. Every Hamiltonian cycle of G must contain all edges of G that are incident with a vertex of the form u_i; we call these edges of G *special*. Consequently, every Hamiltonian cycle of G can be obtained as follows:

- Inside each copy H_i of H, we place a copy of either C_1 or C_2.
- Then we delete all edges of the form $v_{i,0}$—$v_{i,1}$ from these copies, and we replace them with the $2k$ special edges of G so as to connect everything up.

Therefore, G has exactly $2^k \geq 2020$ Hamiltonian cycles.

On the other hand, every Hamiltonian path of G can be obtained in one of the following ways:

(i) By omitting one special edge from a Hamiltonian cycle of G.

(ii) By omitting one special edge from a Hamiltonian cycle of G, and then replacing the copy of C_1 or C_2 in the copy H_i of H incident with that special edge by any Hamiltonian path of H_i one of whose endpoints is also a vertex of the other special edge incident with H_i.

(iii) In the exact same way as the Hamiltonian cycles of G, except that in one copy H_i of H we must take a Hamiltonian path of H_i that contains edge $v_{i,0}$—$v_{i,1}$, instead of a copy of either C_1 or C_2.

Since Hamiltonian cycles C_1 and C_2 are not equivalent in H, from this description of the Hamiltonian paths of G we derive that no two Hamiltonian cycles of G are equivalent, either. This concludes the solution.