Extending the Restricted Lie Algebra Structure on the Homology of a Double Loop Space

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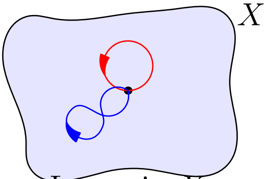
Introduction

In this talk, we introduce some algebraic structures on a double loop space $Y = \Omega^2 X$ and examine how they can help us recover the space X.

Loops

Definition

A **loop** in a space X is a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = \gamma(1)$.

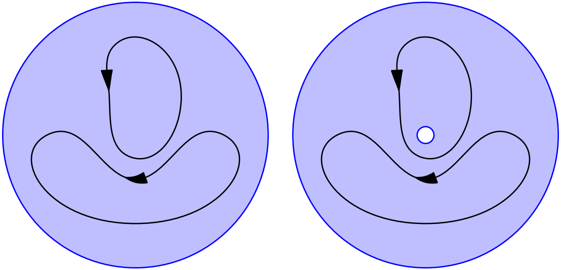


Loops in X

Homotopy Equivalence

Definition

We say two loops are **homotopy equivalent** if we can deform one loop into the other.



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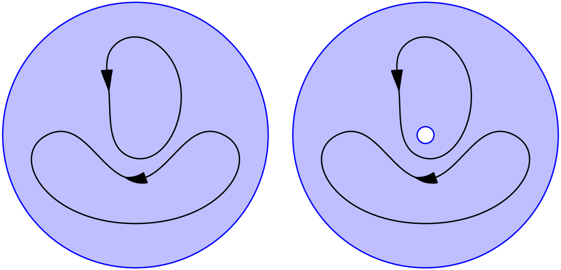
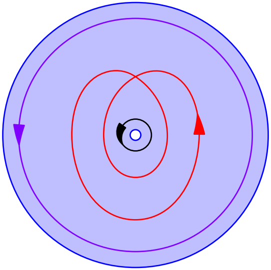


In $\mathbb{R}^2$ any two loops are homotopy equivalent to the constant loop at a point. But if we remove the origin from $\mathbb{R}^2$, the loop encompassing the origin is not homotopy equivalent to the other.

More Examples of Homotopy Equivalence



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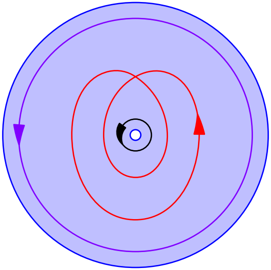

The purple loop is homotopy equivalent to the black loop since you can shrink the purple loop. The red loop is homotopy equivlaent to neither, since you cannot “unravel” it.

Homology

An important question in topology is to determine when two *spaces* are **homotopy equivalent** i.e., one space can be "deformed" into the other without breaking. The algebraic topology way of studying this is to construct invariants.

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Definition (Homology)

For a space X and a ring R, the *nth homology group* $H_n(X; R)$ measures the number of boundaries of $(n+1)$-dimensional balls in X. The *homology* $H_*(X; R)$ is defined as the direct sum of all the homology groups

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Example

For $n \geq 1$, the nth sphere S^n, the only nontrivial homology groups are $H_0(S^n; R) = H_n(S^n; R) = R$.

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S^1 is not homotopy equivalent to S^2, since $H_1(S^1) \cong R \cong H_2(S^2)$ and $H_1(S^2) \cong 0$. Intuitively, the circle bounds a 2-dimensional disk while a sphere bounds a 3-dimensional ball.

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Theorem (cf. Hatcher)

If there is a continuous map $f : X \to Y$, then there is an induced map $f_* : H_*(X) \to H_*(Y)$ in homology

For the remainder of the presentation we fix $R = \mathbb{F}_2$.

Loops

Definition

For a space X with basepoint $p \in X$, we define the **loop space** ΩX to be the space of loops γ with $\gamma(0) = \gamma(1) = p$. Each point in this space is a loop in X.

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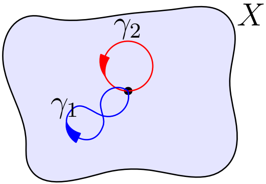
The multiplication on the loop space is given by concatenation.



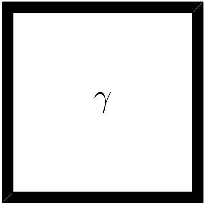
For instance, $\gamma_1 \cdot \gamma_2$ is given by tracing γ_1 first, and then γ_2.

Double Loop Space

Similarly, we can defined the double loop space on a space X with basepoint p.

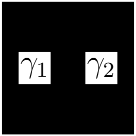
Definition

The **double loop space** $\Omega^2 X$ is the space of all maps $\gamma : [0, 1] \times [0, 1] \to X$ that map the boundary of the square to the base point p.



Multiplication in a Double Loop Space

Given two double loops $\gamma_1, \gamma_2 : [0,1]^2 \to X$, their product $\gamma_1 \cdot \gamma_2$ is given by first placing the squares in a bigger square γ and then mapping γ to X. Everything outside of γ_1 and γ_2 is mapped to the base point p of X.



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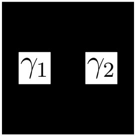
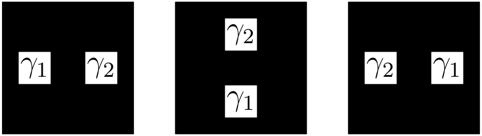


What makes the double loop space special is that the product is surprisingly commutative!

Commutativity of Multiplication

Using homotopy, we can rotate the small squares within the big square.



Therefore $\gamma_1 \cdot \gamma_2$ is the same as $\gamma_2 \cdot \gamma_1$ up to homotopy.

Binary Operations

The multiplication $\Omega^2 X \times \Omega^2 X \to \Omega^2 X$ induces a product on the homology

$$H_*(\Omega^2 X) \otimes H_*(\Omega^2 X) \cong H_*(\Omega^2 X \times \Omega^2 X) \to H_*(\Omega^2 X).$$

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Theorem (F.Cohen)

There is a Browder bracket $[-,-] : H_p(\Omega^2 X) \otimes H_q(\Omega^2 X) \to H_{p+q+1}(\Omega^2 X)$, *satisfying certain properties, which makes* $H_*(\Omega^2 X)$ *a Lie algebra.*

The Dyer-Lashof Operations

The **Dyer-Lashof operations** on $H_*(\Omega^2 X)$ are Q_0 and Q_1. The Q_0 operation is the same as squaring: $Q_0 x = x^2$. The Q_1 operation is defined as "half" the self bracket $[x, x]$.

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Theorem (cf. BMMS)

The Dyer-Lashof operations on $H_*(\Omega^2 X)$ satisfy the following identities:

- *(Top Additivity)* $Q_1(x + y) = Q_1(x) + Q_1(y) + [x, y]$;
- *(Adjoint Identity)* $[x, Q_1 y] = [y, [y, x]]$;
- *The Cartan Formula.*

The first two identities make Q_1 a **restriction** map for the Browder bracket, making $H_*(\Omega^2 X)$ a **restricted** Lie algebra.

Spectral Sequences

The Dyer-Lashof operations are important because they can tell you whether or not a space Y is the double loop space of another space. That is, if you can define operations Q_0, and Q_1 on Y satisfying the identities mentioned before, **then Y must be a double loop space**.

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However, it is often difficult to find a space X such that $Y = \Omega^2 X$. Still, we can get information about X by studying its homology.
We use the **bar spectral sequence** to compute $H_*(\Omega X)$ from $H_*(Y)$ and $H_*(X)$ from $H_*(\Omega X)$.

The Bar Spectral Sequence

The **bar spectral sequence** of $H_*(\Omega^2 X)$ is a sequence of approximations $E^0, E^1, E^2, \ldots$ converging to the E^∞-page, which contain pieces that can be used to reassemble $H_*(\Omega X)$.

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We have complete knowledge of what the E^0 and E^1-page look like. The E^1-page is the bar construction that we will introduce later. Each successive page E^r of the spectral sequence can be obtained from the previous page E^{r-1}.

E^1-page: the Bar Construction

Let $*$ denote the space consisting of a single point. The collapse map $\Omega^2 X \to *$ induces a map $\epsilon : H_*(\Omega^2 X) \to H_*(*) \cong \mathbb{F}_2$. Set

$$\overline{H_*(\Omega^2 X)} = \ker \epsilon.$$

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The **bar construction**, $B_{*,*}(H_*(\Omega^2 X))$ of $H_*(\Omega^2 X)$, is the chain complex defined as

$$B_{s,*} = \overline{H_*(\Omega^2 X)} \otimes \cdots \otimes \overline{H_*(\Omega^2 X)},$$

repeated s times for all s.

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We write an element $x_1 \otimes x_2 \otimes \cdots \otimes x_s$ as $[x_1|x_2|\cdots|x_s]$. The differential $D : B_{s,*} \to B_{s-1,*}$ is given by

$$D[x_1|\cdots|x_s] = \sum_{i=1}^{s-1} [x_1|\cdots|x_i x_{i+1}|\cdots|x_s].$$

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Extending the Operations on Homology

It is often possible to define structures on the E^1-page that survive to the E^∞-page. We defined a multiplication and the Browder bracket on $H_*(\Omega^2 X)$. Can we extend these operations to the bar construction?

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Yes! In fact, it turns out that multiplication and bracket can be extended to the bar construction. Both survive to the E^∞-page, where they give rise to the product and the Browder bracket on $H_*(\Omega X)$ as proved by Xianglong Ni.

Extending the Operations on Homology

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Recall that in $H_*(\Omega^2 X)$ we defined the Dyer-Lashof operations Q_1, which is a restriction map for the Browder bracket. Some natural questions to ask are

1. Is there a way to extend Q_1 to an operation ξ on the bar construction such that ξ is the restriction of the extended bracket?
2. Does that construction survive to the E^∞-page?
3. Does it survive to the squaring operation Q_0 on $H_*(\Omega X)$, which is the restriction for the Browder bracket on $H_*(\Omega X)$?

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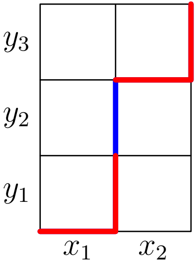
2. Does that construction survive to the E^∞-page?
3. Does it survive to the squaring operation Q_0 on $H_*(\Omega X)$, which is the restriction for the Browder bracket on $H_*(\Omega X)$?

This is what my project is about. So far, we have answered Question 1. To define ξ we'll first discuss the construction of the bracket on the bar construction.

Bracket on the bar construction (Ni)

Next we'll want to define the bracket of $x = [x_1|\cdots|x_p]$ and $y = [y_1|\cdots|y_q]$.
Again, label a $p \times q$ grid as follows, with x on one side and y on the other and consider the paths from $(0,0)$ to (p,q).
For a given path, mark all the places where it changes direction from horizontal to vertical, and insert a bracket.



The above path gives terms $[[x_1, y_1]|y_2|x_2|y_3] + [x_1|y_1|y_2|[x_2, y_3]]$.

Restriction on the Bar Construction

Intuition: this is "half the extended bracket $[x, x]$".

Construction: Let $x = [x_1|\cdots|x_s]$. If $s = 1$, we set $\xi(x) = [Q_1(x_1)]$. For $s > 1$, we take

$$\xi(x) = \sum_{\substack{(s,s)-\text{shuffles } \varphi \\ \text{with } \varphi^{-1}(1)=1}} \sum_{\substack{\varphi^{-1}(i)\leq s \\ \varphi^{-1}(i+1)>s}} [a_{\varphi^{-1}(1)}|\cdots|[a_{\varphi^{-1}(i)}, a_{\varphi^{-1}(i+1)}]|\cdots|a_{\varphi^{-1}(2s)}],$$

where

$$a_i = \begin{cases} x_i & \text{if } i \leq s \\ x_{i-s} & \text{if } i > s \end{cases}$$

We extend ξ to the entire bar construction via top additivity

$$\xi(x+y) = \xi(x) + \xi(y) + [x, y],$$

for any x and y.

Restriction on the Bar Construction

Theorem (S.)

The operation ξ satisfies these identities:

- *Top additivity* $\xi(x+y) = \xi(x) + \xi(y) + [x,y]$;
- *Adjoint identity* $[x, \xi y] = [y,[y,x]]$;
- $D\xi x = [x, Dx]$.

Hence the bar construction $B_{*,*}(H_*(\Omega^2 X))$ is a restricted Lie alegbra with the extended Browder bracket and restriction map ξ.

These all follow from some combinatorial arguments.

Acknowledgments

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- Prof. Haynes Miller for suggesting the project idea
- my parents for supporting me
- and finally PRIMES USA for giving me this opportunity.

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