Topological Entropy of Simple Braids

Luke Robitaille
mentor: Dr. Minh-Tâm Trinh

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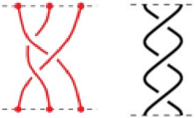
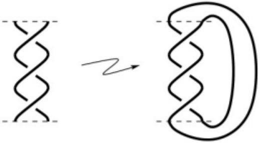
What is a braid?

We can think of a braid as formed by n strands (think of pieces of string) that can cross over and under one another.



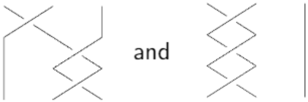
Braids are related to other topological objects, including knots and links.



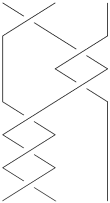
[images from https://arxiv.org/abs/1103.5628]

What is a braid?

The braids on n strands form a group B_n. For example, the product of



is



[images made using https://users.math.msu.edu/users/wengdap1/filling_to_cluster.html]

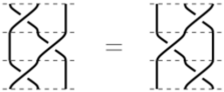
What is a braid?

The multiplication operation in B_n is not commutative in general.

The group B_n is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$.

$\sigma_i =$ 

They satisfy the relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$; they also satisfy $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$.

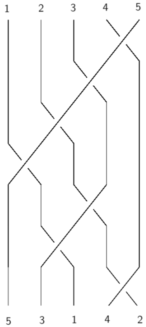


[images from https://arxiv.org/abs/1103.5628]

What is a simple braid?

There is a natural map from B_n to S_n (the group of permutations of $\{1, \ldots, n\}$) where σ_i maps to the transposition swapping i and $i + 1$.

The *simple braids* are natural preimages of the $n!$ elements of S_n.



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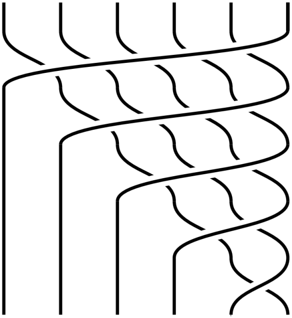
What is a simple braid?

One notable simple braid is the *half twist*, which corresponds to the permutation $i \mapsto n + 1 - i$. For $n \geq 3$, the square of the half twist generates the center of B_n.



[image from https://arxiv.org/abs/1302.6536]

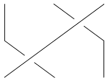
The Nielsen–Thurston classification

Braids can be classified as

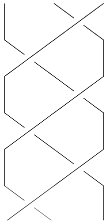
- periodic,
- reducible and not periodic, or
- pseudo-Anosov.

The Nielsen–Thurston classification

A braid is periodic if it can be raised to some power to equal some power of the full twist. For example, cubing this braid



gives the full twist.



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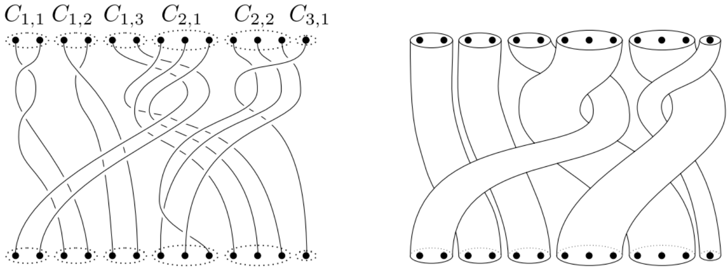
The Nielsen–Thurston classification

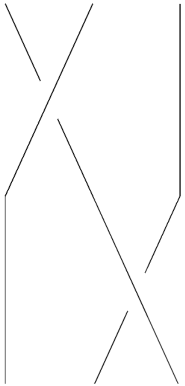
A braid is reducible if it is possible to draw some loops to get something like the image below.



[image from Juan González-Meneses, "The *n*th root of a braid is unique up to conjugacy"]

The Nielsen–Thurston classification

A braid that is not periodic or reducible is pseudo-Anosov.



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https://users.math.msu.edu/users/wengdap1/filling_to_cluster.html]

Topological entropy

If a braid is periodic or is reducible with all components periodic, it's "orderly" and has topological entropy zero. Otherwise (if it is pseudo-Anosov or is reducible with at least one pseudo-Anosov component), it's "chaotic" and has positive topological entropy.

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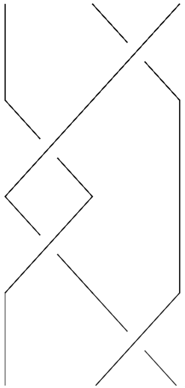

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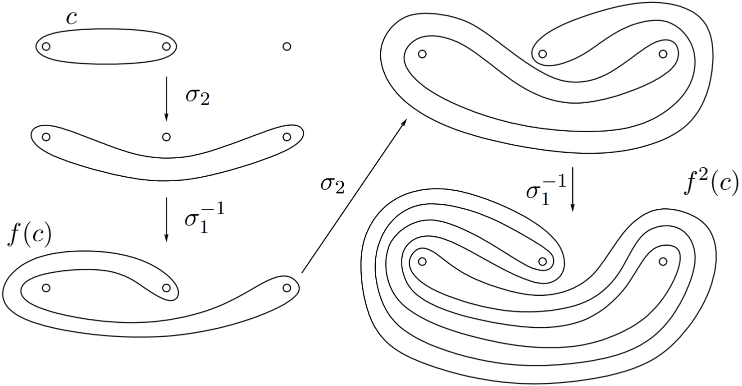

[image from Benson Farb and Dan Margalit, *A Primer on Mapping Class Groups*]

Topological entropy

Topological entropy of braids has applications in real life to the mixing of fluids.

The property of having topological entropy zero is preserved under raising to a power.

The Burau representation

There is a useful homomorphism from B_n to the group of invertible $(n-1) \times (n-1)$ matrices whose entries are polynomials with integer coefficients in t and t^{-1}.

$$\sigma_1 \mapsto \begin{bmatrix} -t & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \sigma_2 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & -t & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_3 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & -t & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \sigma_4 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t & -t \end{bmatrix}$$

Kolev found a relationship between the topological entropy of a braid and the eigenvalues of its image under the Burau representation, for t on the unit circle in the complex numbers.

Simple braids and the Burau representation

Theorem (R.–Trinh)

The images of simple braids obey certain patterns, as the example below illustrates.

1	2	3	4	5	6	7	8
4	7	2	5	8	6	1	3

1 2 3 4 5 6 7 8

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0	0	0	0	0	$-t$	1
0	$-t^2$	t	0	0	$-t$	1
0	$-t^3$	t^2	0	0	$-t^2$	0
t^3	$-t^3$	t^2	0	0	$-t^2$	0
t^4	$-t^4$	0	t^2	0	$-t^2$	0
t^5	$-t^5$	0	t^3	$-t^3$	0	0
0	0	0	t^3	$-t^3$	0	0

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t^3	$-t^3$	t^2	0	0	$-t^2$	0
t^4	$-t^4$	0	t^2	0	$-t^2$	0
t^5	$-t^5$	0	t^3	$-t^3$	0	0
0	0	0	t^3	$-t^3$	0	0

Main theorem

Theorem (R.–Trinh)

The proportion of simple braids in B_n that have positive topological entropy goes to 1 as n goes to infinity.

Acknowledgments

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