Bounds on Symmetric Numerical Semigroups

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A Familiar Problem

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Example

Given only coins worth 3 and 4 cents, the largest value that we cannot obtain is $3 \cdot 4 - 3 - 4 = \boxed{5}$.

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The set $\langle 3, 5, 7 \rangle = \{3a + 5b + 7c \mid a, b, c \in \mathbb{N}_0\} = \{0, 3, 5, 7, 8, 9, 10, \ldots\}$.

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The smallest set $\{a_1, a_2, \ldots, a_n\}$ s.t. $\Gamma = \{a_1 x_1 + \cdots + a_n x_n \mid x_i \in \mathbb{N}_0\}$ consists of the **minimal generators** of Γ.

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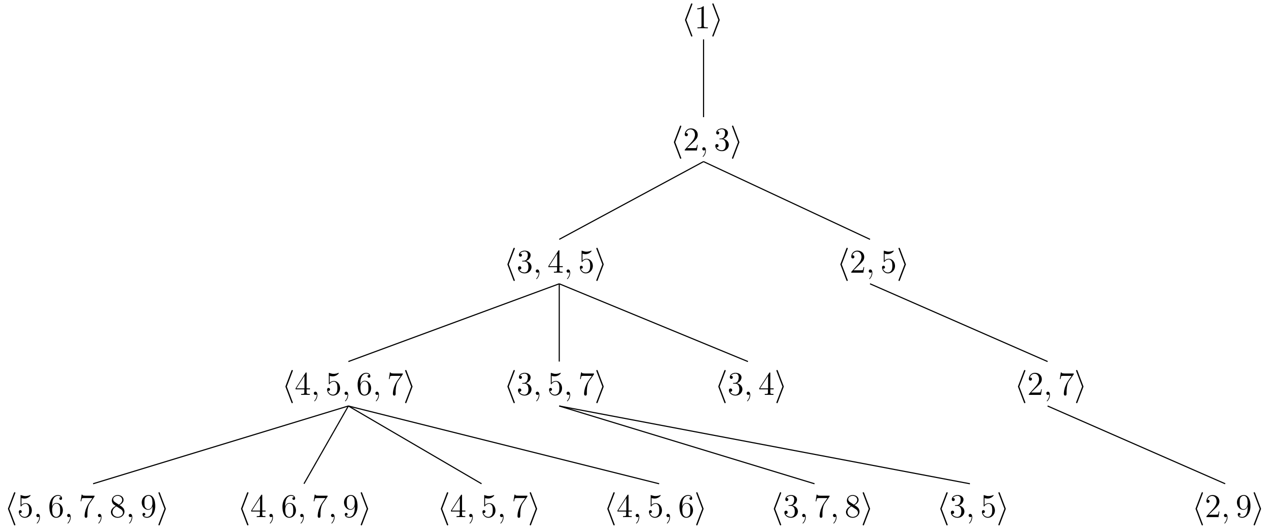

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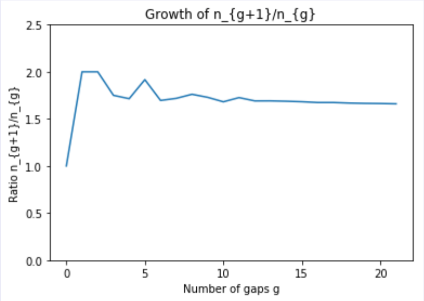

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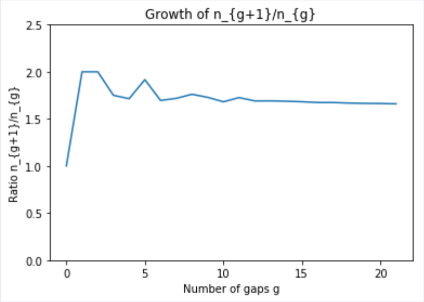

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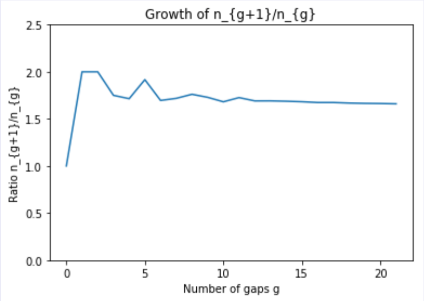

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Motivation

- Any semigroup contains at most one of $(k, F-k)$. Thus, a symmetric semigroup contains the **maximum** number of elements below its Frobenius number.

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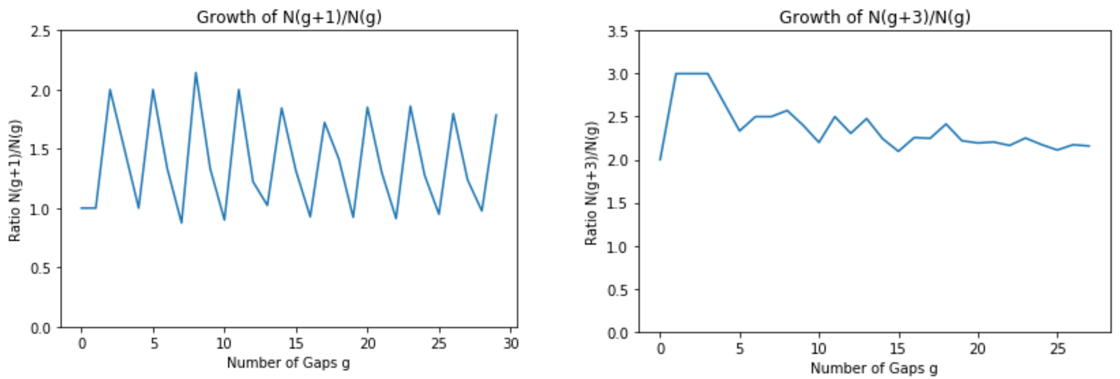
Conjecture



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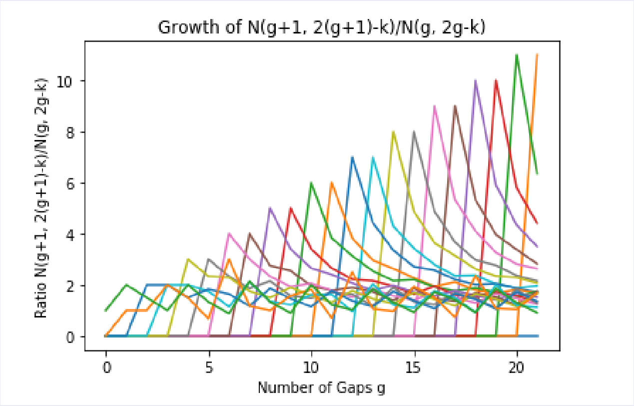

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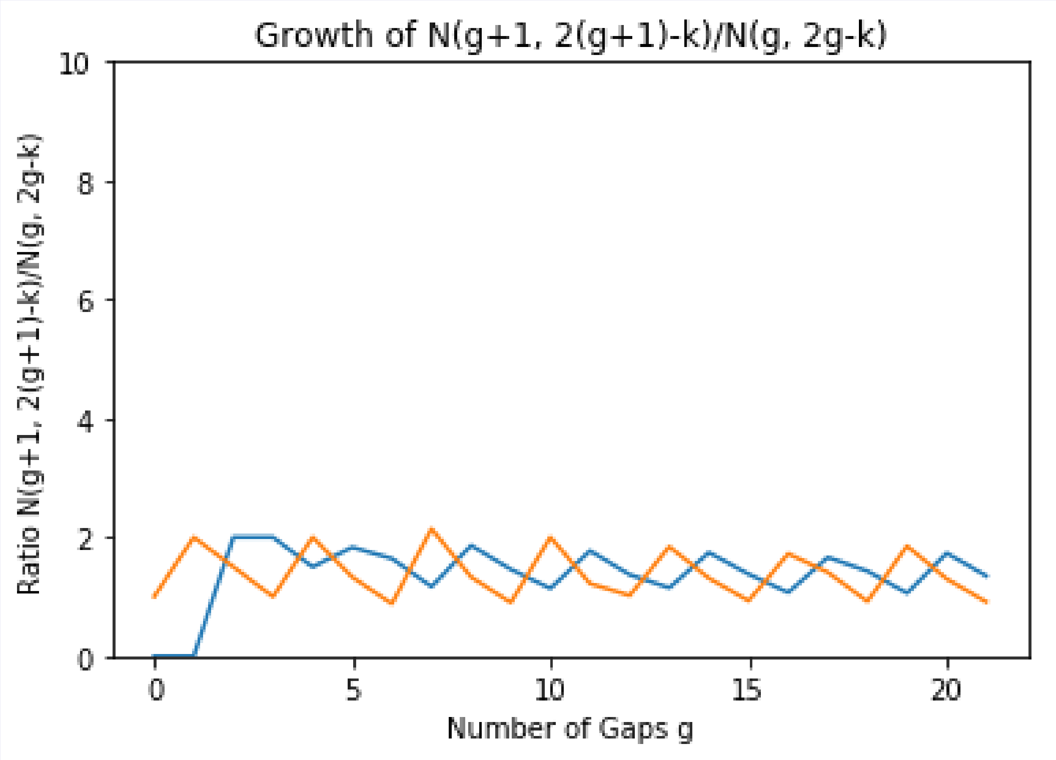

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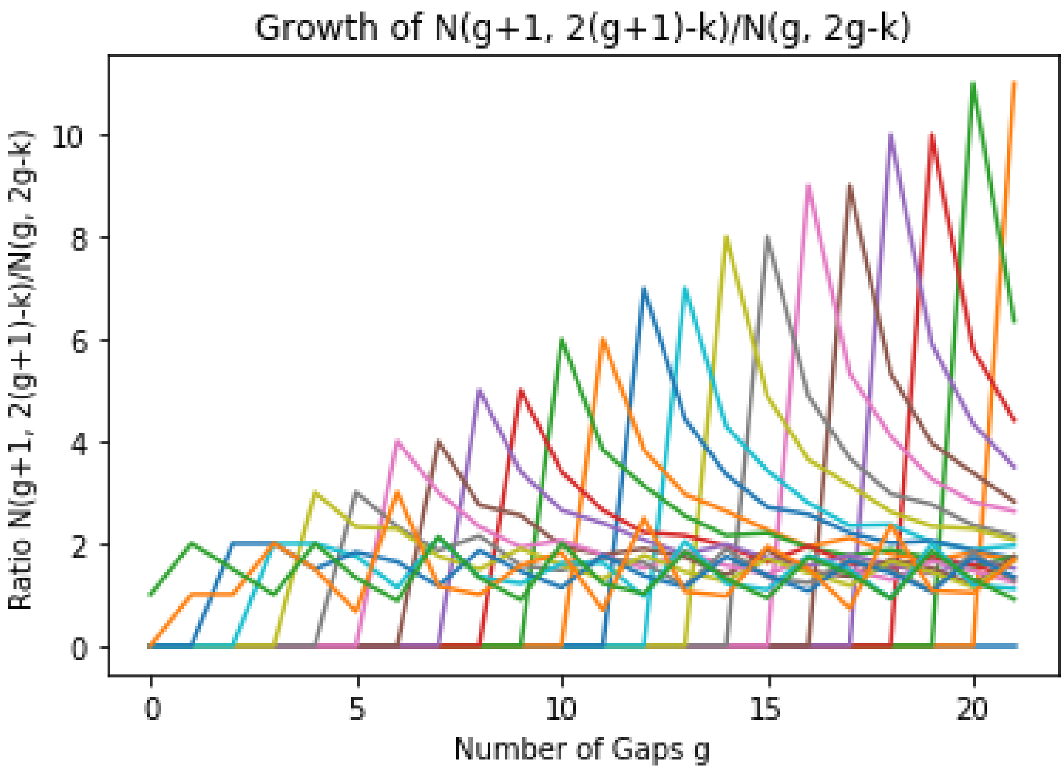

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Bibliography

Maria Bras-Amoros.
Bounds on the number of numerical semigroups of a given genus, 2008.

Maria Bras-Amoros and Stanislav Bulygin.
Towards a better understanding of the semigroup tree, 2008.

Maria Bras-Amorós.
Fibonacci-like behavior of the number of numerical semigroups of a given genus.
Semigroup Forum, 76(2):379–384, Oct 2007.

Nathan Kaplan.
Counting numerical semigroups, 2017.

Alex Zhai.
Fibonacci-like growth of numerical semigroups of a given genus, 2011.