Strichartz Estimates and Well-Posedness for the One-dimensional Periodic Dysthe equation

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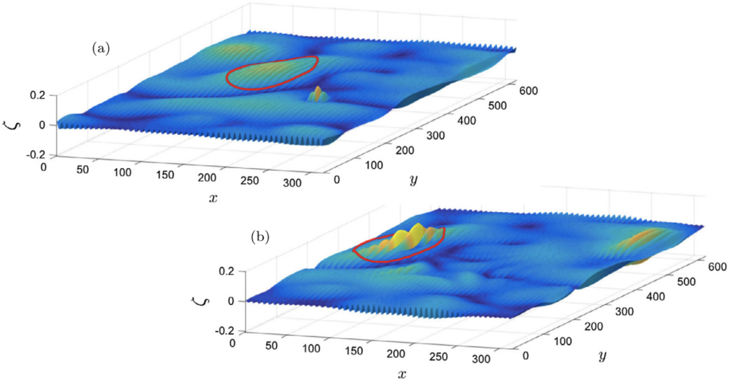
Rogue Waves

- What are rogue waves?

Rogue Waves

- Approximation of the water waves system



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- Quasilinear dispersive PDE with cubic nonlinearity
- Spatial periodicity: $u(x,t) \equiv u(x+2\pi, t)$, or equivalently $x \in \mathbb{T}$. The periodic setting is more applicable to numerical studies.

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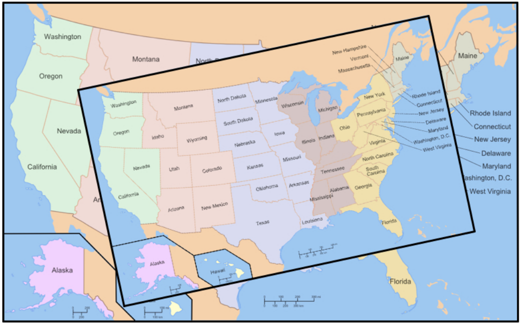
- Banach Fixed Point Theorem



Duhamel Form

- Duhamel (integrated) form for Dysthe equation:

$$u(t) = \eta(t)e^{it\mathcal{L}}u_0 - \eta(t)\int_0^t d\tau e^{i(t-\tau)\mathcal{L}}\mathcal{N}(u(\tau))$$

where

$$\mathcal{N}(u) = \frac{i}{2}|u|^2 u + \frac{3}{2}|u|^2\partial_x u + \frac{1}{4}u^2\partial_x u^* - \frac{1}{2}iu|\partial_x||u|^2$$

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- Physical significance: well-posedness or vibration modes correspond to numeric modelling and prediction of rogue waves.
- Philosophical significance: well-posedness = deterministic nature of the system, and ensures that algorithms give the correct results (that converge to the genuine solution).

Fourier Series

- For a sqaure-summable periodic function, we can always uniquely represent it with Fourier series:

$$u(x) = \sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx}, \quad \widehat{u}(n) = \int_0^{2\pi} e^{-inx} u(x) dx.$$

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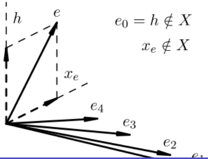
- Property: $\widehat{u'(n)} = -in\widehat{u}(n)$
- Parseval Theorem-Pythagorean Theorem for an infinite dimensional space: function-vector, Fourier modes -projections on orthogonal directions

$$\|u\|_{L^2_x(\mathbb{T})} = \left(\sum_{n\in\mathbb{Z}} |\widehat{u}(n)|^2\right)^{\frac{1}{2}}$$



Function Norms

- L^p spaces are metric function spaces for $1 \leq p \leq \infty$ equipped with the norm

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- The Sobolev space H^s is equipped with the norm

$$\|f\|_{H^s} = \|\langle n \rangle^s \widehat{f}(n)\|_{l_n^2} = \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\widehat{f}(n)|^2\right)^{\frac{1}{2}}.$$

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- Sobolev Embedding Theorem: weakly differentiable functions exhibit some regularity properties.

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- Linearized equation after scaling x direction:

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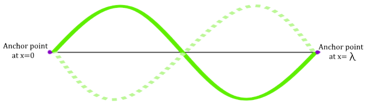
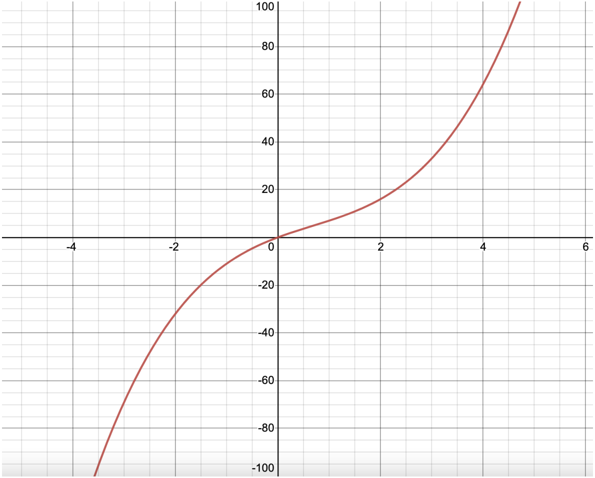
- Solution given by $\widehat{u}(n) = e^{-itP(n)}\widehat{u_0}(n)$, so

$$u(x,t) = \sum_{n\in\mathbb{Z}} e^{ixn - itP(n)}\widehat{u_0}(n)$$



Dispersive Relation Graph



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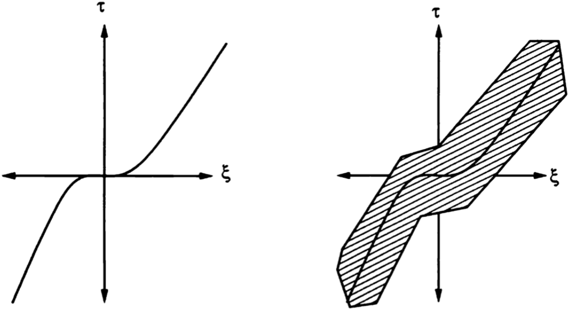
- The Bourgain space is denoted $X^{s,b}$ and is equipped with the norm:

$$\|u\|_{X^{s,b}} = \|\langle n\rangle^s \langle \tau - P(n)\rangle^b \widehat{u}(n,\tau)\|_{l^2_{n,\tau}}$$



Diophantine Equation in L^6 Strichartz Estimate

Theorem 1

The solution u to the linearized Dysthe equation with initial condition u_0 satisfies

$$\|u\|_{L^6_{x,t}} \lesssim \|u_0\|_{H^\epsilon_x}.$$

- Expand the L^6 norm with Parseval's Identity

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- Expand the L^6 norm with Parseval's Identity
- Control the resonances by bounding the number of solutions to the Diophantine equation $P(n_1)+P(n_2)+P(n-n_1-n_2)=j$ for fixed n, j.
- Result: We have an upper bound of $O(n^\epsilon)$ solutions to the previous diophantine equation.

L^4 Strichartz Estimate

Theorem 2

For the Bourgain space $X^{s,b}$ corresponding to the dispersive relation of the Dysthe equation, there holds

$$\|f\|_{L^4_{x,t}} \lesssim \|f\|_{X^{0,\frac{1}{3}}}.$$

- Main idea: control the L^2 norm of two dyadic frequency regions of f based off the $\langle \tau - P(n) \rangle$ term.

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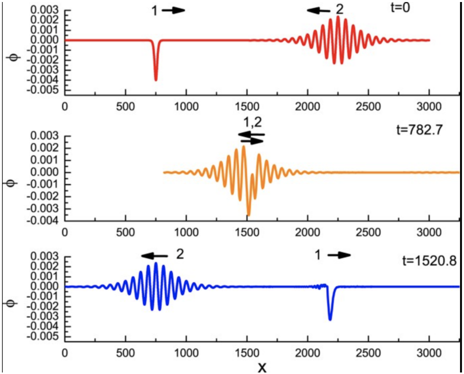
- Main idea: control the L^2 norm of two dyadic frequency regions of f based off the $\langle \tau - P(n) \rangle$ term.
- Why are Strichartz type estimates important?

Multilinear Estimates

Theorem 3

With $s \geq \frac{1}{2}$, $\mathbb{P}$ *being the orthogonal projection to zero-mean-value,*

$$\|\mathbb{P}(u_1)\mathbb{P}(u_2)\|_{Z^{s,\frac{-1}{2}}} \lesssim \|u_1\|_{Z^{s-1,\frac{1}{2}}}\|u_2\|_{Z^{s-1,\frac{1}{3}}} + \|u_1\|_{Z^{s-1,\frac{1}{3}}}\|u_2\|_{Z^{s-1,\frac{1}{2}}}.$$



Obstructions to Low-regularity Well-posedness

- We work a contraction mapping argument using the theorems on prior slides
- One difficulty in proving the well-posedness of the Dysthe equation is that the mean is not conserved unlike the widely studied Korteweg de-Vries equation.
- Thus the terms in the critical case for the bilinear estimate with $n_1 = 0$, $\langle \tau - P(n) \rangle \lesssim 1$, $\langle \tau_2 - P(n_2) \rangle \lesssim 1$, pose issues that we leave as an open question

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