A Topological Centrality Measure for Directed Networks

Linda Fenghuan He
Mentor: Lucy Yang

Commonwealth School

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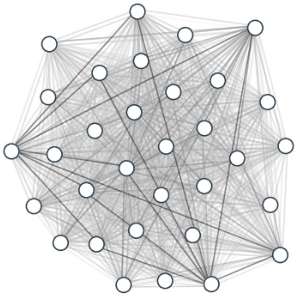

Motivation & Background

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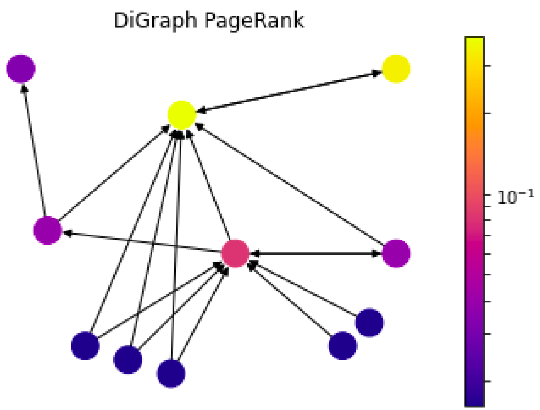

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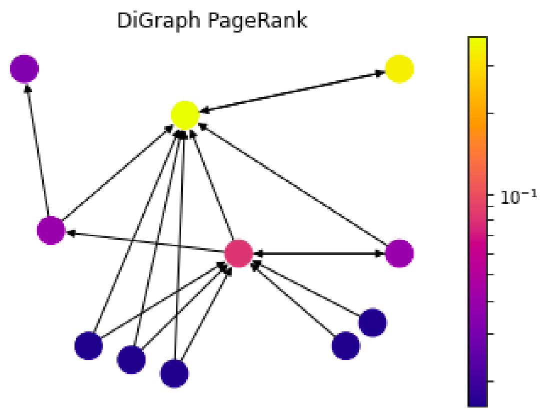

Goal

Define a centrality measure that captures non-local propagating effects and directedness.

Networks

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Definition

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Definition (F.Iannelli, 2017)

Let $G = (X, w_X)$ be a network, define $\gamma(G)$ to be (X, m_X) where $m_X : X \times X \to \mathbb{R}$ is given by:

$$m(x,y) = \begin{cases} 1 - \log \frac{w(x,y)}{\sum_{z \neq y} w(x,z)} \geq 1 & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$$

Two nodes that interact a lot ($w(x,y) \gg 0$) will be closer ($m(x,y) \sim 1$).

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Given a network G and x a node in G, define $f(G,x) = \left(X \setminus \{x\}, w_X|_{X\setminus\{x\}}\right)$, i.e. the sub-network induced by deleting x and all edges incident to x in G.

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Idea

Given x a node in G, we compare the difference in the "[dis]connectivity" of $\gamma(G)$ and $\gamma(f(G,x))$.

TDA in Networks

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Definition (F.Memoli and S.Chowdhury, 2016)

Given a network $G = (X, w_X)$ and $\delta \in \mathbb{R}$, the **Dowker Complex** $\mathcal{D}_{\delta,G}$ is the simplicial complex given by:

$$\mathcal{D}_{\delta,G} := \{\sigma \subseteq X : \exists p \in X \text{ s.t. } w(x,p) \leq \delta \ \forall\, x \in \sigma\}.$$

Example

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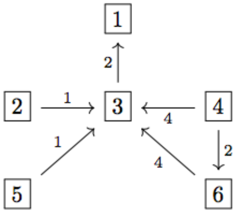

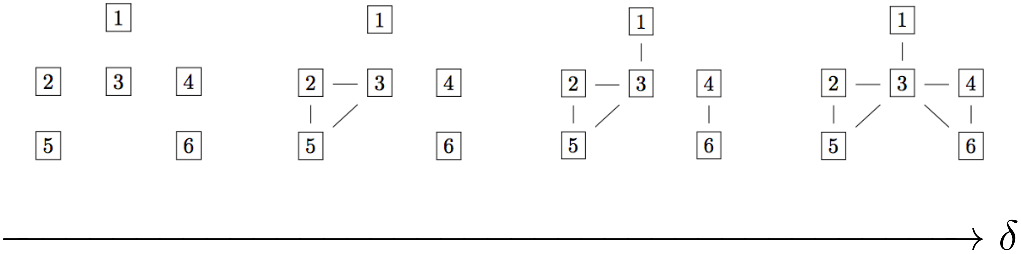

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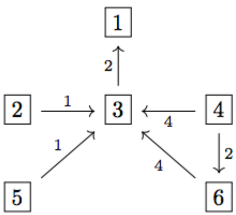
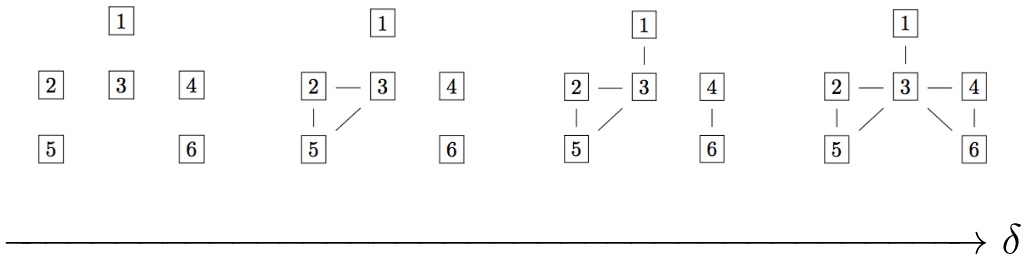
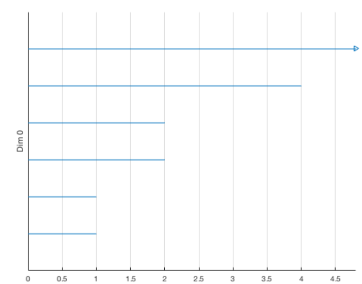




1. As $\delta \nearrow$, number of path components $\searrow$.
2. This data is recorded on a **persistence diagram**.
3. We denote $\mathbf{P}_0(G)$ as the set of 0-dimensional barcodes for the Dowker complex $\mathcal{D}_{\cdot,G}$.



Quasi-centrality

For a node $x \in X$, let $\mu(x) := d$ to be the value of δ for which x merges into another connected component.

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Definition

Let G be a network. The **quasi-centrality** $C(x)$ for node $x \in X$ is:

$$C(x) = \sum_{c \in \mathbf{P}_0(f(\gamma(G),x))} length(c) - \sum_{c \in \mathbf{P}_0(\gamma(G))} length(c) + d$$

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Theorem

For a network $G = (X, w_X)$, $C(x)$ is nonnegative for all $x \in X$.

Applications

Goals

- Demonstrate that C is a valid measure of centrality
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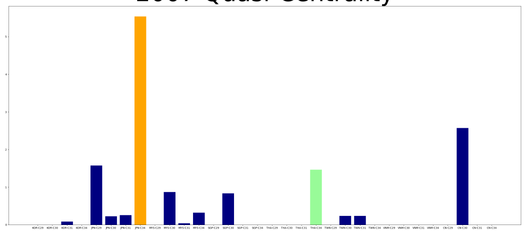
Data

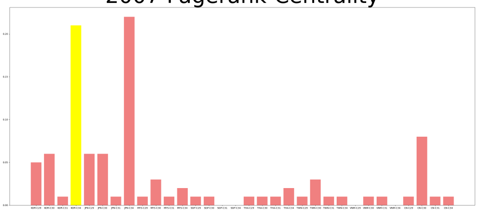
- OECD Inter-Country Input-Output (ICIO) Tables
- Machinery production network in Asia
- Industries: machinery equipment, computer and electronics, electrical machinery, auto machinery

Results

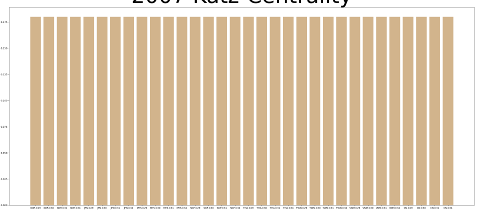
2007 Quasi-Centrality



2007 Pagerank-Centrality

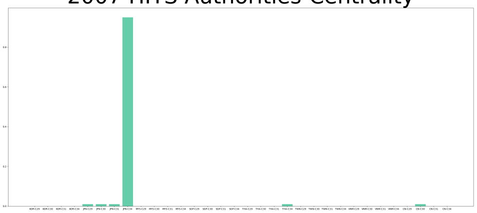


2007 Katz-Centrality



2007 HITS Hubs-Centrality

2007 HITS Authorities-Centrality



Future directions

- Compute the quasi-centrality measure for other asymmetric networks
  - biological networks
  - airflight networks
- Relate higher dimensional homological features in directed networks to real-world phenomena
  - trade flows
  - embargo
- Define other measures in network analysis using TDA
  - connectivity
  - robustness
  - efficiency

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