The binomial theorem and related identities

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Common mistake

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

Common mistake → $(a+b)^2 =$ ~~$a^2 + b^2$~~

The box method:

	a	+b
a	a^2	ab
+b	ab	b^2

The foil method:

$$(a+b)(a+b) = a^2 + 2ab + b^2$$

$(a+b)^4 = ??$ 🤔

Background information- binomial theorem

Help us to expand $(x+y)^n$ expression

Explains how to express the coefficient in $(x+y)^n$

Can prove the result in combinatorics

Explore probability

The binomial theorem

The binomial theorem formula

positive integers

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

any numbers

+ Reminder:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$Expand\,(x+y)^5$

$\binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5$

$\binom{5}{0} = \frac{5!}{0! * 5!} = 1$

Answer: $x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1y^5$

Prove: $0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$

Set X=-1 and y=1 in the binomial theorem

$$(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k}$$

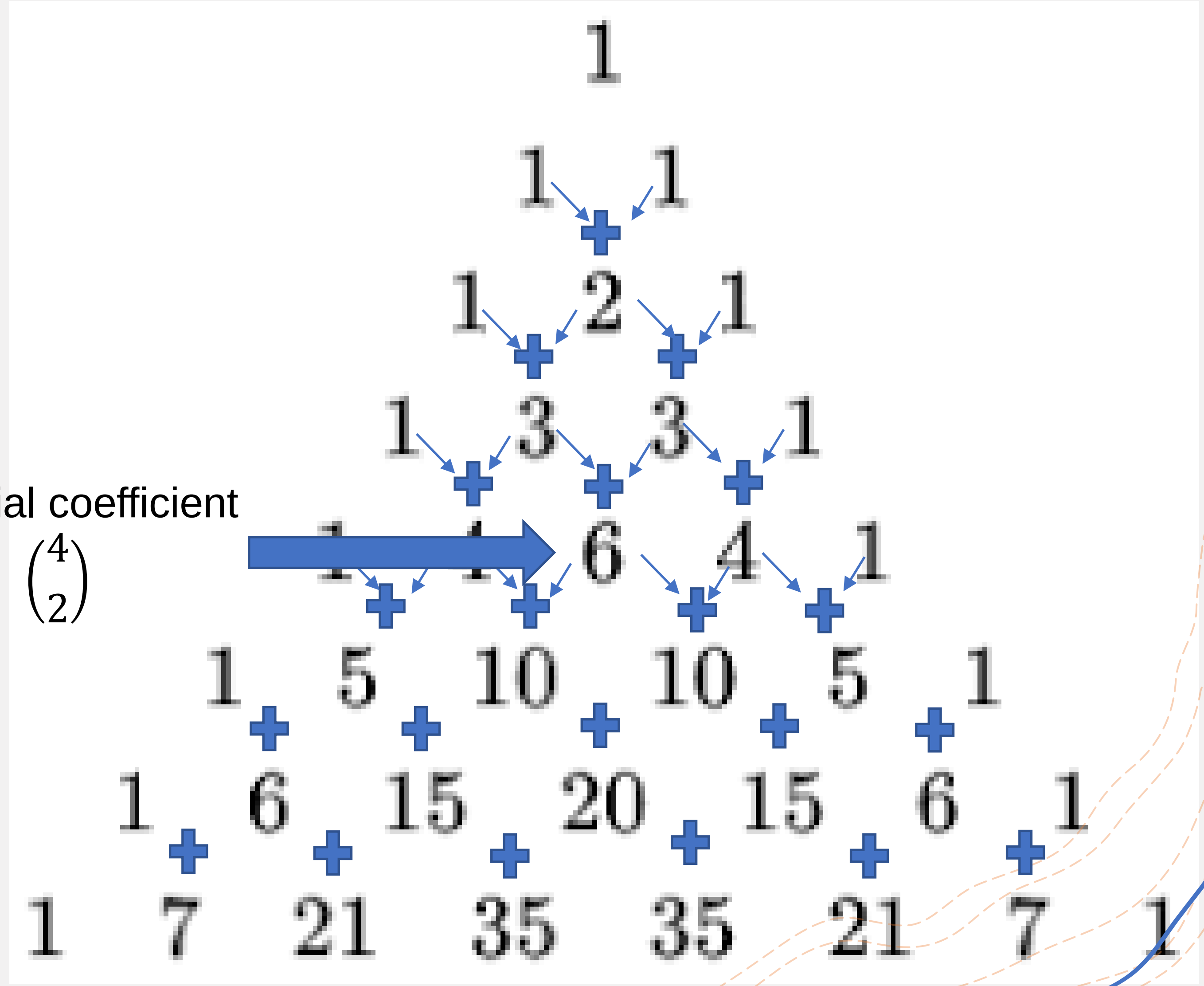
$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

The pascal’s triangle

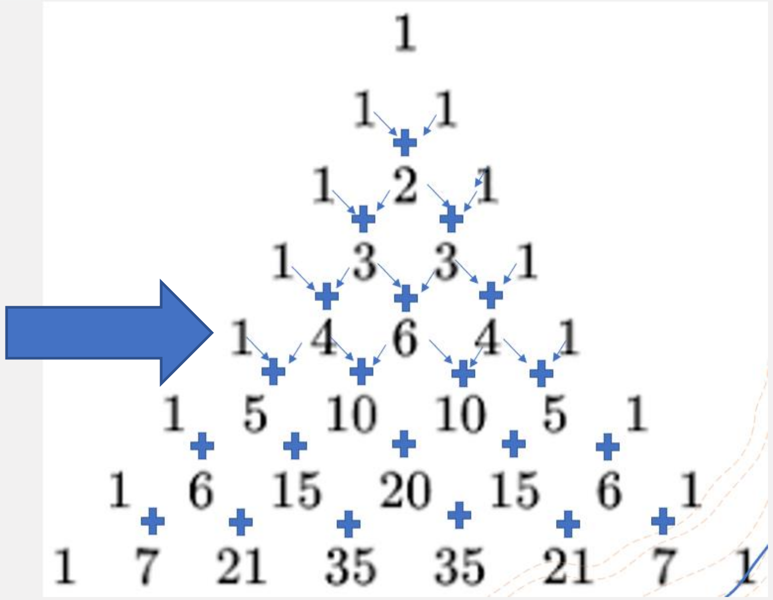
Help you to calculate the binomial theorem and find combinations way faster and easier

We start with 1 at the top and start adding number slowly below the triangular.

Binomial coefficient $\binom{4}{2}$



Example



+ Lets look at an example

$$(3x-6)^4$$

Now let solve this problem by using the pascal's triangle

1 4 6 4 1

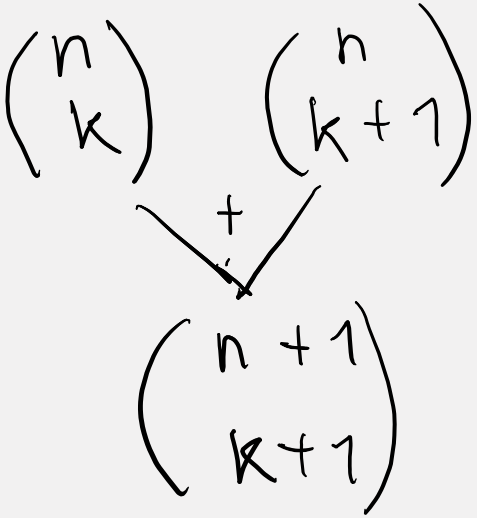
$$1(3x)^{4}(-6)^{0} + 4(3x)^{3}(-6)^{1} + 6(3x)^{2}(-6)^{2} + 4(3x)^{1}(-2)^{3} + 1(3x)^{0}(-6)^{4}$$

Answer: $3x^4 - 648x^3 + 1944x^2 - 96x + 1296$

How can people come up with Pascal’s triangle

All nonnegative integers n and k

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$



Why it is true ?

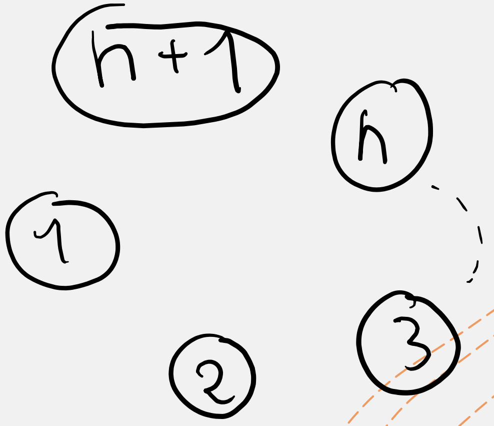
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Numbers of ways to choose k elements from n

Numbers of ways to choose k+1 elements from n

Counts number of ways to choose k+1 elements from n+1 elements

2 options



$\binom{n}{k}$ *choose* $n+1$ *as one of* $k+1$ *elements*

$\binom{n}{k+1}$ *do not choose* $n+1$

Generalized binomial theorem

The binomial theorem is only truth when n=0,1,2..,

So what is n is negative number or factions how can we solve.

The binomial theorem: $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$

The generalized binomial theorem:

$$(1+b)^r = \sum_{k=0}^{\infty}\binom{r}{k} b^k, r \in \mathbb{R}$$

Example

What does r choose k mean when r is not positive interger $\binom{r}{k}$?

$$\binom{r}{k} = \frac{r(r-1)(r-2)(r-3)..\big(r-(k-1)\big)}{k!}$$

$$\binom{-3}{6} = \frac{(-3)(-4)(-5)(-6)(-7)(-8)}{6*5*4*3*2*1} = 28$$

Example

b = -x

$$\sum_{k=0}^{\infty} \binom{-1}{k} (-x)^k = (1 - x)^{-1}$$

Trinomial theorem

$$(a+b+c)^n = \sum_{i+j+k} \binom{n}{i,j,k} a^i b^j c^k$$

Where i,j,k will be non-negative number

$$\binom{n}{i,j,k} = \frac{n!}{i!\,j!\,k!}$$

Example

How many terms in $(a + b + c)^7$?

i+j+k=7

I=2

J=0

K=5

$\binom{9}{2}$=36

Background information- multinomial theorem

How can we expand (x_1 + … + x_k)^n?

Willian L Hosch created the multinomial theorem

Multinomial theorem originally take from binomial theorem

It consist of the sum of many terms

Multinomial theorem

+ $(x_1 + \cdots + x_k)^n = \sum \frac{n!}{e_1!e_2!\ldots e_k!} x_1^{e1} x_2^{e2} \ldots x_k^{ek}$

Where: $e_1, e_2 \ldots e_k \geq 0$,

e_i is exponent of x_i in a monomial

$$e_1 + \cdots + e_k = n$$

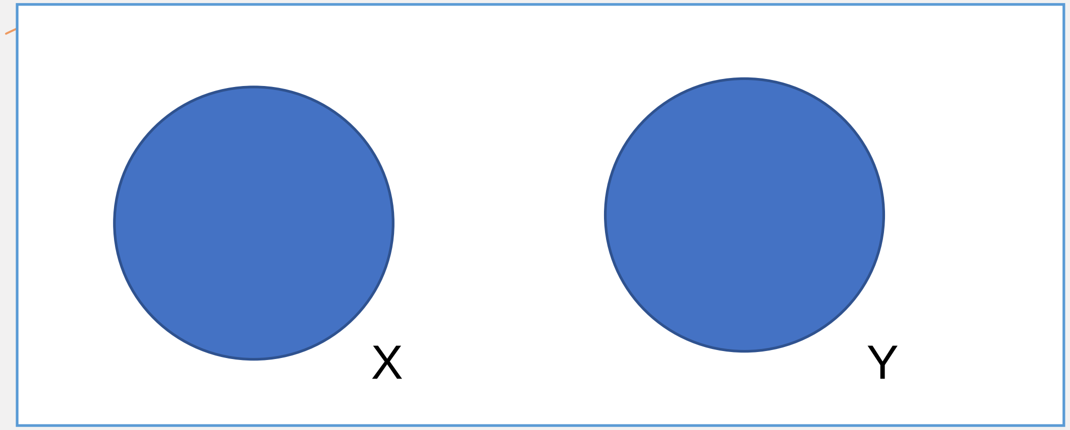
Vandermonde's identity

m,n, and k are non-negative integer with k≤min (m,n)

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} \quad r,m,n \in \mathbb{N}_0$$

Example

Prove Vanderdoes's identity using combinatoric



$$|X| = n$$

$$|Y| = m$$

$$|X \cap Y| = |X| + |Y| - 0$$

$$|X \cup Y| = |X| + |Y| = m + n$$

One side: $\binom{m+n}{r}$

Other side: let $0 \leq k \leq r$

Choose K elements from X $\binom{n}{k}$

Choose r-k elemts from Y $\binom{m}{r-k}$

We get $\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$

Binomial theorem, general version

Formula:

$$(1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n$$

Where m must be any real number

Sum taken all non-negative integer n

Example

Find the power series expansion of $\sqrt{1-4x}$

$$(1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n$$

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2} * \frac{-1}{2} * \frac{-3}{2} \dots \frac{-2n+3}{2}}{n!} = (-1)^{n-1} * \frac{(2n-3)!!}{2^n * n!}$$

Continue the example

$$\sqrt{1-4x} = 1 - 2x - \sum_{n\geq 2} \frac{2^n * (2n-3)!!}{n!} * x^n$$

$$\frac{2^n * (2n-3)!!}{n!} = 2 * \frac{(2n-2)!}{n!\,(n-1)!}$$

We got

$$\sqrt{1-4x} = 1 - 2x - \frac{2}{n}\sum_{n\geq 2} \binom{2n-2}{n-1} x^n$$

Source

A walk-through combinatorics

An introduction to enumeration and graph theory

Fourth edition

Miklos Bona