The Post-Correspondence Problem: Minions and Matching Strings

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An Interesting Puzzle

We have a collection of dominos:

$$\left\{\left[\frac{onsb}{sb}\right], \left[\frac{a}{an}\right], \left[\frac{nas}{as}\right], \left[\frac{i}{ni}\right], \left[\frac{ni}{on}\right], \left[\frac{m}{mi}\right], \left[\frac{n}{\epsilon}\right]\right\}.$$

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Goal: to make a *match* (repetition allowed) where the string with all the symbols on the top is the same as the string on the bottom.

Solution

$$\{[\frac{m}{mi}], [\frac{i}{ni}], [\frac{ni}{on}], [\frac{onsb}{sb}], [\frac{a}{an}], [\frac{n}{\epsilon}], [\frac{a}{an}], [\frac{nas}{as}]\}$$

Post Correspondence Problem Definition

We have the collection of dominos $\{[\frac{t_1}{b_1}], [\frac{t_2}{b_2}], \ldots, [\frac{t_k}{b_k}]\}$.

Definition

A *match* is $i_1, i_2, \ldots, i_\ell$ where $t_{i_1} t_{i_2} \cdots t_{i_\ell} = b_{i_1} b_{i_2} \cdots b_{i_\ell}$.

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Definition (Post Correspondence Problem)

PCP is the language of collections of dominos with a match:

$$PCP = \{ \langle P \rangle \mid \text{the set } P \text{ of dominos has a match} \}.$$

Post-Correspondence Problem Cont.

Can you find the match?

Example

$\{[\frac{b}{ca}], [\frac{a}{ab}], [\frac{ca}{a}], [\frac{abc}{c}]\}$

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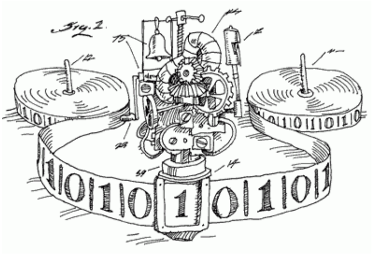
Definition

A language is *undecidable* if there is **no** programmable algorithm that

- accepts all inputs in the language, and
- rejects all inputs not in the language.

Turing Machines!



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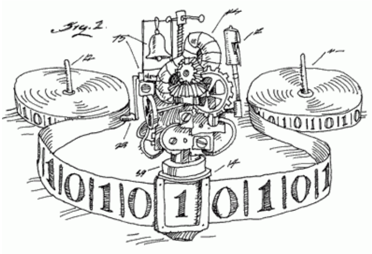

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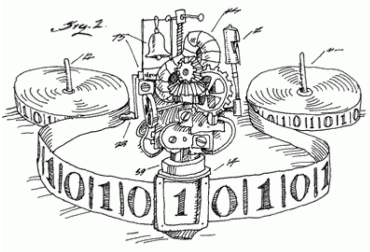

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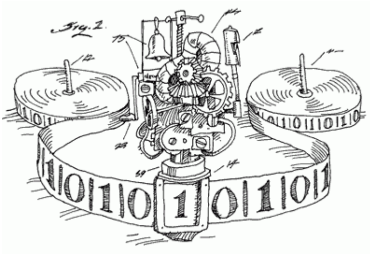

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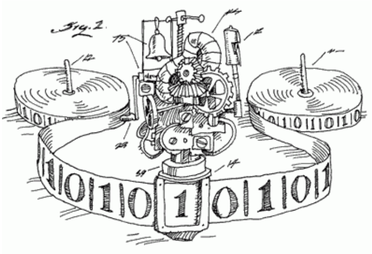

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- An *infinite tape*
- A *tape head* to navigate, read, and edit the elements according to the *transition function*.

Example of a Turing Machine

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Below is an example of a Turing machine that recognizes the language $B = \{0^{2^n} | n \in \mathbb{Z}_{\geq 0}\}$. Let's work with this input:

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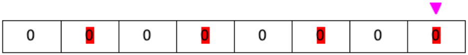

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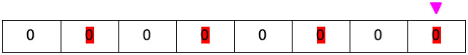

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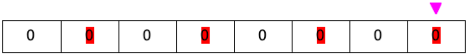

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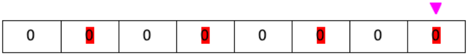

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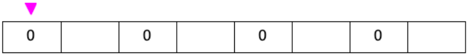

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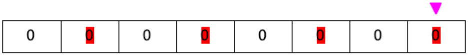

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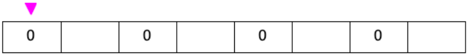

5. Return back to step 1, then repeat.

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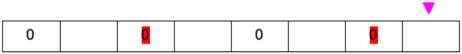

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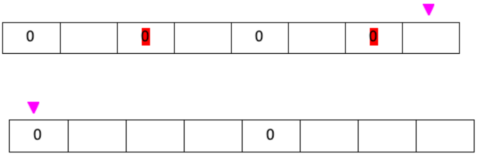

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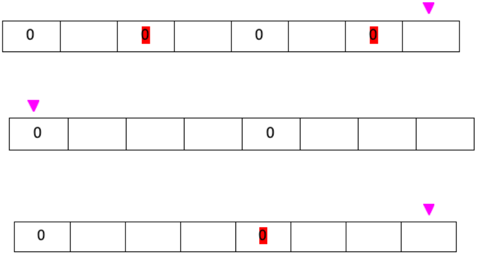

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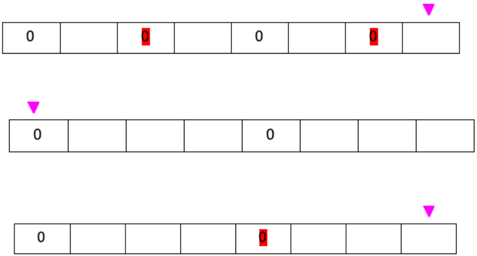

Computation History of a Turing Machine

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A *configuration* of a Turing machine is a setting of three elements:

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The history of changes in configurations is called a *computation history*.

How to write a configuration

We represent a configuration as uqv where

- uv is the current tape contents,
- q is the current state, and
- the first symbol of v is the location of the current tape head.

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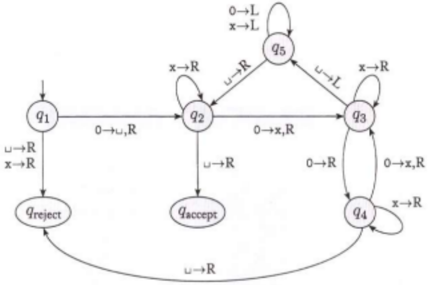
Example

Tape: $\sqcup\, 0\, \mathbf{0}\, 0\, 0$

State: q_2

Configuration: $\sqcup 0\ q_2\ 000$

Computation History Example



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Post-Correspondence Problem

Turing Machines

A_{TM} and Reducibility

Reducing A_{TM} to PCP

Conclusion

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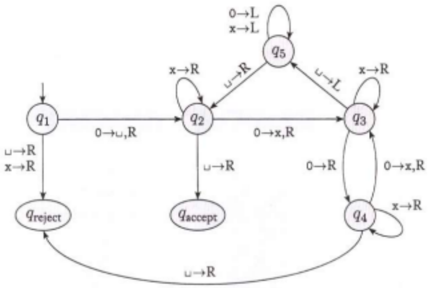

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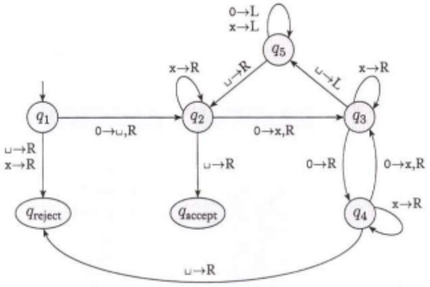

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$$q_1\,00,\ \sqcup\, q_2\, 0,\ \sqcup x q_3 \sqcup,\ \sqcup q_5 x \sqcup,\ q_5 \sqcup x \sqcup,\ \sqcup q_2 x \sqcup,\ \sqcup x q_2 \sqcup,\ \sqcup x \sqcup q_{accept}$$

A_{TM}

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The language $A_{TM} = \{\langle M, w\rangle | M \text{ is a TM that accepts } w\}$

IMPORTANT

A_{TM} is undecidable. We can prove this with diagonalization.

Reduction

Definition

A *reduction* converts one problem to another such that a solution to the second problem can be used to solve the first problem.

[Sipser, Corollary 5.23]

If A is undecidable and reducible to B, B must be undecidable as well.

Despicable Me and Reducibility

In *Despicable Me*, the protagonist Gru wants to steal the moon.

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Then, he can shrink the moon, and steal the new mini moon instead.

You might think stealing the moon would be impossible.
If so, then stealing a shrink ray must be impossible too.
(Corollary 5.23)

The Story of Urg

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2. In Urg's world, however, it is proven that stealing the moon is impossible.
3. Therefore, stealing a shrink ray must be impossible as well.

Despicable Me and Reducibility

Our Reduction

To show PCP is undecidable,

we simulate an accepting Turing machine configuration through a PCP match

and prove A_{TM} is undecidable.

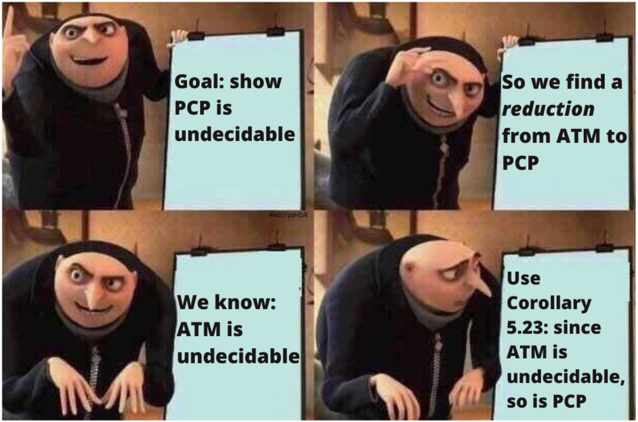
Urg's Reduction

To show he can't steal a shrink ray,

Urg hypothetically shrinks the moon

and proves stealing the moon is impossible.

New Mission: Proving PCP is Undecidable



Reducing A_{TM} to PCP

Reduction Strategy:

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- We use B to create a Turing machine N that decides A_{TM}:

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  1. Make a set of dominos that has a match iff M accepts w.
  2. Run B on this set of dominos.
  3. Accept if B accepts; reject otherwise.
- Construct the set of dominos in 7 steps.

MPCP

We must first convert PCP to Modified PCP (MPCP).

Definition

MPCP is the language of instances of *PCP* whose matches start with $[\frac{t_1}{b_1}]$:

$$MPCP = \{ \langle P' \rangle \mid P' \text{ is a collection of dominos with a match starting with the first domino} \}.$$

The Start Domino - Step 1

STEP 1:

Insert the domino $[\frac{\#}{\#q_0 w_1 w_2 \ldots w_n}]$.
This is the start of the match, $[\frac{t_1}{b_1}]$.

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In the next steps (2,3,4) we simulate the computation.

The Transition Function - Steps 2 and 3

Steps 2 and 3 involve the dominos for transition functions (how the current state and current tape head location changes).

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STEP 2: Tape head moves right.

Read a in state q:

- Turing machine replaces a with b and moves to state q'
- put domino $\left[\frac{q\mathrm{a}}{\mathrm{b}q'}\right]$ into P'.

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- Turing machine replaces $\mathtt{a}$ with $\mathtt{b}$ and moves to state q'
- put domino $[\frac{cq\mathtt{a}}{q'c\mathtt{b}}]$ into P'.

Symbols and Separations - Steps 4 and 5

Now we add the tape alphabet: these dominos are used in the portion of the configurations outside the transition functions.

STEP 4:

For every a in the tape alphabet, insert $\left[\frac{\mathtt{a}}{\mathtt{a}}\right]$ into P'.

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For every a in the tape alphabet, insert $[\frac{\mathtt{a}}{\mathtt{a}}]$ into P'.

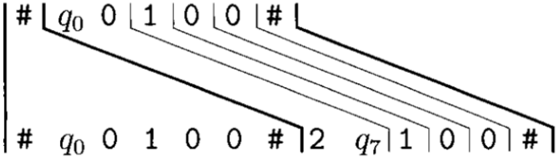
To separate each configuration, we add #s.

STEP 5:

Insert $[\frac{\#}{\#}]$ and $[\frac{\#}{\sqcup\#}]$ into P'.

The second domino allows us to simulate the infinite number of $\sqcup$s.

Example of This Reduction



Accept States - Steps 6 and 7

Add in accept states with symbols to "eat" the remaining characters left on the tape after the machine accepts.

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STEP 7:

The last domino! $[\frac{q_{accept}\#\#}{\#}]$

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Since P' (MPCP) is $\{[\frac{t_1}{b_1}], [\frac{t_2}{b_2}], [\frac{t_3}{b_3}], \ldots [\frac{t_k}{b_k}]\}$,

we can let P (PCP) be $\{[\frac{\star t_1}{\star b_1 \star}], [\frac{\star t_2}{b_2 \star}], [\frac{\star t_3}{b_3 \star}], \ldots [\frac{\star t_k}{b_k \star}], [\frac{*\Diamond}{\Diamond}]\}$, where $[\frac{*\Diamond}{\Diamond}]$ is there to add the extra $*$.

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4. PCP is undecidable because we can reduce A_{TM} to MPCP which we can then reduce back to PCP.

Summary

1. A *match* occurs when the string with all the symbols on the top is the same as the string on the bottom.
2. A_{TM} is a set of all Turing machines M and input string w in which the machine M accepts w.
3. A_{TM} is undecidable! (diagonalization)
4. PCP is undecidable because we can reduce A_{TM} to MPCP which we can then reduce back to PCP.
5. Our reduction is essentially simulating an accepting Turing machine computation through an MPCP match.

Acknowledgements

We would like to thank...

- PRIMES Circle for this wonderful opportunity
- Alexandra for being a great mentor :)
- You for coming!!!

References

- [la] Sipser, M. (2013). *Introduction to the theory of computation* (3rd ed.). Cengage Learning.
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A_{TM} is undecidable

Theorem

A_{TM} is undecidable.

Proof

First, let's assume that A_{TM} is decidable. Then, by the definition of decidable Turing machines, there must exist some decider H, where $L(H) = A_{TM}$, such that for an input string $\langle M, w \rangle$, it

1. accepts if M accepts w, and
2. rejects if M rejects w.

Proof

Now, consider a new Turing machine D, which takes Turing machines $\langle M \rangle$ as inputs. It

1. accepts when H rejects $\langle M, \langle M \rangle \rangle$, and
2. rejects when H accepts $\langle M, \langle M \rangle \rangle$

Proof

Consider a case when D takes in itself, $\langle D \rangle$, as an input. Then by construction, D will

1. reject $\langle D \rangle$ when H accepts $\langle D, \langle D \rangle \rangle$, and
2. accept $\langle D \rangle$ when H rejects $\langle D, \langle D \rangle \rangle$

However, because H accepts $\langle D, \langle D \rangle \rangle$ if and only if D accepts $\langle D \rangle$, we see that there exists a contradiction either way:

1. D rejects $\langle D \rangle$ if and only if D accepts $\langle D \rangle$, or
2. D accepts $\langle D \rangle$ if and only if D rejects $\langle D \rangle$.

Therefore, such a decider H cannot exist, so A_{TM} is undecidable.