Asymptotics for Iterating the Lusztig-Vogan Bijection for GL_n on Dominant Weights

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Abstract

In this paper, we iterate the explicit algorithm computing the Lusztig-Vogan bijection in Type A (GL_n) on dominant weights, which was proposed by Achar and simplified by Rush. Our main result focuses on describing asymptotic behavior between the number of iterations for an input and the length of the input; we also present a recursive formula to compute the slope of the asymptote. This serves as another contribution to understanding the Lusztig-Vogan bijection from a combinatorial perspective and a first step in understanding the iterative behavior of the Lusztig-Vogan bijection in Type A.

Keywords: Lusztig-Vogan bijection, Asymptote, General Linear Group, Dominant Weights.

Contents

1 Introduction

The Lusztig-Vogan bijection, conjectured independently by Lusztig in [3] and Vogan in [5], is an important tool in the modular representation theory: in characteristic 0, the Lusztig-Vogan bijection shows a connection to 2-sided cells in the affine Weyl group [3] and is crucial for proving the Humphreys conjecture on the support varieties of tilting modules for quantum groups at unity [1]. Precisely defined, the Lusztig-Vogan bijection is a correspondence between dominant weights of a reductive algebraic group G and the set of G-equivariant vector bundles on nipotent orbits: let

$$\Omega_{\mathbf{k}} := \{(x, V) | x \in \mathcal{N}_{\mathbf{k}}, V \in \text{Irr}(Z_{G_{\mathbf{k}}}(x))\} / (G_{\mathbf{k}}\text{-conjugacy}),$$

where $\mathbf{k}$ is an algebraically closed field, $G_{\mathbf{k}}$ is a connected reductive algebraic group over $\mathbf{k}$, $\mathbf{X}^+$ is the set of dominant weights for $G_{\mathbf{k}}$, and $\mathcal{N}_{\mathbf{k}}$ is the nilpotent cone of $\mathbf{X}^+$. The Lusztig-Vogan bijection for $G_{\mathbf{k}}$ is a bijection between $\mathbf{X}^+$ and $\Omega_{\mathbf{k}}$.

In [2], Bezukavinkov established this correspondence in full-generality and proved its bijectivity, though he used highly non-elementary and non-explicit means. Recent efforts have been focusing on understanding the Lusztig-Vogan bijection in a more explicit way and in the combinatorial context. Achar, Hardesty, and Riche proved in [1] that the Lusztig-Vogan bijection is independent of the characteristic, which generalized the result of characteristic 0 to the general case and proposed an algorithm for computing the Lusztig-Vogan bijection for GL_n (Type A). In [4], Rush greatly simplified Achar's algorithm and presents a combinatorial description of the Lusztig-Vogan bijection for GL_n. Rush's simplification of Achar's algorithm can be viewed as a map that sends an input of a weakly decreasing sequence of integers to a tuple of weakly decreasing sequences of integers, as illustrated below in Figure 1.

Little is known, however, about the properties of this combinatorial perspective of the Lusztig-Vogan bijection in Type A. In this paper, we pursue a new direction: we consider the iteration of Rush's algorithm of computing Lusztig-Vogan bijection for GL_n. Suppose we start with a sequence of weakly decreasing integers σ. On the first iteration, we obtain a tuple of sequences of weakly decreasing integers $(\mu_1, \mu_2, \cdots, \mu_l)$. On the next iteration, we perform the operation on each of the sequences with length at least 2. We repeat this algorithm until each sequence has zero or one number. Consider, for example, we iterate on the sequence in Figure 1b:

$$(10, 10, 9, 7, 4, 2) \to ((6, 5, 5), (), (26)) \to (((), (), (16)), (), (26)).$$

Note that in the output of the first iteration, the second sequence has length 0 and the third sequence has length 1, so we stop iterating on those two sequences. Performing the algorithm on $(6, 5, 5)$ produces $((), (), (16))$. The procedure terminates afterwards because each sequence of the second output has length 0 or 1.

We iterate this map until it terminates and study the behavior of this algorithm for large inputs. We describe the asymptotic behavior between the number of iterations before the procedure terminates and the length of the input (Theorem 22) and obtain a recursive formula to compute the slope of the asymptote (Theorem 23). This paper serves as another contribution to the recent effort of understanding the Lusztig-Vogan bijection from a combinatorial perspective and as a first step to understand the iterative behavior of the Lusztig-Vogan bijection in Type A.

(a) Lusztig-Vogan bijection

(b) Example

Figure 1

In Section 2, we rigorously define the Lusztig-Vogan Bijection, and in Section 3, we present some basic properties of the Type A Lusztig-Vogan bijection itself and also in the context of iteration. In Section 4, we first attempt to provide a general formula of time for all inputs. The complexity of Theorem 17 motivates us to define the average time (Definition 18) and prove the asymptotic behavior of the average time respect to the length of the input. Finally, in Section 5, we propose two possible directions to continue the work in this paper.

2 Definition of the Lusztig-Vogan Bijection Type A and Its Iteration

In [4], Rush simplified the Type A Lusztig-Vogan bijection in terms of three maps: ϕ, E, and κ, and we define them in this section.

Definition 1. Let σ be a sequence of integers $a_1, a_2, \cdots, a_n$. We say σ is *weakly decreasing* if and only if $a_i \leq a_j$ for all $i \geq j$.

Definition 2. Let $X = (X_1, \cdots, X_l)$ be a tuple of weakly decreasing sequences of positive integers, and let $n = \sum_{i=1}^{l} |X_i|$. We say X is a *weighted diagram* of size n. The *shape-class* of this weighted diagram is the partition of n determined by the row lengths of the X_i. We call each X_i a *row*, and we denote $X_{i,j}$ as the jth number in row i. The kth *column* denotes the set of all entries in the from of $X_{i,k}$.

Definition 3. For any positive integer r, consider the r-th column of a weighted diagram X. We define a total order $\prec$ on all entries in the r-th column: we say $X_{j,r} \prec X_{i,r}$ if and only if

1. $X_{j,r} < X_{i,r}$, or
2. $X_{j,r} = X_{i,r}$ and $j > i$.

Definition 4. Let $D_{n,\alpha}$ be the set of all weighted diagrams with size n and shape-class α. Let $E : D_{n,\alpha} \to D_{n,\alpha}$ be defined as follows: for any $X \in D_{n,\alpha}$, perform the following operation on each column r to obtain X'. For each i, if $X_{i,r}$ is preceded in the total order $\prec$ by k elements in that column (i.e., there are exactly k values for which $X_{j,k} \prec X_{i,k}$ and $j \neq i$), we replace $X_{i,r}$ with $X_{i,r} + 2k - (l - 1)$, where l is the number of elements in the rth column. Then we have $E(X) = X'$.

Note that map E does not change the size or the shape-class of the weighted diagram.

Definition 5. Let map κ be defined as follow. Let X be any weighted diagram. For each positive integer i, let μ_i be the non-decreasing sequence of all row-sums of the rows in X with length i. Then we define $\kappa(X) = (\mu_1, \mu_2, \cdots, \mu_l)$, where l is the maximum row lengths in X.

Before explicitly defining map ϕ we need the definition of a "maximal clump" of a sequence of weakly decreasing integers.

Definition 6. Let $\sigma = (a_1, a_2, \cdots, a_n)$ be a sequence of weakly decreasing integers. Define a *clump* of the sequence as a partition of the sequence such that the underlying set of each block is a set of consecutive integers. Define the *maximal clump* of the sequence as the clump with the smallest number of blocks.

To illustrate the above definition, let us consider the following sequence:

$$\sigma = (10, 10, 9, 8, 4, 3, 3, 0, -4, -4, -8).$$

The *maximal clump* of σ is $(10, 10, 9, 8), (4, 3, 3), (0), (-4, -4), (-8)$. With this, we are ready to define the map ϕ.

Definition 7. The map ϕ sends a sequence of weakly decreasing integers σ to a weighted diagram X. We construct X column by column. Let us construct a sequence of weakly decreasing integers $\sigma_1, \sigma_2, \cdots$, with $\sigma_1 = \sigma$ and for any $k \geq 2$, σ_k is obtained by removing elements used to construct the $k-1$th column of X from σ_{k-1}. To build the rth column of X, we assume σ_r is given. Let $\sigma_r = A_1 \cup A_2 \cup \cdots \cup A_l$ be the maximal clump of σ. Let B_i be the underlying set of A_i (with repetition removed) and number the elements of B_i in decreasing order. Define a set of integers $\mathbb{Z}_r$ as follow:

1. For each B_i of odd size, include in $\mathbb{Z}_r$ the odd-numbered entries of B_i.
2. For each B_i of even size,
   (a) If r is odd, include in $\mathbb{Z}_r$ the odd-numbered entries of B_i;
   (b) If r is even, include in $\mathbb{Z}_r$ the even-numbered entries of B_i.

Each column of X consists of all elements of $\mathbb{Z}_r$ arranged in decreasing order. If $r = 1$, the numbers all occupy the first $|\mathbb{Z}_r|$ rows. If $r \geq 2$, place each number in a way that it is adjacent to the entry in the $(r-1)$th column containing either x or $x + (-1)^r$. Repeating this process we can build X with all elements of σ.

Note that the last step (placing elements of $\mathbb{Z}_r$ into the weighted diagram) can be done uniquely because we are choosing every other entry in each block — it is impossible for two equal or consecutive numbers present in the same column. With those three maps, we are ready to define the Lusztig-Vogan bijection Type A.

Definition 8. The map $LV = \phi \circ E^{-1} \circ \kappa$ is the Lusztig-Vogan bijection in Type A. It sends a sequence of weakly decreasing integers to a tuple of sequences of weakly decreasing integers.

We show in Corollary 11 that this map is well-defined. Figure 1 illustrates the definition with an example. Note that in Figure 1b, the second sequence of the output is an empty sequence — this is because the weighted diagram does not have any sequence of length 2.

Finally, we define the time of this algorithm in the context of iteration.

Definition 9. Let t be a map from all sequences of weakly decreasing integers to non-negative integers, such that $t(\sigma)$ denotes the number of iterations needed to apply the algorithm until each sequence has length 0 or 1. We call $t(\sigma)$ the *time* of σ.

3 Preliminaries

In this section, we present several basic properties of Rush's map itself and also in the context of its iteration. Theorem 10 and Corollary 11 show Rush's combinatorial map is well-defined.

Theorem 10. *Let X be any weighted diagram with columns $c_1, c_2, \cdots, c_m$. Then $X \in \mathrm{im}(E)$ if and only if for all $1 \leq i \leq m$, when sorting all elements of c_i in weakly decreasing order,*

1. *all consecutive terms differ by at least 2, and*
2. *shall two terms differ by 2, the larger one must be on the top of the smaller one in the original arrangement.*

In addition, E is injective.

Proof. Because all columns of X are independent to each other, let us consider an individual column. Let us first sort all elements of that column by order $\prec$: $a_1 \prec a_2 \prec a_3 \prec \cdots$. Note that by Definition 4, the difference between consecutive terms increases by 2. Thus, for all columns of $E(X)$, when sorted in non-decreasing order, the difference between consecutive terms are at most 2, so the first item is a necessary condition. To show the second item is necessary, notice that if two terms differ by only two on the image, their corresponding entries must be equal on the original weighted diagram. By Definition 3, the entry on the bottom always precedes the entry on the top. Thus, on the image, the entry on the top must be larger than the entry on the bottom.

To show that those two requirements are sufficient, let us consider any column satisfying those two requirements. If we sort all entries by order $\prec$: $b_1 \prec b_2 \prec \cdots \prec b_n$. We can get the original weighted diagram by replacing b_k with $b_k + 2k - n - 1$.

To show the injectivity of E, we note that map E does not change the order of the elements in the column, and increase each element by an amount related to the number of elements preceding that element, so each weighted diagram in $\mathrm{im}(E)$ can only admit one weighted diagram that produces it. □

Corollary 11. *The map LV is well-defined. In other words, $\mathrm{im}(\phi) \subset \mathrm{im}(E)$.*

Proof. For all weighted diagrams that is in the image of ϕ, by Definition 7, all of their columns are arranged in decreasing order. Further, because we are selecting every *other* term in each block of the maximal clump, the consecutive terms must differ by at least 2. By Theorem 10, the claim is proved. □

Corollary 12 is a basic but important property of the bijection. It governs the behavior of the bijection when the entries are far apart and greatly simplifies our calculation.

Corollary 12. *Let $\sigma = (a_1, a_2, \cdots, a_n)$ be a sequence of weakly decreasing integers, and assume that $a_i - a_{i+1} \geq 2$ for all $1 \leq i \leq n-1$. Then $LV(\sigma) = (a_1 + k_1, a_2 + k_2, \cdots, a_n + k_n)$, where $k_i = 2(i - \frac{n+1}{2})$.*

Proof. Because all consecutive terms of σ differ by at least 2, by Definition 7 $\phi(\sigma)$ has only one column, with the r-th row has only one element a_r. By Definition 3, a_r is preceded by exactly $n-r$ elements. Thus, map E^{-1} sends each a_r to $a_r - 2(n-r) + n = a_r - n + 2r$. Because all sequences are of length 1, $LV(\sigma)$ has only one sequence, $(a_1 - n + 1, a_2 - n + 3, a_3 - n + 5, \cdots, a_n + n - 1)$, as desired. □

Theorem 13. *The time for any sequence of weakly decreasing integers is finite.*

Proof. We use induction on the length of the sequence. The base case is trivial. For inductive step we assume $t(\sigma)$ is finite for all $|\sigma| \leq n$. Now we consider any sequence of weakly decreasing integers σ_1 of length $n+1$: $(a_1, a_2, \cdots, a_{n+1})$. If there exists two consecutive terms that differ by 0 or 1, during the first iteration, the maximal clump of the sequence must have at least one block with length at least 2. Thus, the output must be consisted of multiple sequences, and the length of each sequence is at most n. If the difference between all consecutive terms are at least 2, then by Corollary 12 every iteration reduces each difference by 2, until there are two consecutive terms whose difference is 0 or 1, and we go back to the first case. By our inductive hypothesis, this procedure always terminates in a finite number of turns. □

4 Average Time and Asymptotic Behavior

As pointed out in Definition 9, we are interested in describing the time function, especially for large inputs. We start by considering the following instances of inputs of length 2, 3, and 4, as an attempt to compose a general formula for the time function.

4.1 Inputs of Length 2, 3, and 4

Let us start with inputs of length 2, (x, y) $(x \geq y)$. According to Corollary 12, the larger entry decreases by one and the smaller one increases by one after each iteration, until their difference is at most 1. At that point, the weighted diagram produced from the ϕ map has only one row with those two elements. The final output, therefore, is $((), (x+y))$. For example, if we start with $(10, 3)$:

$$(10,3) \to (9,4) \to (8,5) \to (7,6) \to ((), (13)),$$

This observation leads to Theorem 14.

Theorem 14. *For all weakly decreasing sequence of integers of length 2 (x, y), we have $t(x,y) = \left\lfloor \frac{d}{2} \right\rfloor + 1$, where $d = x - y$. In addition, the final output is always $((), (x+y))$.*

Proof. Each iteration decreases the difference by 2, until the difference is at most 1, at which point the procedure terminates after one more iteration with output $((), (x+y))$. □

Theorem 14 allows us to describe $\operatorname{im}(t)$.

Corollary 15. *The image of t is the set of all non-negative integers.*

Proof. By definition 9, time cannot be negative. Any input of length 1 has a time of 0. For any $n \in \mathbb{Z}^+$, we have by Theorem 14 that $t((2n-1, 0)) = n$. Therefore that $\operatorname{im}(t) = \mathbb{Z}_{\geq 0}$, as desired. □

We find that the time of length 3 inputs has a formula in the same form as the input of length 2 (Theorem 14), shown below in Theorem 16.

Theorem 16. *For all weakly decreasing sequence of integers of length 3 (x, y, z), we have* $t(x, y, z) = \left\lfloor \frac{\min(d_1,d_2)}{2} \right\rfloor + 1$*, where* $d_1 = x - y$ *and* $d_2 = y - z$*.*

Proof. Let us consider the first iteration. If $d_1, d_2 \geq 2$, we have $LV((x, y, z)) = ((x - 2, y, z + 2))$; if $0 \leq d_1 \leq 1, d_2 \geq 2$, we have $LV((x, y, z)) = ((z + 1), (x + y - 1))$; if $0 \leq d_2 \leq 1, d_1 \geq 2$, we have $LV((x, y, z)) = ((x - 1), (y + z + 1))$; if $(d_1, d_2) = (0, 0)$ or $(1, 0)$ or $(0, 1)$, we have $LV((x, y, z)) = ((), (), (x + y + z))$; if $d_1 = d_2 = 1$, we have $LV((x, y, z)) = ((y), (x + z))$. All of these cases can be checked easily by considering the weighted diagram generated by map ϕ and E^{-1}.

We find the pattern is similar to that of length 2: Corollary 12 governs that after each iteration, the difference between the consecutive terms decrease by 2 if each difference between consecutive terms is at least 2; otherwise, the procedure terminates with the next iteration. We have, therefore, that $t(x, y, z) = \left\lfloor \frac{\min(d_1,d_2)}{2} \right\rfloor + 1$. □

Unfortunately, we find the time for length 4 inputs is much more complicated than the one for length 2 and 3.

Theorem 17. *Let* $\sigma = (x, y, z, w)$ *be a weakly decreasing sequence. Assume that* $d_1 = x - y, d_2 = y - z, d_3 = z - w, f_1 = \left\lfloor \frac{d_1}{2} \right\rfloor, f_2 = \left\lfloor \frac{d_2}{2} \right\rfloor, f_3 = \left\lfloor \frac{d_3}{2} \right\rfloor$. *Then we have*

$$t(\sigma) = \begin{cases} \left\lfloor \frac{d_1}{2} \right\rfloor + 1 \text{ if } f_3 < f_1, f_2; \\ -\left\lfloor \frac{d_2}{2} \right\rfloor + \left\lfloor \frac{d_1+d_3+\mathrm{mod}(d_2,2)}{2} \right\rfloor \text{ if } f_2 < f_1, f_3; \\ \left\lfloor \frac{d_3}{2} \right\rfloor + 1 \text{ if } f_1 < f_2, f_3; \\ \left\lfloor \frac{d_2}{2} \right\rfloor + 1 \text{ if } f_1 > f_2 = f_3 \text{ and } 2 \mid d_2d_3; \\ \left\lfloor \frac{d_1}{2} \right\rfloor + 1 \text{ if } f_1 > f_2 = f_3 \text{ and } 2 \nmid d_2d_3; \\ d_2 - \left\lfloor \frac{d_1}{2} \right\rfloor \text{ if } f_2 > f_1 = f_3; \\ \left\lfloor \frac{d_1}{2} \right\rfloor + 1 \text{ if } f_3 > f_2 = f_1 \text{ and } 2 \mid d_1d_2; \\ \left\lfloor \frac{d_3}{2} \right\rfloor + 1 \text{ if } f_3 > f_2 = f_1 \text{ and } 2 \nmid d_1d_2; \\ \left\lfloor \frac{\min(d_1,d_2,d_3)+1}{2} \right\rfloor + 1 \text{ if } f_1 = f_2 = f_3. \end{cases}$$

Proof. If $d_1, d_2, d_3 \geq 2$, by Corollary 12, each iteration decreases each difference by 2. Thus for now we assume that at least one of the three differences is 1 or 0.

Case 1: if $(d_1, d_2, d_3) = (\geq 2, \geq 2, 0/1)$ (note that this notation means "for any (d_1, d_2, d_3) with $d_1 \geq 2, d_2 \geq 2$, and $d_3 = 0$ or 1"), $LV(\sigma) = ((x - 2, y), (z + w + 2))$, and further $\left\lfloor \frac{x-y-2}{2} \right\rfloor + 1$ iterations are needed. Thus, in total, we have $t(\sigma) = \left\lfloor \frac{d_1}{2} \right\rfloor + 1$.

Case 2: if $(d_1, d_2, d_3) = (0/1, \geq 2, \geq 2)$, $LV(\sigma) = ((z, w+2), (x+y-2))$, and further $\left\lfloor \frac{z-w-2}{2} \right\rfloor + 1$ iterations are needed. Thus, in total, we have $t(\sigma) = \left\lfloor \frac{d_3}{2} \right\rfloor + 1$.

Case 3: if $(d_1, d_2, d_3) = (\geq 2, 0/1, \geq 2)$, $LV(\sigma) = ((x - 2, w + 2), (y + z))$, and further $\left\lfloor \frac{x-w-4}{2} \right\rfloor + 1$ iterations are needed. Thus, in total, we have $t(\sigma) = -\left\lfloor \frac{d_2}{2} \right\rfloor + \left\lfloor \frac{d_1+d_3+\mathrm{mod}(d_2,2)}{2} \right\rfloor$. Note that $\mathrm{mod}(a, b)$ gives the remainder when a is divided by b.

Case 4: if $(d_1, d_2, d_3) = (\geq 2, 0, 0)$ or $(\geq 2, 1, 0)$ or $(\geq 2, 0, 1)$, $LV(\sigma) = ((x-1), (), (y+z+w+1))$, and the procedure immediately terminates. Thus, in total, we have $t(\sigma) = \left\lfloor \frac{d_2}{2} \right\rfloor + 1$.

Case 5: if $(d_1, d_2, d_3) = (\geq 2, 1, 1)$, $LV(\sigma) = ((x-2, w+2), (y+z))$, and further $\left\lfloor \frac{x-w-4}{2} \right\rfloor + 1$ iterations are needed. Thus, in total, we have $t(\sigma) = \left\lfloor \frac{d_1}{2} \right\rfloor + 1$.

Case 6: if $(d_1, d_2, d_3) = (0/1, \geq 2, 0/1)$, $LV(\sigma) = ((), (x+y-2, z+w+2))$, and further $\left\lfloor \frac{x+y-z-w-4}{2} \right\rfloor + 1$ iterations are needed. Thus, in total, we have $t(\sigma) = d_2 - \left\lfloor \frac{d_1}{2} \right\rfloor$.

Case 7: if $(d_1, d_2, d_3) = (0, 0, \geq 2)$ or $(0, 1, \geq 2)$ or $(1, 0, \geq 2)$, $LV(\sigma) = ((w+1), (), (x+y+z-1))$, and the procedure immediately terminates. Thus, in total, we have $t(\sigma) = \left\lfloor \frac{d_1}{2} \right\rfloor + 1$.

Case 8: if $(d_1, d_2, d_3) = (1, 1, \geq 2)$, $LV(\sigma) = ((z, w+2), (x+y-2))$, and further $\left\lfloor \frac{z-w-2}{2} \right\rfloor + 1$ iterations are needed. Thus, in total, we have $t(\sigma) = \left\lfloor \frac{d_3}{2} \right\rfloor + 1$.

Case 9: if $(d_1, d_2, d_3) = (0,0,0), (0,1,0), (0,0,1), (1,0,0)$, $LV(\sigma) = ((), (), (), (x+y+z+w))$, and the procedure immediately terminates; if $(d_1, d_2, d_3) = (1,1,0), (1,0,1), (0,1,1)$, $LV(\sigma) = ((w+1), (), (x+y+z-1))$, and the procedure immediately terminates; if $(d_1, d_2, d_3) = (1,1,1)$, $LV(\sigma) = ((), (x+y-2, z+w+2))$, and the procedure terminates in one iteration. Thus, in total, we have $t(\sigma) = \left\lfloor \frac{\min(d_1,d_2,d_3)+1}{2} \right\rfloor + 1$

Considering those nine cases proves the original claim. □

Because we are interested in the behavior of the algorithm for larger inputs, and given the complicated form of the time of inputs of length 4, we introduce the *average time* and instead investigate its asymptotic behavior.

4.2 Definition of Average Time

Definition 18. Let $n \geq 0, k > 0$ be integers. Let $S_{n,k}$ be the set of all length k weakly decreasing sequences whose first (largest) term is n and whose last (smallest) term is 0. Define $\text{avg}_k : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ by

$$\text{avg}_k(x) = \frac{\sum_{y \in S_{x,k}} t(y)}{|S_{x,k}|}, \text{ for all nonnegative integer } x.$$

In other words, $avg_k(x)$ gives the average value of $t(y)$ across all $y \in S_{x,k}$. For example, $S_{4,3}$ has five elements: $(4,4,0), (4,3,0), (4,2,0), (4,1,0), (4,0,0)$, with time of $1,1,2,1,1$, respectively. Thus, $\text{avg}_3(4) = \frac{6}{5}$.

4.3 Average Time for Length 2, 3, and 4

Here, we briefly revisit the conclusions in Section 4.1, and describe the asymptotic behavior of avg_k for $k = 2, 3$, and 4. The cases for $k = 2$ and 3 immediately follow from Theorem 14 and Theorem 16.

Corollary 19. *The average time for inpute of length* 2 *is given by* $\text{avg}_2(x) = \left\lfloor \frac{x}{2} \right\rfloor + 1$.

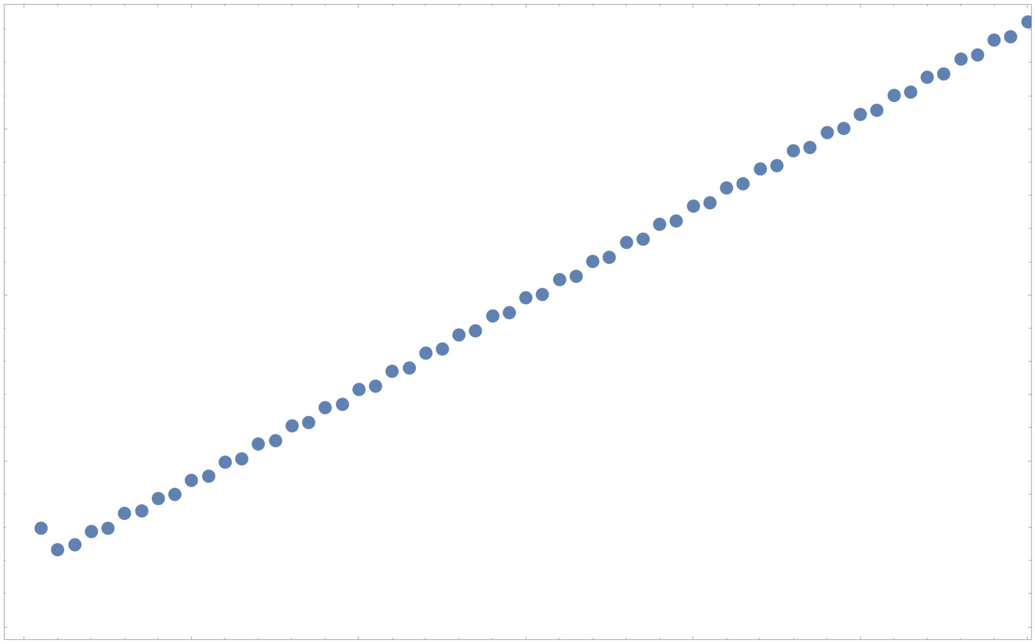


Figure 2: Average Time for Length 4
$\text{avg}_4(x), 1 \le x \le 60$

Proof. By Definition 18, we have $S_{2,x} = \{(x, 0)\}$; by Theorem 14, we have $t(x, 0) = \left\lfloor \frac{x}{2} \right\rfloor + 1$. The corollary then immediately follows. □

Corollary 20. *The asymptotic behavior of the average time for inputs of length* 3 *is given by* $\lim_{x \to \infty} \text{avg}_3(x) = \frac{x+6}{8}$.

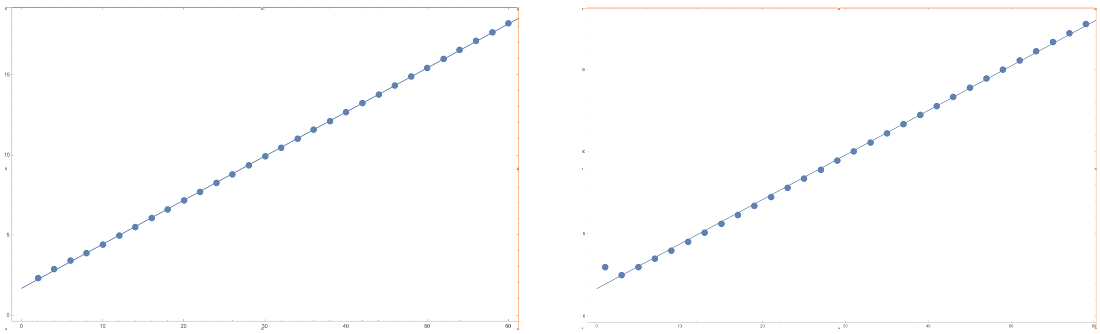
Proof. Given a large integer n and consider all weakly decreasing sequence of integers in the form of $(n, m, 0)$, we let $d_1 = n - m$ and $d_2 = m$. We find that $E(\min(d_1, d_2)) = \frac{1}{4}n$. Thus, we have by Theorem 16 that

$$\lim_{x \to \infty} \text{avg}_3(x) = E\left(\left\lfloor \frac{\min(d_1, d_2)}{2} \right\rfloor + 1\right) = \frac{1}{2}E(\min(d_1, d_2)) - \frac{1}{4} + 1 = \frac{n+6}{8},$$

as desired. □

From the above two corollaries, we observe that $\text{avg}_2(x)$ is linear for both odd and even x, and the slope is always $\frac{1}{2}$; $\text{avg}_3(x)$ shows a linear asymptote for all x. In fact, $\text{avg}_4(x)$ also has a linear asymptote for even and odd x, respectively, as shown below in Figure 2 and 3. Observe that the constant term (1.468 and 1.626) differs in the lines of best fit of different parity, the slopes are the same up to three significant figures (0.277). With this observation, we prove the following theorem.

Theorem 21. *The graphs of* $\text{avg}_4(x), 2 \mid x$ *and* $\text{avg}_4, 2 \nmid x$ *approach two lines, both of which have slope* $\frac{5}{18}$.



(a) $\text{avg}_4(x), 2 \mid x$ and Line of Best Fit $y = 0.277x + 1.626, R^2 = 0.9999$

(b) $\text{avg}_4(x), 2 \nmid x$ and Line of Best Fit $y = 0.277x + 1.468, R^2 = 0.9980$

Figure 3: Average Time for Length 4 by Parity

Proof. We recall the nine cases of Theorem 17. For both odd and even n, the expected value of each case is linear with respect to n. Thus, $\text{avg}_4(x)$ approaches two lines based on the parity of n. We claim that none of the last six cases contribute to the slope of the asymptote.

Let us take, for example, the sixth case. The number of weakly decreasing sequences satisfying $f_2 > f_1 = f_3$ (or $\left\lfloor \frac{d_2}{2} \right\rfloor > \left\lfloor \frac{d_1}{2} \right\rfloor = \left\lfloor \frac{d_3}{2} \right\rfloor$) is $O(n)$. The total number of elements of $S_{n,4}$ is $\binom{n+1}{2}$, which is $O(n^2)$. Furthermore, the expected value of time for sequences in this case is $O(n)$ (because it approaches a line). Therefore, this case contributes $\frac{O(n)O(n)}{O(n^2)} = O(1)$ to the overall asymptote, which is only the constant term.

The same logic applies to all of the last six cases. Thus, we only need to consider the first three cases: namely, $f_3 < f_1, f_2$; $f_2 < f_1, f_3$; and $f_1 < f_2, f_3$, all of which are equally likely with a probability of $\frac{1}{3}$. Note that the first and third cases are symmetric. We assume that n is sufficiently large.

Case 1: Given that $f_3 < f_1, f_2$, we have

$$E(d_3) = \frac{\int_0^{\frac{n}{3}} x(n-x)dx}{\int_0^{\frac{n}{3}} (n-x)dx} + k_1 = \frac{1}{9}n + k_1,$$

where k_1 is a constant. Thus, $E(\left\lfloor \frac{d_1}{2} \right\rfloor) = \frac{2}{9}n - \frac{k_1+1}{4}$, and the slope is $\frac{2}{9}$.

Case 2: Given that $f_2 < f_1, f_3$, we have $E(d_2) = \frac{1}{9}n + k_1$. Thus,

$$E\left(-\left\lfloor \frac{d_2}{2} \right\rfloor + \left\lfloor \frac{d_1 + d_3 + \text{mod}(d_2, 2)}{2} \right\rfloor\right) = \frac{7}{18}n + k_2,$$

in which k_2 is a constant that takes two different values based on the parity of n, and the slope is $\frac{7}{18}$.

In conclusion, the slope of the asymptote is $\frac{1}{3}(\frac{2}{9} + \frac{7}{18} + \frac{2}{9}) = \frac{5}{18}$, as desired. □

Note that the constant of the two asymptotes differ because of the second case ($f_2 < f_1, f_3$). In particular, they differ by $\frac{1}{6}$.

4.4 Asymptotic Behavior of Average Time of Length n

Note that in the proof for Theorem 21, we determined that all cases besides the first three major ones only contribute to the constant term. Further, while computing the first three cases, we eventually used our conclusions for inputs of length 2. Using this idea, we may prove that the asymptotic behavior of avg_n is linear for any positive integer n. Further, we may prove a recursive formula to compute the slope of the asymptote of the avg_n.

Theorem 22. *For any positive integer $k \geq 4$, the graphs of $\text{avg}_k(n)$, $2 \mid n$ and $2 \nmid n$ approach two lines, both of which have the same slope.*

Proof. We use induction on the length. The base cases are proven in Corollary 19, Corollary 20, and Theorem 21. By Corollary 12, each iteration reduces each difference by 2 until at least one difference is 0 or 1. Consider inputs of length $n : \{a_1, a_2, \cdots, a_n\}$. For $1 \leq k \leq n-1$, assume $d_k = a_{k+1} - a_k$. Further assume $f_k = \left\lfloor \frac{d_k}{2} \right\rfloor$. Let $S = \{d_i \mid 1 \leq i \leq n-1, \text{ for any } 1 \leq j \leq n-1, f_i \leq f_j\}$; in other words, S is the set of differences that are first reduced to one or zero in the same iteration. Let us consider S_1, a possible value for S. Let e be the number of turns before any differences reduce to zero or one, and g be the remaining number of turns before the procedure terminates. Because the expected value of any element of S is $O(n)$, the expected value of e is also $O(n)$. Further, after a difference is reduced to zero, the sequence is reduced to a tuple of sequences, all of which have length at most $n-2$. The expected maximum time of all sequences in the tuple, by the inductive hypothesis, is also $O(n)$. Thus, we have

$$E(S_1) = E(e) + E(g) = O(n) + O(n) = O(n).$$

Because the average time of all cases are $O(n)$, $\text{avg}_n = \sum p_i E(S_i)$ is also $O(n)$, where p_i is the probability that $S = S_i$ (we know $\sum p_i = 1$), as desired. □

Theorem 23. *For positive integer n, let c_n be the slope of the asymptote of $\text{avg}_n(x), 2 \mid x$. We claim that $c_2 = \frac{1}{2}, c_3 = \frac{1}{8}$. Let $c_{2,1} = c_2$ and $c_{3,1} = c_{3,2} = c_3$. For $n \geq 4$, we define a sequence $\{c_{n,1}, c_{n,2}, \cdots, c_{n,n-2}, c_{n,n-1}\}$ recursively:*

$$c_{n,1} = c_{n,n-1} = \frac{n-3}{n-1} c_{n-2} + \frac{1}{2(n-1)^2};$$

$$c_{n,k} = \frac{n-2}{n-1} \left(\frac{n-2}{n-3} c_{n-2} - \frac{1}{n-3} c_{n-2,k-1} \right) + \frac{1}{2(n-1)^2} \text{ for any } 2 \leq k \leq n-2.$$

We claim that $c_n = \frac{1}{n-1} \sum_{i=1}^{n-1} c_{n,i}$.

Proof. Consider inputs of length n. Let $\{a_n\}, \{d_n\}, \{g_n\}$, and S carry the same notation as in Theorem 22. We claim, similar to our claim in Theorem 21, that all sets S except those with exactly one element contribute only to the constant term of the asymptote. Let us consider the situation when $|S| \geq 2$. The probability that S takes this particular value is $O(\frac{1}{n^{|S|}})$, and we have

$$E(t) * p(S) = O\left(n \cdot \frac{1}{n^{|S|-1}}\right) \leq O(1),$$

so it only contributes the constant term. Thus, to compute the slope, it suffices to only consider S with only one element. We claim that for all n and $1 \leq k \leq n-1$, $c_{n,k}$ is the slope of the asymptote of the expected value of the time when $S = \{d_k\}$, and we prove this claim by induction. The base cases when $n = 2$ and $n = 3$ are trivial. For now we assume $n \geq 4$. Assume $\sum d_i = m$. We do casework regarding the value of k.

Case 1: $k = 1$ or $n - 1$. By symmetry we only consider $k = 1$. This implies that d_1 first reduces to zero. Let us consider the weighted diagram in the next iteration: a_1 and a_2 are both in the first row, and the remaining entries each occupies one row. Thus, the output only removes the first two entries, not influencing the remaining entries. We find that the expected value of d_1 given that it first diminishes is

$$E(d_1) = \frac{\int_0^{\frac{m}{n-1}} x \frac{1}{n-3}(m-x)^{n-3} dx}{\int_0^{\frac{m}{n-1}} (m-x)^{n-3} dx} + k_1 = \frac{1}{(n-1)^2} m + k_1,$$

where k_1 is a constant. Thus, it is expected to take

$$E\left(\left\lfloor \frac{d_1}{2} \right\rfloor \right) = \frac{1}{2(n-1)^2} m + \frac{k_1}{2} - \frac{1}{4}$$

number of turns for d_1 to diminish. Further, the remaining sequence is one with $n - 2$ entries, and the expected value of the largest entry to that of the smallest one is

$$E = m - (n-1)E(d_1) = \frac{n}{n-1}.$$

By inductive hypothesis, the expected number of turns after d_1 diminishes for the procedure to terminate is $\frac{n}{n-1} c_{n-2}$. In total, the slope of the expected time when $S = \{d_1\}$ is $\frac{n-3}{n-1} c_{n-2} + \frac{1}{2(n-1)^2} = c_{n,1} = c_{n,n-1}$.

Case 2: $2 \leq k \leq n - 2$. Similarly, we find that

$$E(d_k) = \frac{1}{(n-1)^2} m + k_1.$$

And it is expected to take $\frac{1}{2(n-1)^2} m + \frac{k_1}{2} - \frac{1}{4}$ for d_k to diminish. Now, at the iteration in which d_k diminishes, the first $k - 1$ lines of $\phi(\sigma)$ consist of the first $k - 1$ entries, the next line consist of the next two entries, and the remaining entries each occupy a line. Thus, in the output, d_{k-1} and d_{k+1} sum into one difference, with all other differences not influenced by the removal of d_k. The probability that this combined difference is the smallest difference is

$$\frac{\int_0^{\frac{m_0}{n-3}} \frac{1}{n-5} x (m_0 - (n-3)x)^{n-5} dx}{\int_0^{m_0} \frac{1}{n-3} x (m_0 - x)^{n-5} dx} = \frac{1}{(n-3)^2},$$

where m_0 is the difference between the largest and smallest entry after d_k diminishes. Therefore, the expected number of iterations after d_k diminished before the procedure terminates is given by

$$\frac{n-2}{n-1} m \left(\frac{n-2}{(n-3)^2} \sum_{i=1}^{n-3} c_{n-2,i} - \frac{1}{(n-3)} c_{n-2,k-1} \right)$$

n	c_n
2	1/2=0.500
3	1/8=0.125
4	5/18=0.278
5	7/64=0.109
6	11/50=0.220
7	19/192=0.099
8	93/490=0.190
9	187/2048=0.091
10	193/1134=0.170

Table 1: First 9 Slopes of Asmptotes of avg_n

$$= \frac{n-2}{n-1}\left(\frac{n-2}{n-3}c_{n-2} - \frac{1}{n-3}c_{n-2,k-1}\right)m.$$

In total, the slope is given by

$$\frac{n-2}{n-1}\left(\frac{n-2}{n-3}c_{n-2} - \frac{1}{n-3}c_{n-2,k-1}\right)m + \frac{1}{2(n-1)^2} = c_{n,k}.$$

We have shown that the slope of the expected time when $S = \{d_k\}$ is equal to $c_{n,k}$. Because each case is equally likely to happen with a probability of $\frac{1}{n-1}$, we have that $c_n = \frac{1}{n-1}\sum_{i=1}^{n-1} c_{n,i}$, as desired. □

Table 1 summarizes the value of c_n when n ranges from 2 to 10. Figure 4 plots the first 200 values of c_n. We observe that the slopes form two curves, one for even n and one for odd n. This is intuitive since in the induction of the proof to Theorem 23, we use c_{n-2} to prove the conclusion for c_n. Note that the curve for even n is above that for odd n. In addition, we observe that both curves are strictly decreasing, which gives the following corollary:

Corollary 24. *For any $n \geq 4$, $c_n < c_{n-2}$.*

Proof. By Theorem 23, we have

$$c_n = \frac{1}{n-1}\sum_{i=1}^{n-1} c_{n,i}$$

$$= \frac{1}{n-1}\left(\frac{2(n-3)}{n-1}c_{n-2} + (n-3)\frac{(n-2)(n-2)}{(n-1)(n-3)}c_{n-2} - \frac{n-2}{(n-1)(n-3)}c_{n-2}(n-3)\right) + \frac{1}{2(n-1)^2}$$

$$= \frac{n^2-3n}{(n-1)^2}c_{n-2} + \frac{1}{2(n-1)^2} < c_{n-2},$$

as desired. □

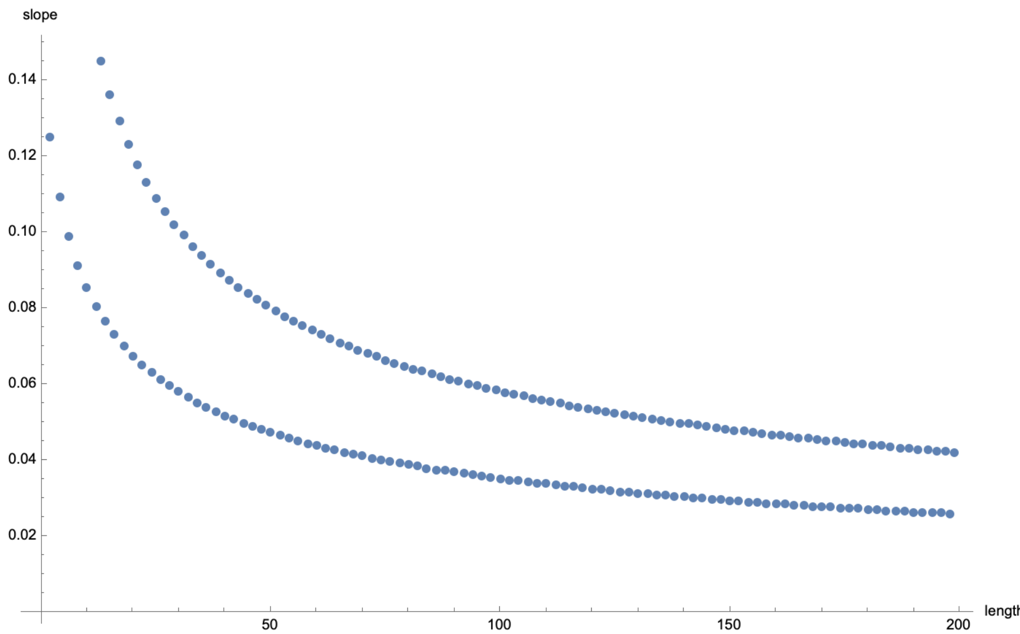


Figure 4: First 200 Slopes of Asymptotes of avg_n;
$c_n, 2 \le n \le 200$

5 Future Work

We have shown a recursive formula for computing c_n and proved that c_n is strictly decreasing for both $2 \mid n$ and $2 \nmid n$, respectively. We have not, however, obtained an explicit formula for c_n, neither have we shown any asymptotic behavior of c_n. From Figure 4, we observe that the derivative of the curve $c_n, 2 \mid n$ and $c_n, 2 \nmid n$ are both increasing. Since we have shown the derivative is always negative, we conjecture that both curves have a horizontal asymptote:

Conjecture 25. *There exists positive constant d_1, d_2 such that $\lim_{n\to\infty} c_n = d_1$ for $2 \mid n$ and $\lim_{n\to\infty} c_n = d_2$ for $2 \nmid n$. Further, $d_1 > d_2$.*

In addition, we mention in Definition 2 that the weighted diagram of size n may be considered as a partition of n (its shape-class). While iterating the Type A Lusztig-Vogan bijection, one can track all weighted diagrams obtained and have a sequence of partitions of n for each input (weakly decreasing sequence of integers) of length n. A possible direction is to explore this correspondence — for example, to reverse this correspondence — given a sequence of partitions, is it possible to find the set of weakly decreasing sequence of integers whose elements would produce this sequence of partition?

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