The Stembridge Equality for Skew Dual Stable Grothendieck Polynomials

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Outline

- Schur polynomials
- Dual stable Grothendieck polynomials
- The Stembridge equality for skew dual stable Grothendieck polynomials

Partitions

Definition

Let $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ be integers summing to n. Then the sequence $(a_1, a_2, \ldots, a_k)$ is a **partition** of the integer n.

Example

The partitions of the positive integer 4 are (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).

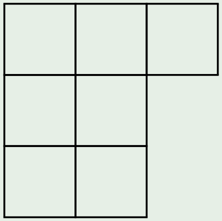
Young Diagrams

Definition

The **Young diagram** of a partition $\lambda = (a_1, a_2, \ldots, a_k)$ is a left-aligned array of boxes such that the ith row from the top has a_i boxes.

Example

The following is the Young diagram of the partition $(3, 2, 2)$ of 7.



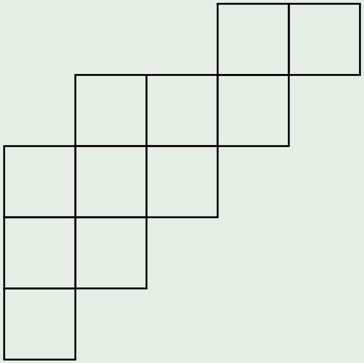
Young Diagrams of Skew Shapes

Definition

Given Young diagrams $\mu \subseteq \lambda$, the **Young diagram** of the skew shape λ/μ consists of the squares in λ but not μ.

Example

The following is the Young diagram of $(5, 4, 3, 2, 1)/(3, 1)$.



Semi-Standard Young Tableaux

Definition

A **semi-standard Young tableau** (SSYT) contains numbers that strictly increase in each column and weakly increase in each row. Given a SSYT T, define the monomial

$$x^T = \prod_i x_i^{(\text{number of entries of } i)}.$$

Example (SSYT of shape $(4,4,3,2)/(3,1)$)

			2
	1	1	4
1	2	2	
3	4		

$$x^T = x_1^3 x_2^3 x_3 x_4^2$$

Schur Polynomials

Definition

For a skew shape λ/μ, the **skew Schur polynomial** $s_{\lambda/\mu}$ with variables $x = (x_1, x_2, \dots)$ is a sum over all SSYT T of shape λ/μ:

$$s_{\lambda/\mu} = \sum_T x^T.$$

Example

When $\lambda/\mu = (2,2)/(1)$, the following are all SSYT with $i < j < k$.

$$\begin{array}{|c|c|} \hline & i \\ \hline i & j \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & i \\ \hline j & j \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & i \\ \hline j & k \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & j \\ \hline i & k \\ \hline \end{array}$$

Thus,

$$s_{(2,2)/(1)} = \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2 + 2 \sum_{i<j<k} x_i x_j x_k.$$

Schur Polynomials Are Symmetric

Theorem (Stanley)

The skew Schur polynomial $s_{\lambda/\mu}$ is a symmetric polynomial for all skew partitions λ/μ.

A symmetric polynomial stays the same when x_i and x_j are swapped.

Example

The polynomials

$$\sum_i x_i = x_1 + x_2 + x_3 + \cdots$$

and

$$\sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots$$

are symmetric.

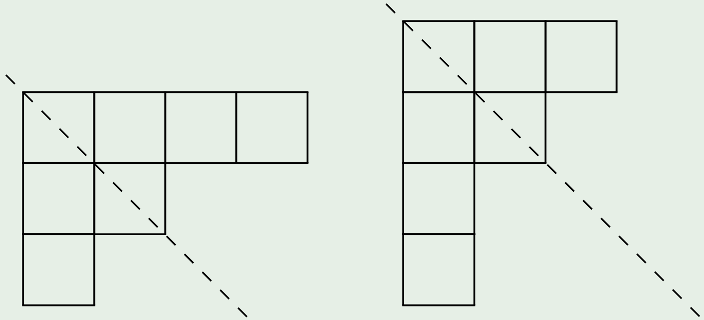
Transposes

Definition

Reflecting a Young diagram μ across the top-left to bottom-right diagonal gives its **transpose** μ^T.

Example

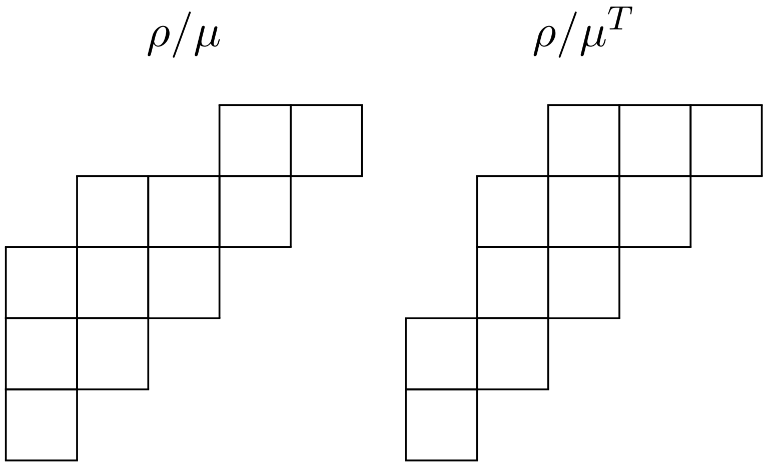
The partitions (4, 2, 1) and (3, 2, 1, 1) are transposes.



Stembridge Equality

Theorem (Stembridge)

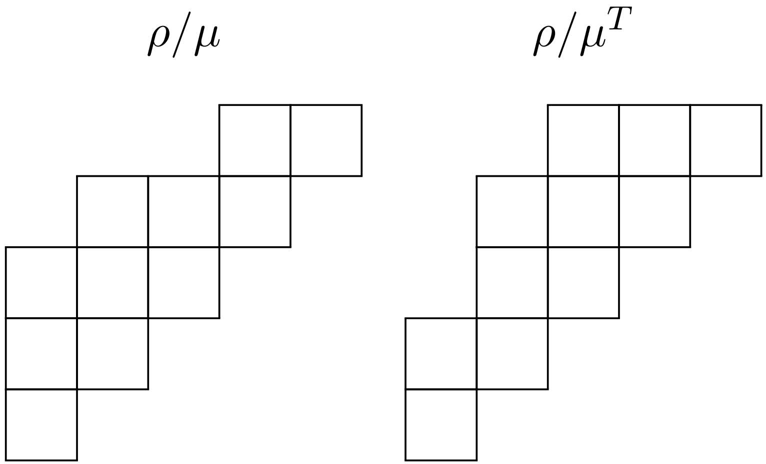
Let $\rho = (n, n-1, \ldots, 1)$. *We have* $s_{\rho/\mu} = s_{\rho/\mu^T}$ *for all* $\mu \subseteq \rho$.



Stembridge Equality

Theorem (Stembridge)

Let $\rho = (n, n-1, \ldots, 1)$. We have $s_{\rho/\mu} = s_{\rho/\mu^T}$ for all $\mu \subseteq \rho$.



Research Project

Is it true that $g_{\rho/\mu} = g_{\rho/\mu^T}$ for skew dual stable Grothendieck polynomials?

Reverse Plane Partitions

Definition

A **reverse plane partition** (RPP) contains numbers that weakly increase in each row and in each column. Given an RPP P, define the monomial

$$x^{\text{ircont}(P)} = \prod_i x_i^{(\text{number of columns that contain } i)}.$$

Example

			2	4
	1	1	2	
1	1	5		
3	3			
3				

$$x^{\text{ircont}(P)} = x_1^3 x_2 x_3^2 x_4 x_5$$

Dual Stable Grothendieck Polynomials

Definition

The **skew dual stable Grothendieck polynomial** $g_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \dots)$ is a sum over all RPPs P of shape λ/μ :

$$g_{\lambda/\mu} = \sum_{P} x^{\mathrm{ircont(P)}}.$$

Example

When $\lambda/\mu = (2,2)/(1)$, the following are all RPPs with $i < j < k$.

$$\begin{array}{cc} & \boxed{i} \\ \boxed{i} & \boxed{i} \end{array} \quad \begin{array}{cc} & \boxed{j} \\ \boxed{i} & \boxed{j} \end{array} \quad \begin{array}{cc} & \boxed{i} \\ \boxed{i} & \boxed{j} \end{array} \quad \begin{array}{cc} & \boxed{i} \\ \boxed{j} & \boxed{j} \end{array} \quad \begin{array}{cc} & \boxed{i} \\ \boxed{j} & \boxed{k} \end{array} \quad \begin{array}{cc} & \boxed{j} \\ \boxed{i} & \boxed{k} \end{array}$$

Thus,

$$g_{(2,2)/(1)} = \sum_{i} x_i^2 + \sum_{i<j} x_i x_j + \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2 + 2 \sum_{i<j<k} x_i x_j x_k.$$

Comparison

Note that the top degree of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$.

Example (Schur)

$$\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{j} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{j} & \multicolumn{1}{c|}{j} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{j} & \multicolumn{1}{c|}{k} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{j} \\ \hline \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{k} \\ \hline\end{array}$$

$$s_{(2,2)/(1)} = \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2 + 2 \sum_{i<j<k} x_i x_j x_k$$

Example (Dual Stable Grothendieck)

$$\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{i} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{j} \\ \hline \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{j} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{j} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{j} & \multicolumn{1}{c|}{j} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i} \\ \hline \multicolumn{1}{|c|}{j} & \multicolumn{1}{c|}{k} \\ \hline\end{array} \quad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{j} \\ \hline \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{k} \\ \hline\end{array}$$

$$g_{(2,2)/(1)} = \sum_{i} x_i^2 + \sum_{i<j} x_i x_j + \left(\sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2 + 2 \sum_{i<j<k} x_i x_j x_k \right)$$

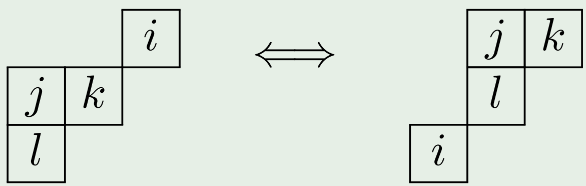
Main Result: Stembridge for Dual Grothendieck Polynomials

Theorem (A., A., N.)

Let $\rho = (n, n-1, \ldots, 1)$. *We have* $g_{\rho/\mu} = g_{\rho/\mu^T}$ *for* $\mu = (k)$ *and transpose* $(1^k) = (1, \ldots, 1)$.

Example

$$g_{(3,2,1)/(2)} = g_{(2,1)}g_{(1)} = g_{(3,2,1)/(1,1)}$$



Sketch of Proof of Main Result

We prove that $g_{\rho/(k)} = g_{\rho/(1^k)}$ in two steps:

- First, translate this to a problem about comparing the Littlewood-Richardson coefficients $c^{\rho}_{\mu\nu}$;
- Then, use a combinatorial description of these coefficients to show that they are equal for $\mu = (k)$ and (1^k).

Littlewood-Richardson Coefficients

Theorem (Buch)

In the expansion

$$g_{\rho/\mu} = \sum_{\nu} c^{\rho}_{\mu\nu} g_{\nu},$$

the coefficient $c^{\rho}_{\mu\nu}$ *is equal to* $(-1)^{|\rho|-|\mu|-|\nu|}$ *times the number of set-valued tableaux* T *of shape* $\nu * \mu$ *such that the reverse reading word of* T *is a lattice word with content* ρ*.*

Theorem (Lam and Pylyavskyy)

The dual stable Grothendieck polynomials g_ν *are symmetric functions and form a basis for the ring of symmetric functions.*

So, we have $g_{\rho/\mu} = g_{\rho/\mu^T} \iff c^{\rho}_{\mu\nu} = c^{\rho}_{\mu^T\nu}$ for all ν.

Set-Valued Tableaux

Definition

A **set-valued tableau** contains sets of positive integers such that the entries weakly increase along rows and strictly increase along columns.

Example

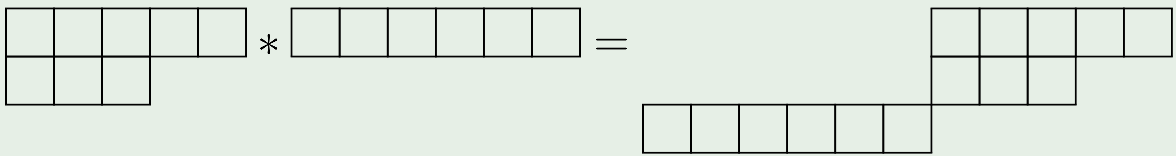
We have $\{1,2,3\} \leq \{3,5\}$ and $\{1,2,3\} < \{4,6,8\}$.

Example

		$1,2$	$2,3,4$	7
	3	$3,5$	5	
2	$4,5,6$	6		

Definitions

Example ($\nu * \mu$)



Definition

A tableau T having **content** $\rho = (n, n-1, \ldots, 1)$ means that there are n 1's, $n-1$ 2's, and so on in T.

Reverse Reading Words

Definition

The **reverse reading word** of a tableau T is read right to left along a row, starting with the top row and moving down, and with the elements within a cell read largest to smallest.

Example

	$1,2$	$2,3,4$
3	$3,5$	5
$4,5,6$		

432215533654

Lattice Words

Definition

A reverse reading word is a **lattice word** if the nth instance of $i+1$ comes after the nth instance of i.

Example

1121322 is a lattice word, but 121221 is not.

Expansion of Skew Dual Stable Grothendieck Polynomials

Theorem (Buch)

In the expansion

$$g_{\rho/\mu} = \sum_{\nu} c^{\rho}_{\mu\nu} g_{\nu},$$

the coefficient $c^{\rho}_{\mu\nu}$ *is equal to* $(-1)^{|\rho|-|\mu|-|\nu|}$ *times the number of set-valued tableaux* T *of shape* $\nu * \mu$ *such that the reverse reading word of* T *is a lattice word with content* ρ*.*

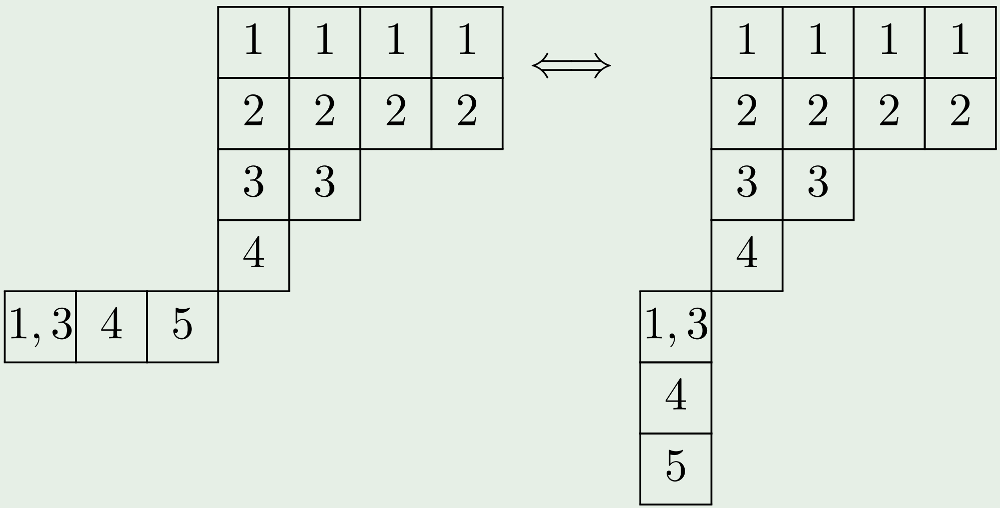
We will show that $c^{\rho}_{(k)\nu} = c^{\rho}_{(1^k)\nu}$ for all ν. In other words, the number of set-valued tableaux T such that its reverse reading word is a lattice word with content ρ is the same for the shapes $\nu * (k)$ and $\nu * (1^k)$.

Bijection

Lemma

If the reverse reading word of T is a lattice word, where T is of a non-skew shape ν, then all cells in the ith row contain only $\{i\}$.

Bijection



So $c^{\rho}_{(k)\nu} = c^{\rho}_{(1^k)\nu}$, and $g_{\rho/(k)} = g_{\rho/(1^k)}$.

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