The Geometry of Harmonic Maps in Genus One

Brian Liu
Mentor: Dhruv Ranganathan

High Technology High School

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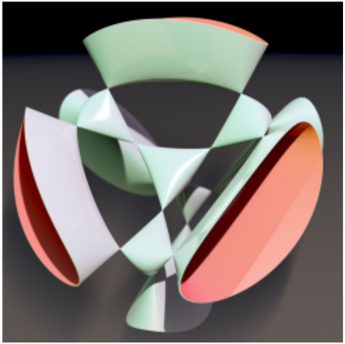
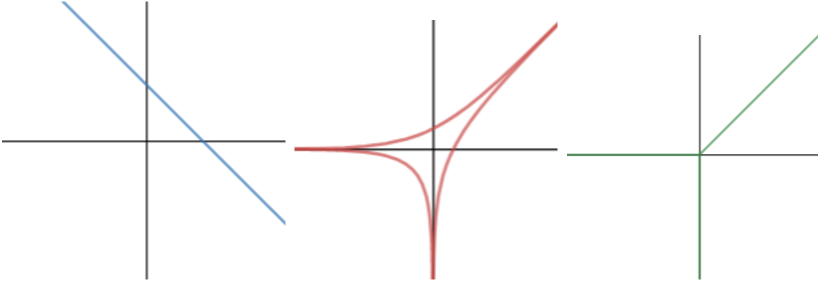
Algebraic Geometry

- Algebraic geometry concerns the structure of algebraic curves.
- Algebraic curves represent the solution sets to multivariable polynomials over the complex numbers.



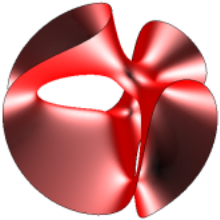
Tropicalization

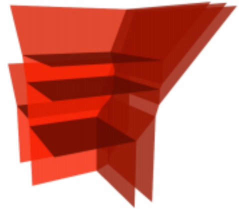
- Instead of taking the entire algebraic curve, we may just look at the logarithm of the **magnitudes** of the coordinates.
- We plot $(\log_t |x|, \log_t |y|)$ for every (x, y) on the curve. We then let t approach ∞.
- This allows us to capture a lot of the structure of the algebraic curve, but it has the advantage that the curve is **piecewise linear**, which can be thought of as a graph.



Analogs

- Just as this operation can convert an algebraic curve into a graph, it can also convert maps of algebraic curves into maps of graphs.
- In particular, it can convert a map from an algebraic curve to the complex numbers into a piecewise linear function on a graph to the reals.



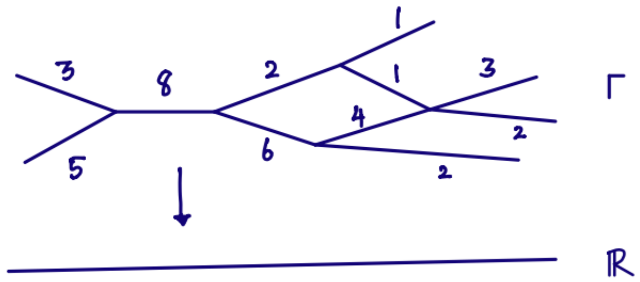


Key Question

- While we may always convert from an algebraic map into a tropical one, it is not always possible to go in the reverse direction.
- Thus, it leads to the following key question:
  - When is it possible for a piecewise linear function on a graph to be the result of a tropicalization of an algebraic map?
- We say that such a function is "liftable" if this is possible.
- Note: This is a very hard problem.

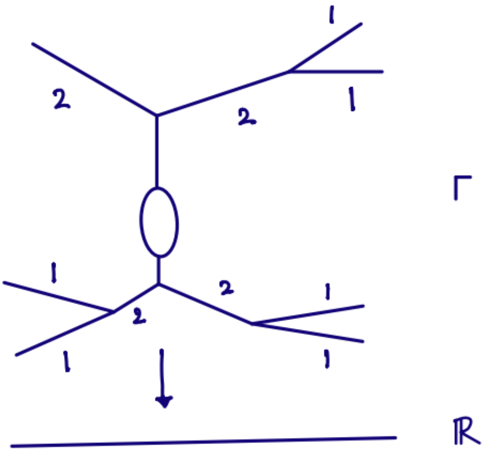
Balanced Condition

- In order for a function to be liftable, it has to be balanced.
- The balanced condition is defined as follows:
  - The sum of the slopes coming into a vertex is equal to the sum of the slopes leaving a vertex.
- Note that slopes are well-defined because the function is linear along every edge.



Example of Non-Liftable Balanced Graph

- However, the balanced condition is not sufficient for liftability.
- For example, we can have a graph such as the one below:

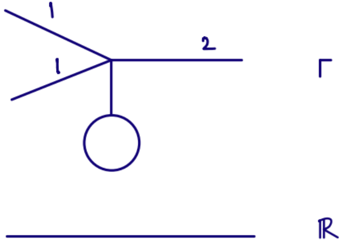


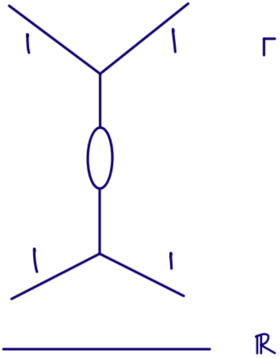
Well-Spacedness

- However, some graphs whose cycles do collapse are still liftable.
- There exists a subtler condition known as **well-spacedness** that is actually equivalent to liftability in genus 1 graphs.
- A proof of this result has been given by Speyer, but it uses sophisticated techniques and is difficult to generalize to higher genuses, while this proof is simpler and is more likely to generalize.

Main Result

- A balanced map from a genus 1 graph to the reals is liftable if and only if the nearest noncollapsing vertex to the cycle has at least 3 noncollapsing edges or there exist at least two such vertices of minimal distance.





Methods

The proof of this result uses the following two main methods:

- Instead of considering a map from a graph to a line (the reals), generalize to a map from a graph to a tree and reduce down.
- The Riemann-Hurwitz formula is a result from algebraic geometry that constrains the shapes of functions on algebraic curves. Similarly, the tropical version constrains the shapes of piecewise linear functions.

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References

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