Group testing via zero-error channel capacity

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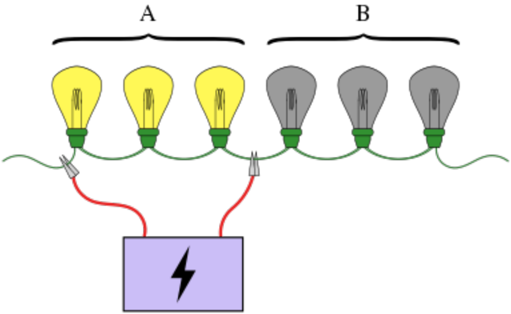
Introduction to Group Testing

- What is group testing? [1]



[1] This image is from Wikimedia Commons, by CheChe

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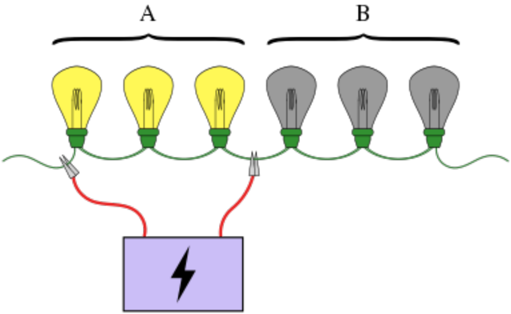

- Origin of group testing

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- This turns out to be optimal.

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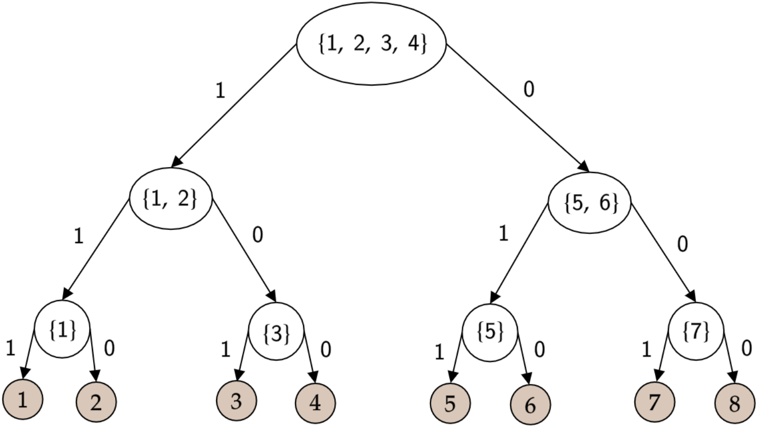
Main result The best algorithm gives about $1.266\log_2(n)$ tests.

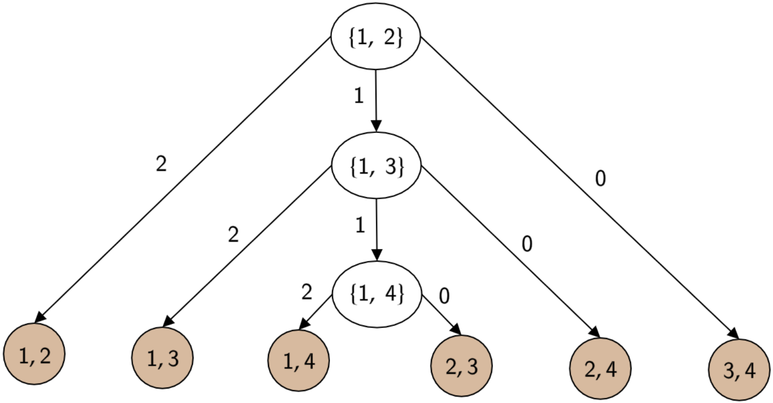
Binary Decision Tree for Single Item Testing

Testing procedure for $\{1, 2, \ldots, 8\}$

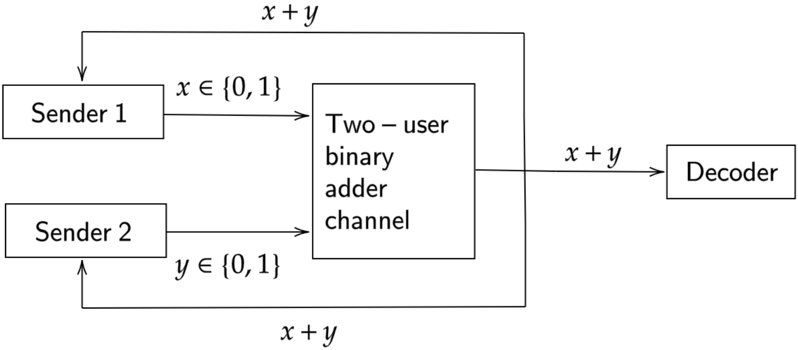


Ternary Decision Tree for Double Item Testing

Testing procedure for $\{1, 2, 3, 4\}$



Two-User Binary Adder Channel with Feedback



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- Computing c is equivalent to an optimization problem involving the entropy function

$$H(x_1, x_2, \ldots, x_n) = -\sum_{i=1}^{n} x_i \log_2 x_i$$

and the related function

$$L(x) = H\left(\frac{1-x}{2}, x, \frac{1-x}{2}\right).$$

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- $0 \leq a_i, b_i, p_i \leq 1,$
- $\sum_{i=1}^{n} p_i = 1$
- $L\left(\sum_{i=1}^{n} p_i(a_i\bar{b}_i + \bar{a}_i b_i + 2c_i)\right) \geq \sum_{i=1}^{n} p_i H\left(a_i b_i - c_i, a_i\bar{b}_i + c_i, \bar{a}_i b_i + c_i, \bar{a}_i\bar{b}_i - c_i\right)$ for all points $\boldsymbol{c} = (c_1, ..., c_n)$ that make the terms inside $H(\cdot, \cdot, \cdot, \cdot)$ non-negative.

Bounding n

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- Is there some n_0 such that, for $n > n_0$, the maximum achieved in the optimization problem doesn't increase?
- Dueck's paper says that $n_0 = 6$ works. But can we do better?

Uniqueness Theorem

We wish to show that $\widetilde{\boldsymbol{c}}$, defined as the $\boldsymbol{c}$ that causes the inequality to be as tight as possible, is uniquely defined. Furthermore, we wish to show it can be defined by taking partial derivatives of the inequality bounding $\boldsymbol{c}$.

Reducing n_0 to 3

In order to reduce n_0 down to 3, we want to show that any possible $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}$ with $n \geq 4$ can be adjusted by changing $\boldsymbol{p}$ to $\boldsymbol{p}^*$ while preserving

- $\sum_{i=1}^{n} p_i^* = 1$
- $\sum_{i=1}^{n} p_i^*(a_i\bar{b}_i + \bar{a}_i b_i + 2\widetilde{c}_i)$ is fixed
- $\sum_{i=1}^{n} p_i^* H\left(a_i b_i - \widetilde{c}_i, a_i\bar{b}_i + \widetilde{c}_i, \bar{a}_i b_i + \widetilde{c}_i, \bar{a}_i\bar{b}_i - \widetilde{c}_i\right)$ is fixed
- $\sum_{i=1}^{n} p_i^*(H(a_i, \bar{a}_i) + H(b_i, \bar{b}_i))$ is non-decreasing

It can be shown that, if $n \geq 4$, then p^* can be created such that $\exists i$ with $p_i^* = 0$.

Conjectures

- We suspect $n = 1$ suffices and that the maximum occurs at $a_1 = b_1 = \frac{\log(2+\sqrt{3})-\log(2)}{2\log(2+\sqrt{3})} \approx 0.23684$ giving a maximum of ≈ 1.57948.

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- This would mean 2 defects out of n could be found with $\left(\frac{2}{H(a_1,\bar{a}_1)+H(b_1,\bar{b}_1)} + O(1)\right)\log_2(n) \approx 1.266\log_2(n)$ tests.

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- The PRIMES program
- Parents
- Sam's cat

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