Refinements of Product Formulas for Volumes of Flow Polytopes

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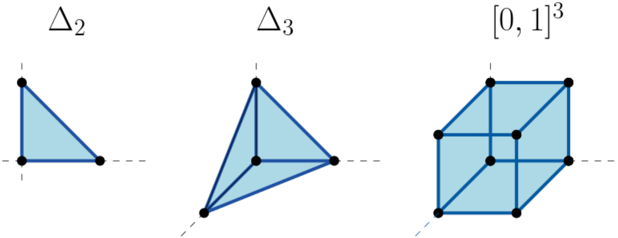
October 17, 2020
MIT PRIMES Conference

Integral Polytopes

- An integral polytope P in $\mathbb{R}^n$ is the convex hull of finitely many vertices v in $\mathbb{Z}^n$.



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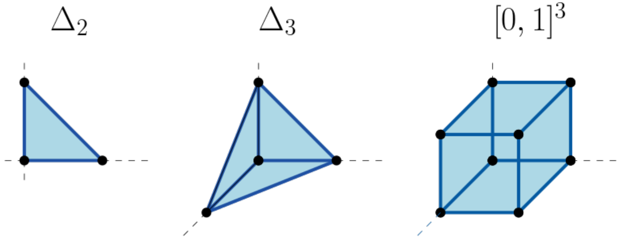
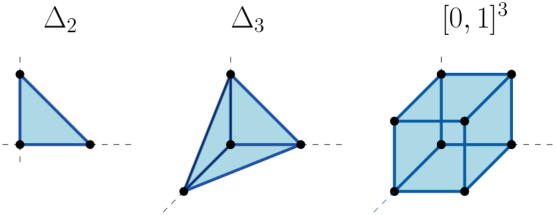


- Equivalently, P is the intersection of finitely many half spaces.

Volume of Polytopes

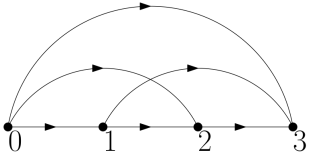
normalized volume of $P := \dim(P)! \cdot (\text{euclidean volume of } P)$



	Δ_2	Δ_3	$[0,1]^3$
Euclidean Volume	$1/2$	$1/6$	1
Normalized Volume	1	1	6

Graphs

- For a loopless graph $G = (\{0, 1, \ldots, n, n+1\}, E)$, we orient edge (i, j) from $i \to j$ if $i < j$.



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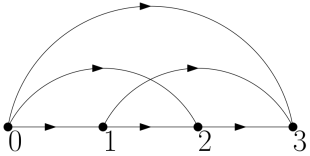
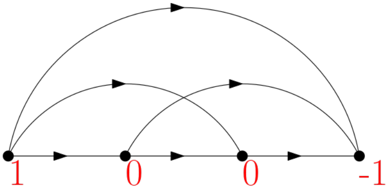
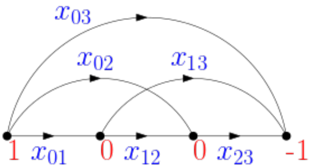


- The source has net flow 1, the sink has net flow -1, and other vertices have net flow 0.

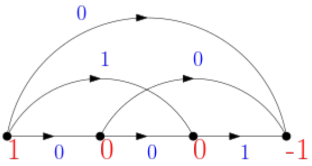


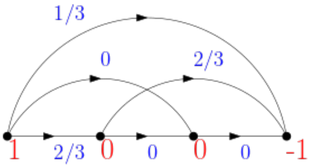
Flows

A *flow* is a function $f : E \to \mathbb{R}^m_{\geq 0}$ that satisfies the net flow vector $(1, 0, \ldots, 0, -1)$.



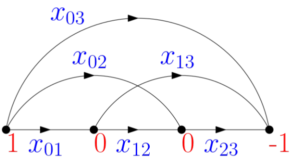
$$x_{01} + x_{02} + x_{03} = 1$$
$$x_{12} + x_{13} - x_{01} = 0$$
$$x_{23} - x_{02} - x_{12} = 0$$





Flow Polytopes

- The *flow polytope* $\mathcal{F}_G$ is the set of all flows on G.



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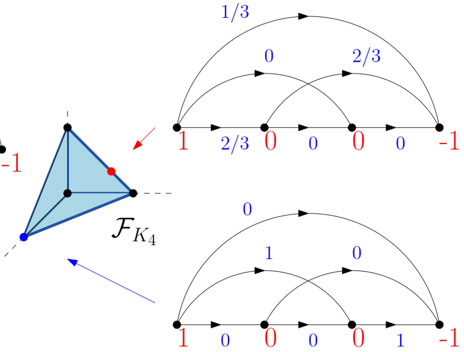

The Chan-Robbins-Yuen Polytope

- The Chan-Robbins-Yuen (CRY) Polytope is defined by

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$$CRY_{n+1} := \mathcal{F}_{K_{n+2}}.$$

Theorem (Zeilberger 1999)

The volume of the CRY polytope is given by

$$\mathrm{vol}\ CRY_{n+1} = \prod_{i=1}^{n-1} C_i,$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ is the ith Catalan number.

The Morris Identity

Theorem (Zeilberger 1999, Baldoni-Vergne 2001)

For $n, a, b \in \mathbb{Z}^+$ *and* $c \in \mathbb{Z}_{\geq 0}$, *define the constant term*

$$M_n(a,b,c) := \mathrm{CT}_x \prod_{i=1}^{n} (1-x_i)^{-b} x_i^{-a+1} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c},$$

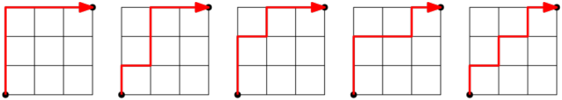
where $\mathrm{CT}_x := \mathrm{CT}_{x_n} \cdots \mathrm{CT}_{x_1}$. *Then*

$$M_n(a,b,c) = \prod_{j=0}^{n-1} \frac{\Gamma(a-1+b+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2}+1)}{\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})\Gamma(\frac{c}{2}(j+1)+1)}.$$

Catalan and Narayana Numbers

- The **Catalan number** $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the lattice paths from $(0,0)$ to (n,n) not passing below the diagonal.

$C_3 = 5$



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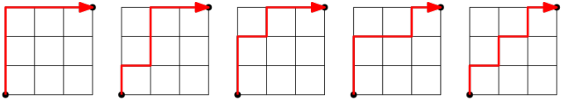

- The Catalan numbers are refined by the **Narayana numbers** $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, which also count the number of peaks.

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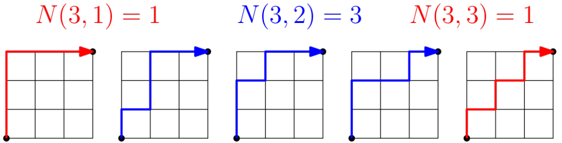

Subdividing the CRY Polytope

- Zeilberger used "Aomoto's extension of Selberg's integral" to refine $M_n(1,1,1)$ as a sum of $N(n-1,k)C_{n-2}\cdots C_1$.

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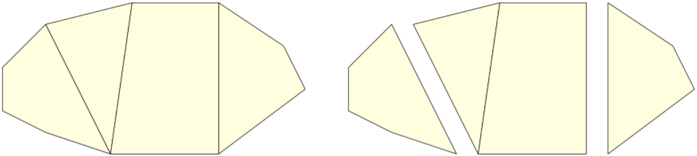
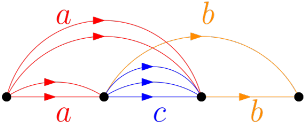
- Mészáros (2011) gave a collection of interior disjoint polytopes with volumes that sum to $N(n-1,k)C_{n-2}\cdots C_1$.



Generalizing the CRY Polytope

- $K_{n+2}^{a,b,c}$ has vertices $\{0, \ldots, n+1\}$ and for $i \in [n]$, edge $(0,i)$ a times, $(i, n+1)$ b times, and (i,j) c times for $i < j \leq n$.



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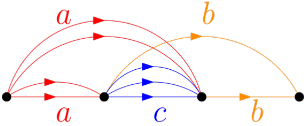

Theorem (Corteel-Kim-Mészáros 2017)

For $n, a, b \in \mathbb{Z}^+$ and $c \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{vol}\, \mathcal{F}_{K_{n+2}^{a,b,c}} = M_n(a,b,c).$$

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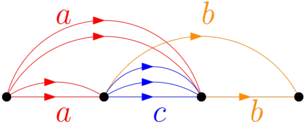

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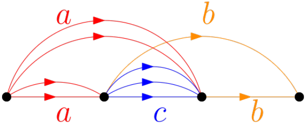

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- **Q1:** Is there a refinement of $M_n(a, b, c)$?
- **Q2:** Does such a refinement have a geometric interpretation?

A New Constant Term Identity

- We define the constant term: $\Psi_n(k,a,b,c) :=$

$$\mathrm{CT}_x\,[t^k]\prod_{i=1}^{n}(1-x_i)^{-b}x_i^{-a+1}\left(1+t\frac{x_i}{1-x_i}\right)\prod_{1\le i<j\le n}(x_j-x_i)^{-c}.$$

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Theorem (Morales-S. 2020)

For $n, a, b \in \mathbb{Z}^+$ and $c, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$, we have

$$\Psi_n(k,a,b,c) = \binom{n}{k} M_n(a,b,c) \prod_{j=1}^{k} \frac{a-1+(n-j)\frac{c}{2}}{b+(j-1)\frac{c}{2}}.$$

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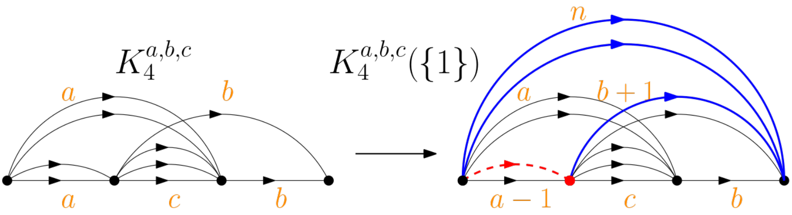
- The proof uses several recurrence relations.

Generalizing $K_{n+2}^{a,b,c}$

For $S \subseteq [n]$, the graph $K_{n+2}^{a,b,c}(S)$ takes $K_{n+2}^{a,b,c}$, adds n edges $(0, n+1)$, and for each $i \in S$, deletes an edge $(0, i)$ and adds an edge $(i, n+1)$.



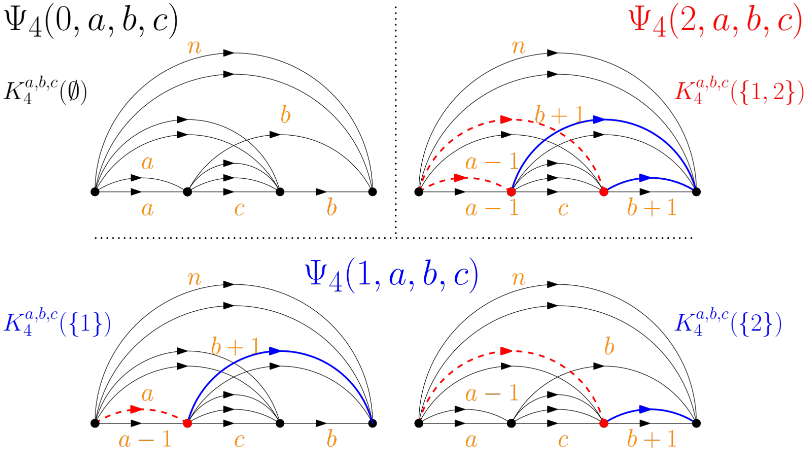
Polytope Interpretation for $\Psi_n(k, a, b, c)$

Theorem (Morales-S. 2020)

For $n, a, b \in \mathbb{Z}^+$ and $c, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$,

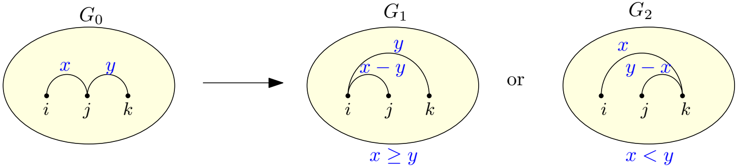
$$\Psi_n(k, a, b, c) = \sum_{S \in \binom{[n]}{k}} \operatorname{vol} \mathcal{F}_{K_{n+2}^{a,b,c}(S)}.$$

Example: Polytope Interpretation



Subdividing $\mathcal{F}_{K_{n+2}^{a,b,c}}$

- The subdivision lemma (Postnikov-Stanley) gives a map that reduces a flow polytope to two interior disjoint polytopes.



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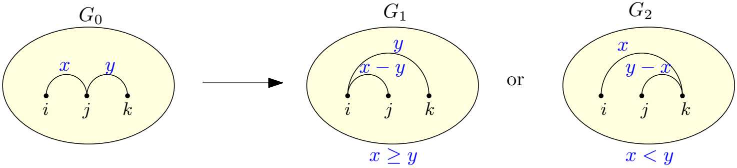
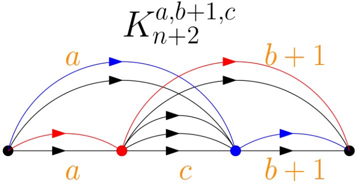


- We apply this to the flow polytope on the graph $K_{n+2}^{a,b+1,c}$.

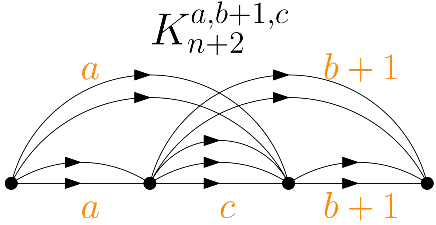


Refining the Morris Identity

Corollary (Morales-S. 2020)

For $n, a, b \in \mathbb{Z}^+$ and $c \in \mathbb{Z}_{\geq 0}$,

$$M_n(a, b+1, c) = \sum_{k=0}^{n} \Psi_n(k, a, b, c).$$



Summary of the Results

$$\prod_{i=1}^{n-1} C_i = \sum_{k=1}^{n-1} N(n-1,k) \prod_{i=1}^{n-2} C_i$$

$$\text{vol } CRY_{n+1} = \prod_{i=1}^{n-1} C_i \longrightarrow \text{vol } \mathcal{F}_{K_{n+2}^{1,1,1}(S)} = N(n-1,k) \prod_{i=1}^{n-2} C_i$$

$$\downarrow \qquad\qquad \downarrow$$

$$\text{vol } \mathcal{F}_{K_{n+2}^{a,b,c}} = M_n(a,b,c) \longrightarrow \text{vol } \mathcal{F}_{K_{n+2}^{a,b,c}(S)} = \Psi_n(k,a,b,c)$$

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Acknowledgements

- My mentor, Prof. Alejandro Morales
- MIT PRIMES-USA Program
- Dr. Tanya Khovanova and Ms. Boya Song
- My family

Thank You!

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