Planar Embeddings of Periodic Time-Dependent Graphs

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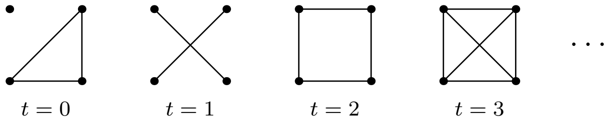
Background

Definition: Temporal Graph

A **Temporal Graph** or a **Time-Dependent Graph** is a graph defined on an underlying graph G_0 and equipped with a function $f : E(G_0) \to 2^{\mathbb{N}}$ such that edge $e \in E(G_0)$ exists at time t iff the tth element of $f(e)$ is 1.

- One can think of a temporal graph as an infinite sequence of graphs on the same set of vertices



- We work with periodic temporal graphs: $\exists p$ such that p is a period of $f(e)$ for all $e \in E(G_0)$

Motivation

- Communication
- Disease Spread Modeling
- Transportation
- Resource Allocation
- Anything which can be modeled as a graph but will change over time

Planar Temporal Graphs

Definition: Planar Temporal Graph

A Temporal Graph is **planar** if its underlying graph G_0 can be embedded in the plane such that, at every time t, no current edges intersect.

- Ex: If G_0 is planar, then any temporal graph on it is planar as well
- Can model applications which occur on a plane (ie. traffic signals)
- Do properties of static graphs extend?
    - Kuratowski's Theorem
    - Fáry's Theorem

Period 2 Planar Temporal Graphs

- We first investigate planarity with $p = 2$
- Suppose the graph is A at $2|t$, B at $2 \nmid t$
- Assume that A, B are planar

- **Claim:** If $E(A) \cap E(B)$ is acyclic, then our temporal graph G is planar
- First draw $E(A) \cap E(B)$. Since it's a forest, it doesn't define any new faces in the plane
- Remaining edges of A, B can be drawn in as their respective embeddings dictate
- So, we can now WLOG:
  - A and B are planar
  - They share a common cycle

Period 2 Planar Temporal Graphs

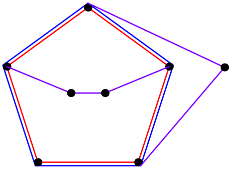
- Let $C \in E(A) \cap E(B)$ be a cycle on $V(G_0)$
- C defines an inside and an outside
- Suppose we want to force the temporal graph to be nonplanar. We can accomplish this by forcing $u, v \in V(G_0)$ to be both on the same side and on different sides of C
- Ex: If $uv \in E(A)$, then u, v on same side of C in planar embedding of A
- Now, consider connected components of $(E(A) \cup E(B)) \setminus C$ after the plane is cut by C
- All vertices of a connected component are on the same side of C
- For a connected component K, denote the feet of K as the vtxs of K which lie on C

Period 2 Planar Temporal Graphs

Intersecting Connected Components

Two connected components K, K' are **intersecting** if there exist feet a, b of K and a', b' of K' such that a, b lie on different sides of chord $a'b'$

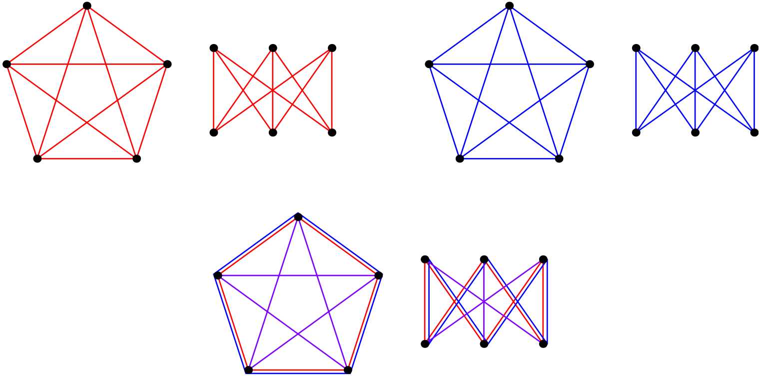
- Intersecting connected components must be on different sides of C



- Construct graph H with connected components as vertices and edges between intersecting components. H must be bipartite
- If H of all shared cycles is bipartite, then we can find embedding
- Forbidden Minor: Shared cycle + connected components (with edges from either A or B) which form an odd cycle

Period 2 Planar Temporal Graphs

- To characterize forbidden minors, we can replicate the proof of Kuratowski's Theorem.
- The structure of the simplified restriction is the same for planar graphs
- Period 2 Planar Temporal Graphs have 6 forbidden minors:



Validity of Robertson-Seymour Theorem

Theorem: Robertson-Seymour

Every family of graphs that is closed under minors can be defined by a finite set of forbidden minors.

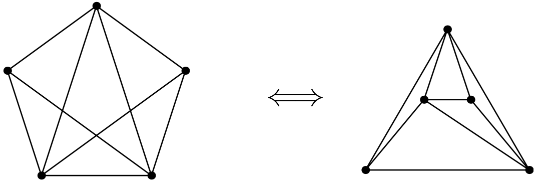
- Ex: Planar static graphs are closed under minors and forbidden minors are K_5, $K_{3,3}$ by Kuratowski
- What do minors mean in temporal graphs?
  - If restricted to $E(A) \cap E(B)$, then R-S is false
  - Instead, we say that we can contract an edge if it exists at some time
- **Future Research:** Prove Robertson-Seymour on Temporal Graphs

Fáry's Theorem

Theorem: Fáry

If a static graph G_0 is planar, then it admits a planar embedding where all edges are line segments

- Ex: $K_5 - K_2$ is planar, and below is an example of a straight-line embedding



- Claim: Fáry's Theorem is false for periodic temporal graphs.

Fáry's Theorem

- We will find a counterexample for $p = 2$
- The maximum number of edges of a planar graph is $3n - 6$
- Maximal planar graphs have $O(n)$ edges, so they're not very dense when $n \gg 1$
- So, for large n, we can find two maximal planar graphs A, B on the same set of vertices, which are edge disjoint
- For any embedding of A, all consecutive vertices of the convex hull must be connected by A's maximality
- Similar for B, but A, B don't share any common edges $\implies$ contradiction

Future Directions

- Generalize our results on period 2 planar graphs to general period
  - Claim: There should be $2 \cdot (2^p - 1)$ classes of forbidden minors, consisting of K_5 and $K_{3,3}$ for all nonempty color combinations
- Robertson-Seymour on Temporal Graphs
- Characterize which planar temporal graphs are Fáry embeddable
- Find diameter bounds for period p planar graphs with n vtxs and m edges

References

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