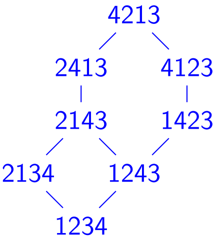
The Sperner Property for 132-Avoiding Intervals in the Weak Order

Katherine Tung
MIT PRIMES Conference
Joint work with Christian Gaetz

October 18, 2020



Weak order interval $[e, 4213]_R$

The weak order on permutations

Let S_n be the $n!$ permutations of $\{1, 2, 3, ..., n\}$.

Weak Bruhat order on S_n:

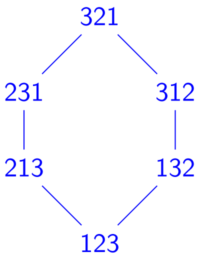
- Least element $e = [1\ 2\ ...\ n]$
- Covering: $\sigma \lessdot \sigma(i \quad i+1)$ if $\sigma_i < \sigma_{i+1}$.
  - $[15243] \lessdot [15423]$.

The weak order on permutations

Let S_n be the $n!$ permutations of $\{1, 2, 3, ..., n\}$.

Weak Bruhat order on S_n:

- Least element $e = [1\ 2\ ...\ n]$
- Covering: $\sigma \lessdot \sigma(i \quad i+1)$ if $\sigma_i < \sigma_{i+1}$.
  - $[15243] \lessdot [15423]$.

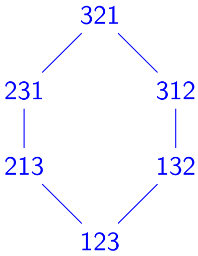


The weak order on permutations

Let S_n be the $n!$ permutations of $\{1, 2, 3, ..., n\}$.

Weak Bruhat order on S_n:

- Least element $e = [1\ 2\ ...\ n]$
- Covering: $\sigma \lessdot \sigma(i \quad i+1)$ if $\sigma_i < \sigma_{i+1}$.
  - $[15243] \lessdot [15423]$.
- Rank function $\ell(\sigma) = \#$ inversions of σ
  - $\ell([312]) = 2$.
- Greatest element $w_0 = [n\ n-1\ ...\ 2\ 1]$ with rank $\binom{n}{2}$.

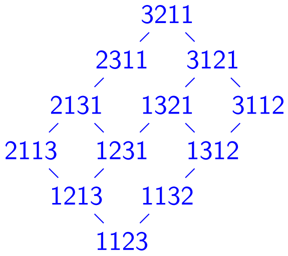


Multisets

We can similarly order the permutations of a set with repetitions such as the 60 orderings of 112333.

The latter set corresponds to the interval $[e, \pi = 456312]$, permutations less than or equal to 456312 in the weak order.

$112333 \longleftrightarrow 123456$
$313231 \longleftrightarrow 415362$
$333211 \longleftrightarrow 456312$



132-avoiding permutations

A permutation π avoids the pattern 132 if for no indices $i < j < k$ is $\pi_i < \pi_k < \pi_j$. So 4312 avoids 132, but 2143 does not avoid 132.

Any permutation corresponding to a greatest permutation of a multiset is 132-avoiding.

There are 2^{n-1} multisets and $C_n = \binom{2n}{n}/(n+1) \sim 4^n/(n^{3/2}\sqrt{\pi})$ 132-avoiding permutations.

We studied intervals $[e, \pi]_R$ with π 132-avoiding, which generalizes the study of permutations of multisets.

Questions about $[e, \pi]_R$

- Are these posets self-dual?
  - Is there an order-reversing self-bijection?

Questions about $[e, \pi]_R$

- Are these posets self-dual?
  - Is there an order-reversing self-bijection?
- Are these posets rank unimodal?
  - Does the rank function increase up to a peak and then fall?

Questions about $[e, \pi]_R$

- Are these posets self-dual?
  - Is there an order-reversing self-bijection?
- Are these posets rank unimodal?
  - Does the rank function increase up to a peak and then fall?
- Are these posets Sperner?
  - Is the size of the largest antichain (pairwise incomparable set) equal to the maximum number of elements with a particular rank?

Lie algebras

- Lie algebra L
  - Lie bracket $[x, y]$

Lie algebras

- Lie algebra L
  - Lie bracket $[x, y]$
    - Bilinear

Lie algebras

- Lie algebra L
    - Lie bracket $[x, y]$
        - Bilinear
        - $[x, x] = 0$ for all $x \in L$

Lie algebras

- Lie algebra L
  - Lie bracket $[x, y]$
    - Bilinear
    - $[x, x] = 0$ for all $x \in L$
    - $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

- Bracket operation $[a, b] = ab - ba$
- Basis elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Relations $[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

- Bracket operation $[a, b] = ab - ba$
- Basis elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Relations $[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f$

- Proctor, Stanley: if there is an $\mathfrak{sl}_2$ representation on $\mathbb{C}P$ respecting the order of P, then P is
  - rank symmetric
  - rank unimodal
  - Sperner

$\mathfrak{sl}_2$ representations

We want a lowering operator F and a raising operator E on $\mathbb{C}[e, \pi]_R$ so that

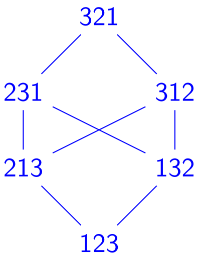
- $F(\sigma)$ is a linear combination of permutations covered by σ
- $E(\sigma)$ is a linear combination of permutations of rank $\ell(\sigma) + 1$
- $[E, F] = H$ is diagonal with $H(\sigma) = (2\ell(\sigma) - \ell(\pi))\,\sigma$.

Given F, there is at most one E that works (Jacobson and Morozov).

Strong order on S_n

Another related order on S_n is the strong order, which has the same rank function with more relations.

The covering relation is that $\sigma \prec \tau$ if $\tau = \sigma(i\ j)$ and $\ell(\tau) = \ell(\sigma) + 1$.



$\mathfrak{sl}_2$ repr. on S_n, weak order (Gaetz and Gao)

$$F\sigma = \sum_{i:\sigma(i\ i+1)\lessdot\sigma} i\sigma(i\ i+1).$$

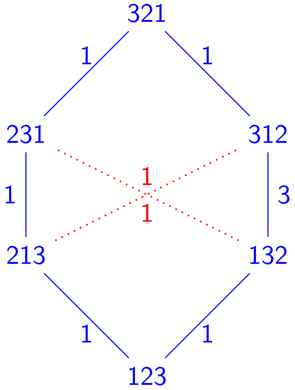
$$E\sigma = \sum_{\sigma \prec\!\!\cdot\, \sigma(i\ j)} \mathrm{wt}\,(\sigma, \sigma(i\ j))\,\sigma(i\ j)$$

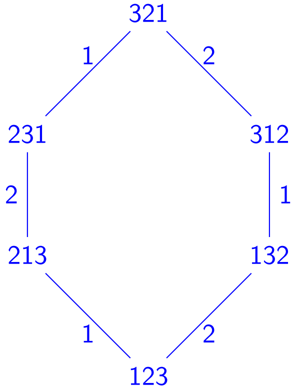
$$H\sigma = (2\ell(\sigma) - \ell(w_0))\,\sigma$$

where

$$\mathrm{wt}(\sigma, \sigma(i\ j)) := 1 + 2|\{k > j \mid \sigma_i < \sigma_k < \sigma_j\}|.$$

$\mathfrak{sl}_2$ repr. on S_n, weak order (Gaetz and Gao)





Above are edge weights for order raising operator E (left) and lowering operator F (right). Example: $E[132] = [231] + 3[312]$ and $F[132] = 2[123]$.

$\mathfrak{sl}_2$ representation on $[e, \pi]_R$

Theorem

We can construct an $\mathfrak{sl}_2$ *representation on* $[e, \pi]_R$ *by*

$$F\sigma = \sum_{i : \sigma(i\ \ i+1) \lessdot \sigma} i\sigma(i\ \ i+1).$$

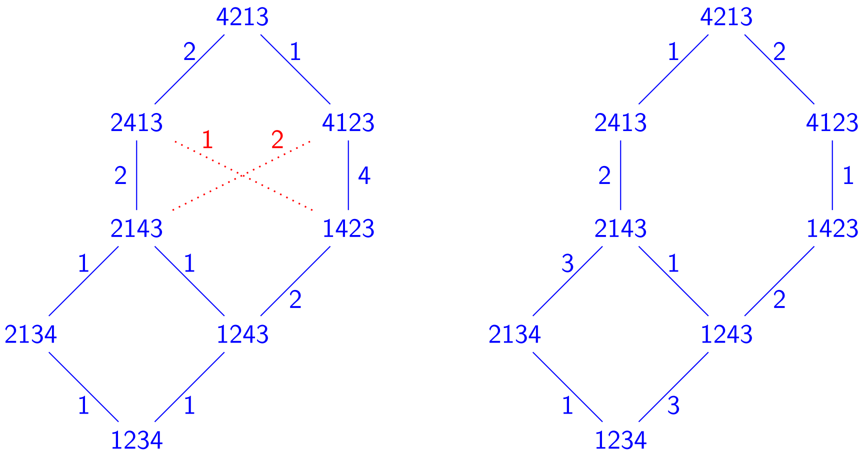
$$E\sigma = \sum_{\sigma \prec\!\!\cdot\, \sigma(i\ \ j) \leq \pi} \mathrm{wt}^{\pi}\left(\sigma, \sigma(i\ \ j)\right) \sigma(i\ \ j)$$

$$H\sigma = \left(2\ell(\sigma) - \ell(\pi)\right) \sigma$$

where

$$\mathrm{wt}^{\pi}(\sigma, \sigma(i\ \ j)) := 1 + |\{k > j \mid \sigma_i < \sigma_k < \sigma_j\}| + |\{k > j \mid \pi^{-1}(\sigma_j) < \pi^{-1}(\sigma_k) < \pi^{-1}(\sigma_i)\}|.$$

$\mathfrak{sl}_2$ representation on $[e, \pi]_R$



Above are edge weights for order raising operator E (left) and lowering operator F (right).

Schubert polynomials

$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} ... x_{n-1}^1$

Let N_i act on a polynomial f by:

$$N_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$$

We have the recursive relation $\mathfrak{S}_{s_i\sigma} = N_i \mathfrak{S}_\sigma$ if $\ell(s_i\sigma) = \ell(\sigma) - 1$.

Schubert polynomials

$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \dots x_{n-1}^1$

Let N_i act on a polynomial f by:

$$N_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$$

We have the recursive relation $\mathfrak{S}_{s_i \sigma} = N_i \mathfrak{S}_\sigma$ if $\ell(s_i \sigma) = \ell(\sigma) - 1$.

Examples:

- $\mathfrak{S}_{3412} = x_1^2 x_2^2$.
- $\mathfrak{S}_{1432} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$.

Principal evaluations of Schubert polynomials

Corollary

If $\sigma \in [e, \pi]_R$ *with* π *132-avoiding, then*

$$\mathfrak{S}_\sigma(1,1,1,...,1) = \frac{1}{(\ell(\pi)-\ell(\sigma))!} \sum_{\sigma \prec \sigma^1 \prec \ldots \prec \pi} \prod_i \mathrm{wt}^\pi(\sigma^i, \sigma^{i+1}).$$

If σ is 132-avoiding, we can use $\pi = \sigma$ which makes the product empty, so $\mathfrak{S}_\sigma(1,1,1,...,1) = 1$.

Principal evaluations of Schubert polynomials

Corollary

If $\sigma \in [e, \pi]_R$ with π 132-avoiding, then

$$\mathfrak{S}_\sigma(1, 1, 1, ..., 1) = \frac{1}{(\ell(\pi) - \ell(\sigma))!} \sum_{\sigma \prec\!\!\cdot\, \sigma^1 \prec\!\!\cdot\, \ldots \prec\!\!\cdot\, \pi} \prod_i \mathrm{wt}^\pi(\sigma^i, \sigma^{i+1}).$$

If σ is 132-avoiding, we can use $\pi = \sigma$ which makes the product empty, so $\mathfrak{S}_\sigma(1, 1, 1, ..., 1) = 1$.

Examples:

- $\mathfrak{S}_{3412}(1, 1, 1, 1) = 1$.
- $\mathfrak{S}_{1432}(1, 1, 1, 1) = 5$.

Acknowledgements

- MIT PRIMES USA
- My mentor Christian Gaetz
- My parents

Bibliography

[1] R. A. Proctor, "Representations of $\mathfrak{sl}(2,\mathbb{C})$ on posets and the Sperner property," *SIAM J. Algebraic Discrete Methods*, vol. 3, no. 2, pp. 275–280, 1982, ISSN: 0002-9890. DOI: 10.1137/0603026. [Online]. Available: https://doi.org/10.1137/0603026.

[2] R. P. Stanley, "Weyl groups, the hard Lefschetz theorem, and the Sperner property," *SIAM J. Algebraic Discrete Methods*, vol. 1, no. 2, pp. 168–184, 1980, ISSN: 0196-5212. DOI: 10.1137/0601021. [Online]. Available: https://doi.org/10.1137/0601021.

[3] V. V. Morozov, "On a nilpotent element in a semi-simple Lie algebra," *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, vol. 36, pp. 83–86, 1942.

[4] C. Gaetz and Y. Gao, "A combinatorial $\mathfrak{sl}_2$-action and the Sperner property for the weak order," *Proc. Amer. Math. Soc.*, vol. 148, no. 1, pp. 1–7, 2020, ISSN: 0002-9939. DOI: 10.1090/proc/14655. [Online]. Available: https://doi-org.libproxy.mit.edu/10.1090/proc/14655.