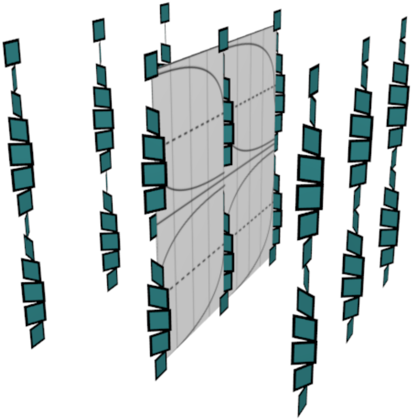
Tight contact structures on the solid torus



Jessica Zhang • October 18, 2020

Mentor: Zhenkun Li • MIT PRIMES Conference

Introduction: Why do we care?

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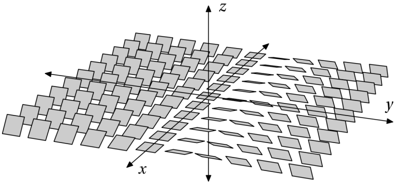
Informally, contact geometry is concerned with contact structures, which are geometric structures defined on odd-dimensional spaces.



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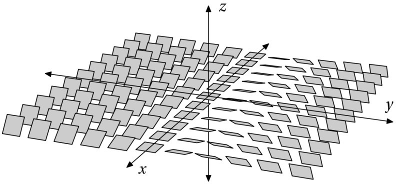

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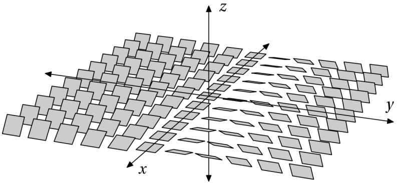


It helps us better understand and prove results in low-dimensional topology, but many fundamental questions still remain unanswered.

Open question

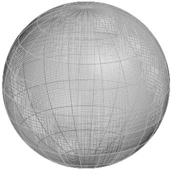
Can we classify the contact structures on a given 3-manifold?

Section 1

Contact geometry background

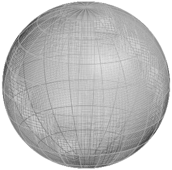
Smooth manifolds

A smooth n-manifold is a (Hausdorff) topological space M without edges or corners which locally looks like an open set of $\mathbb{R}^n$.



The surface of a ball is a 2-manifold without boundary

A smooth n-manifold is a (Hausdorff) topological space M without edges or corners which locally looks like an open set of $\mathbb{R}^n$. If M contains its boundary ∂M, we call it an n-manifold with boundary.



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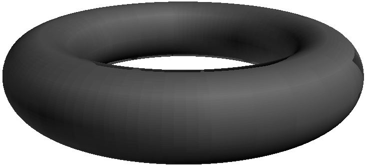
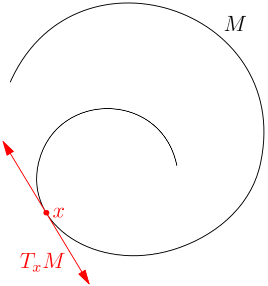
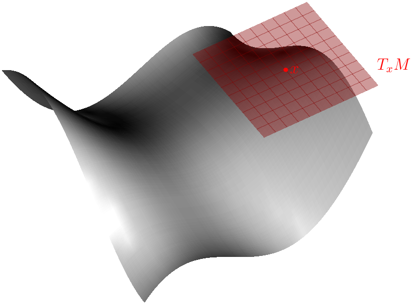


A solid torus is a 3-manifold with boundary

Tangent spaces

At each point $x \in M$, we can define the tangent space T_xM as the n-dimensional vector space consisting of all vectors tangent to M at x.





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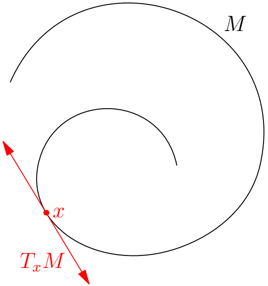
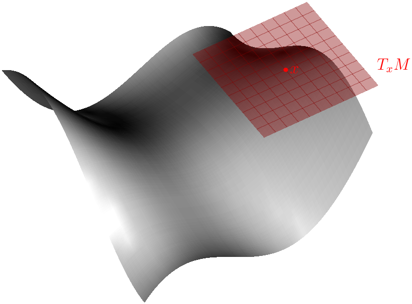


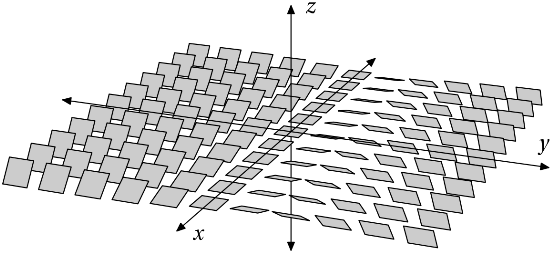


For the rest of this talk, M will be a smooth 3-manifold with boundary ($n = 3$).

Contact structures

Contact structures

A contact structure ξ on M is a way to smoothly assign a plane $\xi_x \subset T_xM$ to every $x \in M$ such that no surface $S \subset M$ is tangent to ξ_x for every $x \in S$.



The standard contact structure ξ_{st} on $\mathbb{R}^3$.

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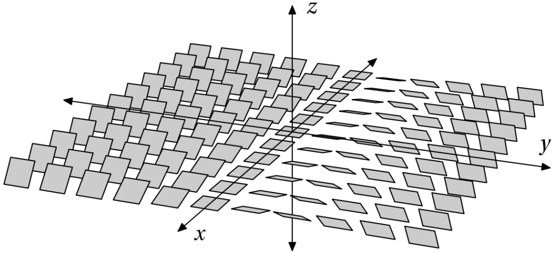


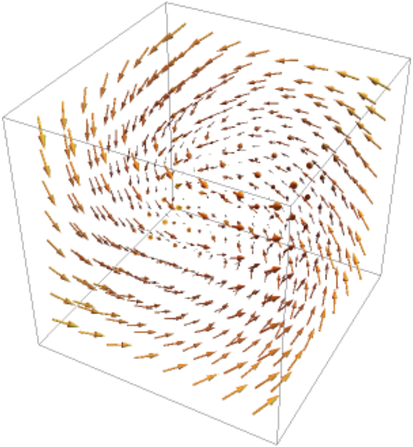
The standard contact structure ξ_{st} on $\mathbb{R}^3$.

A contact structure is just a very "twisty" way to assign a plane to each point.

Vector fields

Vector fields

A vector field X on a 3-manifold M is a way to smoothly assign a tangent vector $X_x \in T_xM$ to every $x \in M$.



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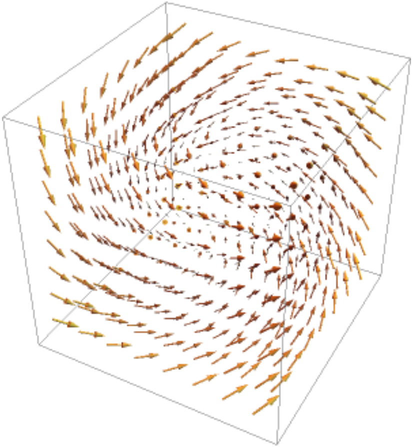
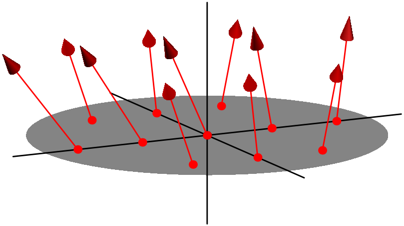


We call X a contact vector field if pushing ξ from x to a nearby point y along X takes ξ_x to ξ_y.

The dividing set

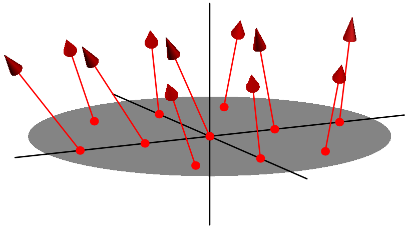
The dividing set

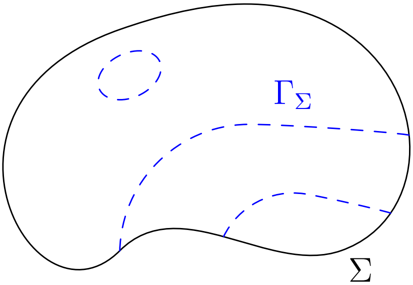
Call $\Sigma \subset M$ be a convex surface if there exists a contact vector field X such that $X_x \notin T_x\Sigma$ for any $x \in \Sigma$.



Call $\Sigma \subset M$ be a convex surface if there exists a contact vector field X such that $X_x \notin T_x\Sigma$ for any $x \in \Sigma$. The dividing set on a convex surface Σ is

$$\Gamma_\Sigma = \{x \in \Sigma : X_x \in \xi_x\}.$$





Giroux flexibility

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Giroux Flexibility Theorem

Let M be a fixed 3-manifold with boundary. If $\Sigma \subset M$ is convex, then Γ_Σ encodes ***all*** *relevant contact topological information about (M, ξ) on the neighborhood of Σ.*

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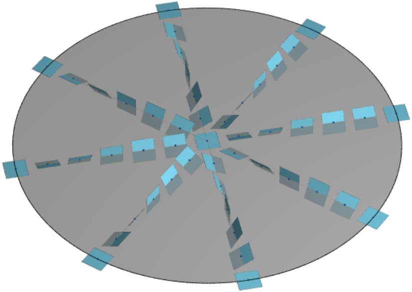
Main question

For every possible dividing set Γ_Σ on the boundary $\Sigma = \partial M$, how many contact structures have the dividing set Γ_Σ on Σ?

Tight contact structures

Call ξ tight if it is not overtwisted, i.e, if it does not contain an overtwisted disk:



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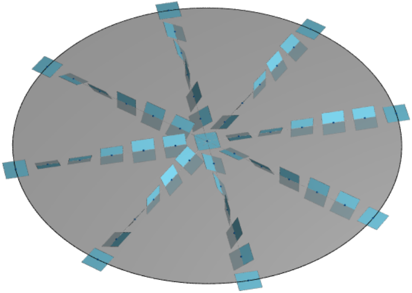

Overtwisted manifolds are fully understood, so we focus on tight contact structures, where the only complete result is on the 3-ball.

Section 2

Research: The solid torus

Dividing sets on the solid torus

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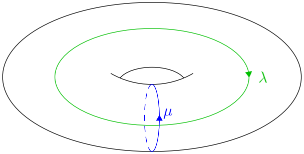
- $2n$ is the number of components of Γ;
- $-p$ is the number of times each component goes around λ;
- q is the number of times each component goes around μ.



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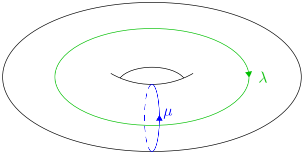

Research question

What is a formula for $N(n, -p, q)$, the number of tight contact structures on the solid torus M with dividing set $\Gamma = (n, -p, q)$ on the boundary ∂M?

Known results on the solid torus

When $(p, q) = (1, 1)$, define $r = 1$. Otherwise, write

$$-\frac{p}{q} = [r_0, r_1, \dots, r_k] = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \dots \frac{1}{r_k}}},$$

where $r_i \leq -2$ are integers and define

$$r = |(r_0 + 1)(r_1 + 1) \dots (r_{k-1} + 1) r_k|.$$

Theorem (Honda)

$$N(n, -1, 1) = C_n = \frac{1}{2n+1}\binom{2n}{n}$$

$$N(1, -p, q) = r$$

Main theorem

When $(p,q)=(1,1)$, define $r=s=1$. Otherwise, write

$$-\frac{p}{q}=[r_0,r_1,\ldots,r_k]=r_0-\frac{1}{r_1-\frac{1}{r_2-\ldots\frac{1}{r_k}}},$$

where $r_i\leq -2$ are integers and define

$$r=|(r_0+1)(r_1+1)\ldots(r_{k-1}+1)r_k|,$$
$$s=|(r_0+1)(r_1+1)\ldots(r_{k-1}+1)(r_k+1)|.$$

Theorem

The number of tight contact structures on $(n,-p,q)$ is

$$N(n,-p,q)=C_n((r-s)n+s),$$

where C_n is the n-th Catalan number.

Bypasses

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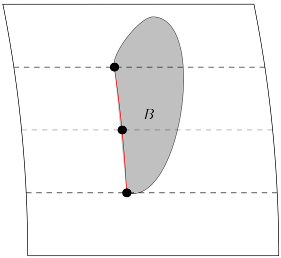
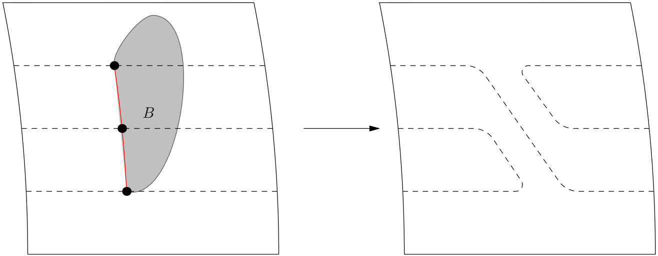
A bypass is a half-disk attached to ∂M and which intersects Γ at three points.



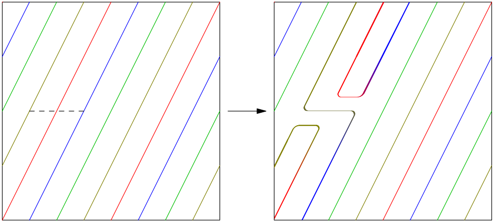
Bypasses

A bypass is a half-disk attached to ∂M and which intersects Γ at three points. It changes Γ based on the bypass attachment lemma.



Recurrence relation

Interior bypasses simplify Γ by decreasing n.



The dividing set goes from $(2, -2, 1)$ to $(1, -2, 1)$

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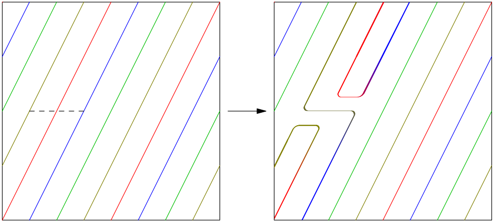

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This lets us use inclusion-exclusion to find a recurrence relation:

$$N(n, -p, q) = \sum_{k=1}^{n} (-1)^{k+1} a_{k,n} N(n-k, -p, q).$$

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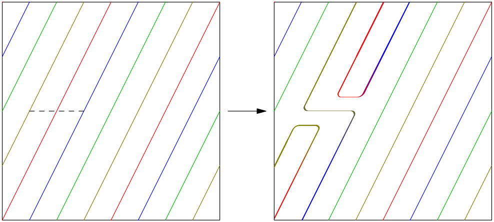

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Theorem

$$N(n, -p, q) = C_n((r-s)n + s)$$

Acknowledgments

- My mentor, Zhenkun Li
- PRIMES
- Professor John Etnyre
- Professor Ko Honda
- My family

Image credits

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- The image of ξ_{st} is due to Wikipedia.
- The image of a vector field was made using Mathematica.
- All other images were made using Asymptote.

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