The Center of the q-Weyl Algebra over Rings with Torsion

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17.-18. October 2020

The Weyl algebra and q-Weyl algebra

Definition

Let q and h be indeterminates. For a commutative ring R we define the **Weyl algebra**, **generalized Weyl algebra**, **the** q**-Weyl Alegbra**, and the **first** q**-Weyl algebra** over R as

$$W(R) = R\langle a, b\rangle/(ba - ab - 1),$$

$$W^h(R) = R\langle a, b\rangle/(ba - ab - h),$$

$$W_q(R) = R\langle a, a^{-1}, b, b^{-1}\rangle/(ba - qab),$$

$$W_q^{(1)}(R) = R\langle x, x^{-1}, y, y^{-1}\rangle/(yx - qxy - 1).$$

respectively.

The Weyl algebra and q-Weyl algebra

The ring of differential and q-differential operators over the affine space $\mathbb{A}^1_R$.

$$W(R) = R\langle x, \frac{\partial}{\partial x}\rangle, W_q^{(1)}(R) = R\left\langle x, \left(\frac{d}{dx}\right)_q \right\rangle.$$

We have $\partial x = x\partial + 1$ since

$$(xf(x))' = xf'(x) + f(x).$$

The q-Weyl algebra: q deformation

In $W(R)$,

$$a : f(x) \mapsto xf(x); b : f(x) \mapsto \frac{\partial f(x)}{\partial x}.$$

In $W_q(R)$,

$$a : f(x) \mapsto e^x f(x); b : f(x) \mapsto f(x + \log q).$$

In $W_q^{(1)}(q)$,

$$a : f(x) \mapsto xf(x); b : f(x) \mapsto \frac{df(x)}{dx}_q.$$

The center of the Weyl algebra

Known Results

Theorem

When R is a field of characteristic p, $Z(W(R))$ is generated by $a^p, b^p, pa^{p-1}, pb^{p-1}, \ldots$.

Theorem

When R is torsion-free and q is a root of unity of order l, $Z(W_q(R))$ is generated by a^l and b^l.

Interpolation: what happens when R is a ring with torsion and q is a root of unity?

The center of the Weyl algebra

Motivation

- A Kaledin's conjecture proven by Stewart and Vologodsky describes the center of the rings of differential operators on smooth varieties over $\mathbb{Z}/p^n\mathbb{Z}$ via Witt vectors.
- "Quantize" Stewart and Vologodsky's result in the simplest case: Weyl algebra $\rightarrow$ q-Weyl algebra.
- Roman Bezrukavnikov raises a question about possible interpolation between the two known results: what if $R = \mathbb{Z}/p^N\mathbb{Z}$ and q is a root of unity.

Witt vectors

Definition

Fix a non-negative integer n and a prime p, a **Witt vector** over a commutative ring R is a vector $(r_0, r_1, r_2, \ldots, r_n)$ with terms in R. Define the **"ghost component map"** from R^{n+1} to R as

$$w_n : (r_0, r_1, r_2, \ldots, r_n) \mapsto \sum_{i=0}^{n} p^i r_i^{p^{n-i}}.$$

We define the **Witt vector ring** $\mathbb{W}_n(R)$ consists of all the Witt vectors over R with addition and multiplication preserving the addition and multiplication of the ghost components.

Our results

We compute the center for

- $W^h(\mathbb{Z}/p^N\mathbb{Z})$;
- $W_q(\mathbb{Z}/p^N\mathbb{Z})/P(q)$ for monic P irreducible in $\mathbb{F}_p$;
- $(W_q(\mathbb{Z}/p^N\mathbb{Z})/(q^{p^n}-1))$ and $W_q(\mathbb{Z}/p^N\mathbb{Z})/(\Phi_{p^n}(q))$;
- $W_q^{(1)}(\mathbb{Z}/p^N\mathbb{Z})/P(q)$.

For simplicity, we write $R = \mathbb{Z}/p^N\mathbb{Z}$.

Our results

On the center of $W^h(R)$

Theorem

Let $h \in \mathbb{Z}/p^N\mathbb{Z}[q]$ be a polynomial of q. Then

$$Z(W^h(R)) \simeq \mathbb{W}_{N-\nu_p(h)}\left(R[\widetilde{a}, \widetilde{b}]\right)[q].$$

Our results

On the center of $W_q(R)$

Definition

For a polynomial $P \in \mathbb{Z}[q]$, define $M(P)$ to be the smallest positive integer such that $x^{M(P)} - 1$ is divisible by P in $\mathbb{F}_p[q]$, and $l(P)$ to be the greatest positive integer such that $x^{M(P)} - 1$ is divided by P in $\mathbb{Z}/p^{l(P)}\mathbb{Z}$.

Theorem

When monic polynomial $P \in R[q] = \mathbb{Z}/p^N\mathbb{Z}[q]$ is irreducible polynomial in $\mathbb{F}_p$, we have

$$Z(W_q(R)/P(q)) \simeq \mathbb{W}_{N-l(P)}(R[\widetilde{a}^{M(P)}, \widetilde{a}^{-M(P)}, \widetilde{b}^{M(P)}, \widetilde{b}^{-M(P)}])[q]/P(q).$$

Our results

On the center of $W_q(R)$, when q is a p^n-th root of unity

Theorem

$$Z(W_q(\mathbb{Z}/p^N\mathbb{Z})/(q^{p^n}-1)) \simeq \sum_{i=0}^{n} \frac{q^{p^n}-1}{q^{p^i}-1} R[\widetilde{a}^{p^i}, \widetilde{a}^{-p^i}, \widetilde{b}^{-p^i}, \widetilde{b}^{p^i}][q]/(q^{p^n}-1)).$$

Theorem

$$Z(W_q(\mathbb{Z}/p^N\mathbb{Z})/(\Phi_{p^n}(q)))$$
$$\simeq \left(\sum_{i=0}^{n-1} p^{N-1} \cdot \frac{\Phi_{p^n}(q)-p}{q^{p^i}-1} R[\widetilde{a}^{p^i}, \widetilde{a}^{-p^i}, \widetilde{b}^{p^i}, \widetilde{b}^{-p^i}][q]/(\Phi_{p^n}(q))\right)$$
$$+R[\widetilde{a}^{p^n}, \widetilde{a}^{-p^n}, \widetilde{b}^{p^n}, \widetilde{b}^{-p^n}][q]/(\Phi_{p^n}(q))).$$

Our results

On the center of $W_q^{(1)}(R)$

Theorem

When $P(1)$ is not a multiple of p, we have

$$Z(W_q^{(1)}(R)/P(q)) \simeq Z(W_q(R)/P(q)).$$

Corollary

If P is monic and irreducible modulo p, we have

$$Z(W_q^{(1)}(R)/P(q)) \simeq \mathbb{W}_{N-l(P)}(R[\widetilde{a}^{M(P)}, \widetilde{b}^{M(P)}, \widetilde{a}^{-M(P)}, \widetilde{b}^{-M(P)}])[q]/P(q).$$

Acknowledgments

We would like to thank:

- Mr. Calder Oakes Morton-Ferguson for mentorship,
- USA-PRIMES for this research opportunity,
- Prof. Roman Bezrukavnikov for helpful suggestions.

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