π-ish: Continued Fractions

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Outline

We'll cover...

- What are continued fractions?
- How do we represent rationals?
- How can we use continued fractions to approximate irrationals?

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Question

How do we approximate π*?*

π-ish

Question

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- 3

π-ish

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- 3.14

π-ish

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- $\frac{22}{7}$

π-ish

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$$\pi - \frac{314}{100} \approx .0015$$
$$\frac{22}{7} - \pi \approx .0013$$

Question

How does a fraction with a denominator of only 7 get so close?

A Weird Fraction

Example

Consider the number

$$a = 4 + \cfrac{1}{3 + \frac{1}{2+\frac{1}{3}}}.$$

We call this representation a continued fraction.

We evaluate this as follows:

$$a = 4 + \frac{1}{3 + \frac{3}{7}} = 4 + \frac{7}{24} = \frac{103}{24}.$$

The Other Direction

Question

Given a rational number $\frac{a}{b}$, how can we convert it into a continued fraction?

$$\frac{a}{b} = c + \frac{1}{\square}$$

1. Begin by taking the floor of $\frac{a}{b}$. Call it c. Take the reciprocal of the fractional part.

2. Take the floor of the fraction represented by the blue square. Call it d.

$$\frac{a}{b} = c + \frac{1}{d + \frac{1}{\square}}$$

The Other Direction: Continued

$$\frac{a}{b} = c + \frac{1}{d + \frac{1}{\square}}$$

3. Repeat the process for the fraction in the red square. Continue until there are only integers and proper fractions in the continued fraction representation.

$$\frac{a}{b} = c + \frac{1}{d + \frac{1}{e + \frac{1}{\square}}}$$

Continuing Fractions

Example

Consider the fraction $\frac{103}{24}$.

In order to convert it to a continued fraction, we can follow the process outlined in the previous slide:

$$\frac{103}{24} = 4 + \frac{1}{\frac{24}{7}}$$

Continuing Fractions

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Do We Terminate?

Proposition

The above process terminates no matter what positive rational number we start with.

$$\frac{103}{24} = 4 + \frac{1}{\frac{24}{7}}$$
$$= 4 + \frac{1}{3 + \frac{1}{\frac{7}{3}}}$$
$$= 4 + \frac{1}{3 + \frac{1}{2 + \frac{1}{\frac{3}{1}}}}.$$

Looking at this example, we see that the 24 in the denominator, moves to the numerator of the following iteration. We then take the floor, so the numerator becomes 7. There is never an opportunity for this number to get larger, because we take out the integral part when we floor.

Terminator

In the previous example, as we move through successive iterations, the denominator is strictly decreasing.

When we are converting any rational number to a continued fraction, we subtract a multiple of the denominator from the numerator. When we take the reciprocal, this numerator becomes the denominator of the next iteration.

Therefore, in every continued fraction representation, the denominators are strictly decreasing.

Eventually we must reach a denominator of 1, meaning our continued fraction terminates.

What Happens With Irrationals?

If our number is irrational, we can still take the floor and the reciprocal, so we follow the same process.

$$
\begin{aligned}
\pi &= 3 + .14159265358979323846264338327 9 \\
&= 3 + \frac{1}{7.0625133} \\
&= 3 + \frac{1}{7 + \frac{1}{15.996594}} \\
&= \cdots \\
&= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{\cdots}}}}
\end{aligned}
$$

This continued fraction is infinite.

Approximating Irrational Numbers

$$\pi = 3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{\cdots}}}}.$$

We can cut off this infinite fraction to get finite approximations:

$$3 = 3.0000$$

$$3 + \frac{1}{7} = \frac{22}{7} = 3.1428571$$

$$3 + \cfrac{1}{7 + \frac{1}{15}} = \frac{333}{106} = 3.14150943$$

$$3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{133} = 3.141592920$$

We call these convergents.

Good Convergents

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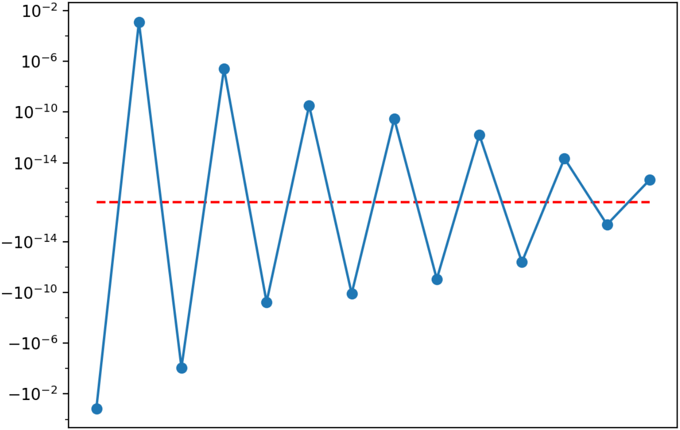

Convergents are the Best Approximations

It turns out that if we are using the convergent $\frac{7}{9}$ to approximate an irrational number, we can be confident that $\frac{7}{9}$ is a better approximation than $\frac{6}{8}$ or $\frac{5}{7}$.

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Theorem

If the following conditions hold:

- $\frac{a}{b}$ *is a fraction*
- α *is an irrational number*
- $\frac{h_n}{k_n}$ *is a convergent of* α
- $|\alpha - \frac{a}{b}| > |\alpha - \frac{h_n}{k_n}|$

then $b < k_n$.

Wow! These approximations are good.

From this theorem, we realize why our approximations are so good. They have to be the best of many options. For the convergent for pi, $\frac{333}{106}$, we can look at all the fractions between 3.14 and 3.15 with smaller denominators.

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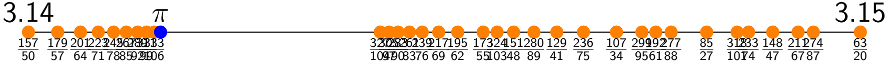

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The continued fractions of roots of quadratics have a really interesting property. Let's look at some examples.

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$$

$$\sqrt{2} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$$

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They always repeat! eventually.
It turns out that any repeating continued fraction can be expressed as the solution to the equation $ax^2 + by + c = 0$.

Repetition

$$1 + \cfrac{1}{\boxed{1 + \cfrac{1}{1 + \frac{1}{\cdots}}}}$$

We see that the two blue boxes are equal, if we set them equal to x we can solve the equation

$$x = 1 + \frac{1}{x}$$
$$0 = -x^2 + x + 1$$
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Regardless of the length of the beginning section of a continued fraction, if we know it repeats, we can guarantee that we can draw similar boxes.

Acknowledgements

Theorem

We are deeply grateful for all the help and support we have received.

Proof.

Maya, our mentor, has been the best, most patient teacher we could have asked for. We also want to thank Peter Haine for proof-reading our paper, organizing the program, and helping with this presentation. We couldn't have done it without PRIMES Circle and our parents. □