ON THE GENERATIONAL BEHAVIOUR OF GAUSSIAN BINOMIAL COEFFICIENTS AT ROOTS OF UNITY

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Abstract. The generational behavior of Gaussian binomial coefficients at roots of unity shadows the relationship between the reductive algebraic group in prime characteristic and the quantum group at roots of unity. In this paper, we study three ways of obtaining integer values from Gaussian binomial coefficients at roots of unity. We rigorously define the generations in this context and prove such behavior at primes power and two times primes power roots of unity. Moreover, we investigate and make conjectures on the vanishing, valuation, and sign behavior under the big picture of generations.

Introduction

In [Wil20], Williamson describes a "philosophy of generations," a conceptual and computational tool used to study the relationship between various groups in representation theory. In particular, the relationship between "Pascal's triangle mod p" (formed from binomial coefficients taken modulo a prime p) and the "q-binomial Pascal's triangle" (formed from q-binomial coefficients evaluated at a pth root of unity) is cited as motivation for a representation-theoretic correspondence between the modular representation theory of algebraic groups and the representation theory of Lusztig's quantum groups at roots of unity. In [Lus89], Lusztig exploits the combinatorial properties of q-binomial coefficients directly (particularly their vanishing) to study the representation theory of quantum groups. Furthermore, this generational approach proves fruitful in [LW18], [LW] in the study of tilting characters. Despite the success of this generational philosophy as a guiding principle in representation theory, it remains somewhat of a mystery, as [Wil20] explains that "we still lack a rigorous definition of generations (aside from the case of quantum groups at roots unity, which provide 'generation 1')."

In this project, we take a reverse approach: setting representation theory aside for a moment, we seek to focus only on the central combinatorial example of q-binomial coefficients at a pth root of unity which motivates this philosophy of generations. We seek to describe explicitly the generational behaviour present in this example, and to extend the combinatorial results which appear in [Lus89] with the aim of elevating this basic example of generational behaviour from a philosophical blueprint which motivates representation theory to a working prototype for the philosophy of generations where the objects involved, their generations, and the relationship between them are precisely defined and described. Along the way, we examine the vanishing and p-valuations of q-binomial coefficients at p^kth roots of

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unity, describing related fractal-like patterns which arise naturally from the study of generations.

We organize our paper as follow. In §1, we review some known results and present a definition of "integral q-binomial coefficients" built from Gaussian binomial coefficients at pth roots of unity, which directly generalizes the example (for $p = 3$) given in Section 1.8 of [Wil20] in an integer-valued way which is conducive to our study of generations; additionally, we present some related definitions which serve as useful tools in relating integral q-binomial coefficients with usual binomial coefficients as well as interesting combinatorial objects in their own rite. In §2, we present our main result, formulating and proving a generational relationship between these integral q-binomial coefficients and usual binomial coefficients mod p. In §3, we discuss related combinatorial properties of integral q-binomial coefficients and their cousins from §1.

1. Integral q-binomial coefficients

We first recall the definition of q-binomial coefficients.

Definition 1.1. Let q be an indeterminate. For any integer $N \geqslant 0$ we define

$$[N]_q! = \prod_{s=1}^{N} \frac{q^s - q^{-s}}{q - q^{-1}}.$$

For non-negative integers $m \geqslant n$, we define the *Gaussian binomial coefficients*

$$\binom{m}{n}_q = \frac{[m]_q!}{[n]_q! \cdot [m-n]_q!}.$$

Additionally, we define $\binom{m}{n}_1 = \binom{m}{n}$, and when $m < n$, we define $\binom{m}{n}_q = 0$.

In [Wil20], Pascal's triangle mod 3 is presented alongside the "Pascal's triangle" which is obtained by replacing the binomial coefficients in Pascal's triangle with their q-analogues (we call this the q-binomial Pascal's triangle) and setting q equal to a 3rd root of unity.

This example suggests a connection between these two combinatorial objects; at a basic level, Figure 2 seems to serve as a first approximation to Figure 1. In the next section, we prove a more precise formulation of this basic observation, and observe that this pattern does not stop with evaluating at a 3rd root of unity, but rather evaluating at 3^kth roots of unity reveals even more "approximations" of the picture on the left picture.

We are lucky in this example that evaluating a q-binomial coefficients at a 3rd root of unity always yields an integer. This is not the case for primes greater than 3. Accordingly, we define "integral q-binomial coefficients" to address this issue, a family of coefficients built from q-binomial coefficients which both generalize this example and always take integer values.

The first pattern we can see in the figures shown is that the zero entries in Figure 1 are also zero entries in Figure 2. Further, the non-zero entries in Figure 1 can be viewed as a collection of small "sub-triangles" with three entries along their base. These small triangles imitate the small triangle at the very top by multiplying its entries by a usual binomial coefficient. This observation is slightly different when we set q to be an even root of unity. Another way to state this observation is the following theorem, part (a) of which appears in [Lus89].

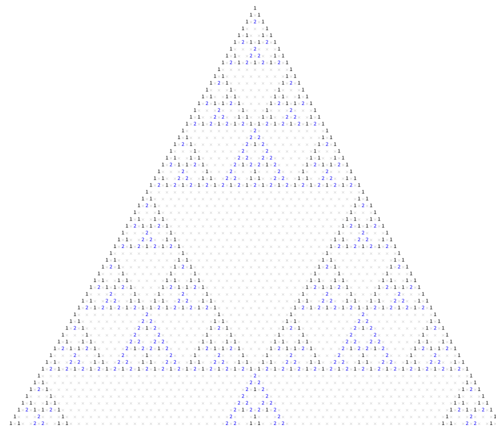

FIGURE 1. $\binom{m}{n} \pmod 3$

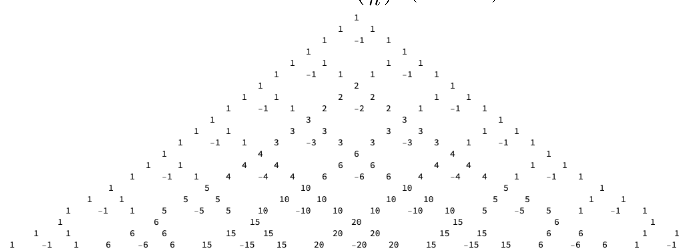

FIGURE 2. $\binom{m}{n}_{e^{\frac{2\pi i}{3}}}$

Theorem 1.2 ([Lus89]). *(a) Let s be an odd positive integer. If v is a primitive s-th root of unity, and $0 \leqslant m, n \leqslant s-1$, then*

$$\binom{m+as}{n+bs}_v = \binom{a}{b} \cdot \binom{m}{n}_v.$$

(b) Let $s = 2r$ be an even positive integer. If v is a primitive s-th root of unity, and $0 \leqslant m, n \leqslant r-1$, then

$$\binom{m+ar}{n+br}_v = (-1)^{b(a-b)} \binom{a}{b} \binom{m}{n}_v.$$

We also now recall Kummer's theorem, which is an essential tool for proving our main result.

Theorem 1.3 (Kummer). *For p prime, let m, n be two non-negative integers. Then $v_p\binom{m}{n}$ is the number of carriers when n is added on $m-n$ in base p. Namely,*

$$v_p \binom{m}{n} = \frac{1}{p-1} \left(S_p(n) + S_p(m-n) - S_p(m) \right),$$

where $S_p(k)$ denotes the sum of digits when the number k is written in base p.

The relationship between the Gaussian polynomial and the theory of partition is also well-known. Explicitly, it is stated as follows. The coefficients of the Gaussian polynomial is the number of partitions to k in a $(m-n) * n$ rectangle. For a proof see, e.g. [Sta11].

Theorem 1.4. *The Gaussian binomial coefficients can be written as follows:*

$$\binom{m}{n}_v = \sum_{k=0}^{(m-n)n} p(m-n,n;k)v^{2k-(m-n)n},$$

where $p(M,N;k)$ is number of partitions of k with at most N parts and each part no greater than N.

1.1. **Integral q-binomial coefficients.** We consider now three possible ways to define q-binomial variants at roots of unity by summing in a way that takes values in the integers.

Definition 1.5. Let q be a positive integer. Denote

$$\binom{m}{n}_q^{\bullet} = \sum_{\gcd(k,q)=1, 1\leqslant k\leqslant q-1} \binom{m}{n}_{e^{2\pi i\frac{k}{q}}};$$

$$\binom{m}{n}_q^{*} = \sum_{k=1}^{q-1} \binom{m}{n}_{e^{2\pi i\frac{k}{q}}}; \text{ and}$$

$$\binom{m}{n}_q^{\dagger} = \sum_{k=0}^{q-1} \binom{m}{n}_{e^{2\pi i\frac{k}{q}}}.$$

It is a simple check that each of these constructions takes integer values for q, m, n positive integers. The main player in our description of generational behaviour is the second definition, $\binom{m}{n}_q^{*}$, which we call an integral q-binomial coefficient; however, the other two definitions are useful in technical ways - in particular, the third has a more combinatorial interpretation which we now exploit.

1.2. **The relationship between these three constructions.** We consider the relationships between the three ways of summing in Definition 1.5. They are easy to prove but provide an important set-up for our future discussions.

Proposition 1.6. *Let q be a positive integer, then*

(a) We have the identity

$$\binom{m}{n}_q^{\dagger} = q \sum_{q|(2k-(m-n)n)} p(m-n,n;k),$$

where $p(m-n,n;k)$ denotes the number of partitions of k which have no more than $m-n$ parts and such that each part does not exceed n.

(b) $\binom{m}{n}_q^{\dagger} = \sum_{d|q} \binom{m}{n}_d^{\bullet}$.

(c) $\binom{m}{n}_q^{\dagger} = \binom{m}{n}_q^{*} + \binom{m}{n}$.

Proof. The first statement follows from

$$\binom{m}{n}_q^{\dagger} = \sum_{\alpha=-n(m-n)}^{n(m-n)} [v^{\alpha}]\binom{m}{n}_v \sum_{k=0}^{q-1} e^{2\pi i\frac{k}{q}\alpha}$$

and the property of $\sum_{k=0}^{q-1} e^{2\pi i\frac{k}{q}\alpha}$ directly.

The second statement can be proved considering that the term of $e^{2ki\pi/q}$ appears only once on the right hand side.

Recall that a Gaussian binomial coefficient may be factored into a product of cyclotomic polynomials. By Definition 1.5, we have

$$\binom{m}{n}_q^{\dagger} = \binom{m}{n}_q^{*} + \binom{m}{n}_1 = \binom{m}{n}_q^{*} + \binom{m}{n}.$$

So the third statement follows. □

Proposition 1.7. *For any odd integer q,*

$$\binom{m}{n}_q^{*} \equiv -\binom{m}{n} \pmod{q}.$$

Proof. By Proposition 1.6, we have

$$\binom{m}{n}_q^{*} \equiv \binom{m}{n}_q^{\dagger} - \binom{m}{n} \equiv -\binom{m}{n} \pmod{q},$$

as desired. □

2. Generational behaviour

We now come to our main result. We describe the triangle formed by the coefficients $\binom{m}{n} \pmod{p}$ as being built from "building blocks": the family of triangles formed by taking the p-valuation of $\binom{m}{n}_{p^k}^{*}$. We can observe and describe this behaviour graphically.

Let us return to $p = 3$ as an example. The following figure gives the 3-valuation of the Pascal Triangle. The first small layers of the graph coincide partially with the zero-entries and 3-valuations of $\binom{m}{n}_{3^k}^{*}$, where k is an integer, as shown in Figures 4, 5, and 6.

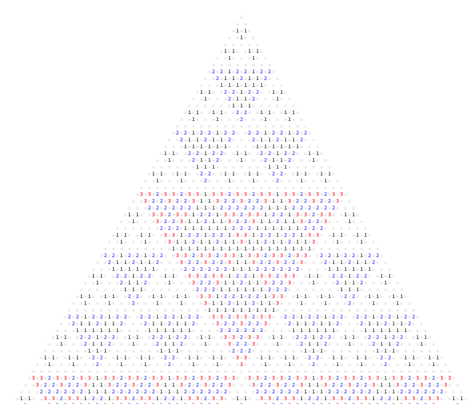

Figure 3. 3-valuations of $\binom{m}{n}$

In this section we formulate and prove this behaviour in more precise terms for all odd primes p.

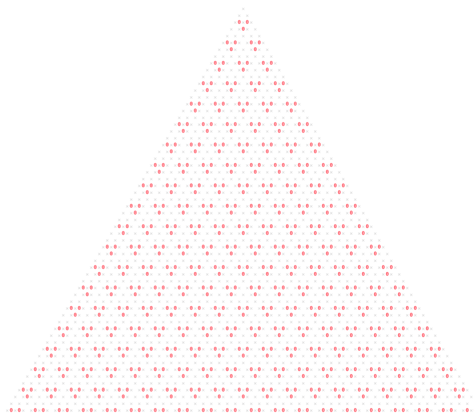

FIGURE 4. Zeros of $\binom{m}{n}_3^*$

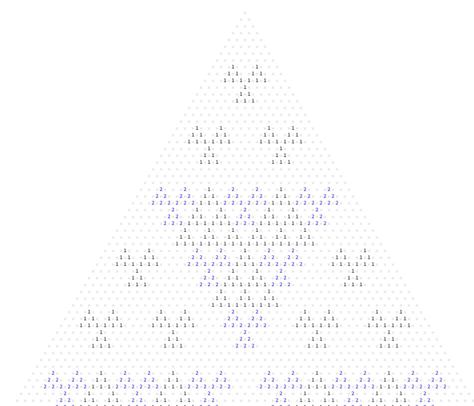

FIGURE 5. 3-valuations of $\binom{m}{n}_3^*$

Definition 2.1. Fix an odd prime p. For a nonnegative integer k, we define the *kth generation vanishing* in the mod p Pascal's triangle to be the set of pairs (m, n) for which there are carriers the last k digits of its base p expansion, which is equivalent to

$$\binom{p^k + m \,(\mathrm{mod}\ p^k)}{n \,(\mathrm{mod}\ p^k)} \equiv 0 \pmod{p^k}.$$

The equivalence is proven in Corollary 2.4.

Using this definition, we now can formulate our main result.

Theorem 2.2. *Fix an odd prime p. The kth generation vanishing in the mod p Pascal's triangle corresponds exactly to the vanishing of $\binom{m}{n}_{p^k}^*$; in other words, $\binom{m}{n}_{p^k}^*$ vanishes if and only if (m, n) belongs to the kth generation vanishing of the mod p Pascal's triangle. Furthermore, if (m, n) does not belong to the kth generation vanishing, then*

$$v_p \binom{m}{n}_{p^k}^* = v_p \binom{m}{n}.$$

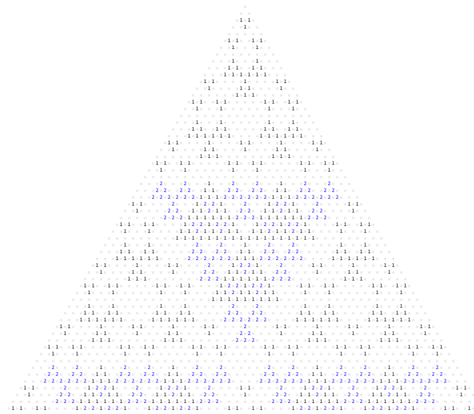

FIGURE 6. 3-valuations of $\binom{m}{n}^*_{3^2}$

2.1. **Proof of Theorem 2.2.**

Lemma 2.3. *For integer k, if $1 \leqslant n \leqslant m \leqslant p^k - 1$, then*

$$v_p \binom{m}{n} < k.$$

Proof. Since $1 \leqslant n \leqslant m \leqslant p^k - 1$, each of m, n, and $m-n$ must have at most k digits in their base p expansions. Thus, when n is added to $m-n$, only the 2nd, 3rd, $\cdots$, k-th digits from the left in their base p expansions permit carriers. Then by Kummer's theorem, $v_p\binom{m}{n}$ is the number of such carriers, which is no greater than $k-1$. □

This lemma immediately gives the following corollary, which provides additional motivation for Definition 2.1.

Corollary 2.4. *For $k \geqslant 1$, The following conditions are equivalent.*

(a) $v_p\binom{p^k+m \pmod{p^k}}{n \pmod{p^k}} \geqslant k$.
(b) $v_p\binom{m \pmod{p^{k+1}}}{n \pmod{p^{k+1}}} = k$.
(c) $\binom{m}{n}$ *belongs to the kth generation vanishing of the mod p Pascal's triangle.*
(d) When n is added to $m-n$, there are carriers the last k digits of its base p expansion.
(e) Write $m = \sum_{i\geqslant 0} m_i p^i, n = \sum_{i\geqslant 0} n_i p^i$; then $m_j < n_j$ for all $j = 0\ldots, k-1$.

Proof. (d) and (e) are equivalent by base-p operations. (a) and (b) are equivalent due to Lemma 2.3. (c) and (e) are equivalent due to Definition 2.1. (b) and (e) are equivalent due to Theorem 1.3. □

Note that a direct implication of (e) and Kummer's theorem is $v_p\binom{m}{n} \geqslant k$. So all the (m,n) in the k-th generation satisfies $v_p\binom{m}{n} \geqslant k$.

We now proceed with the proof of Theorem 2.2.

Proof of Theorem 2.2. We perform induction on k, then on m. When $k=0$, we note that all (m,n) belong to the 0th generation vanishing, while all integral p^k-binomial coefficients are zero (the empty sum) by definition. Similarly this occurs when $m=0$.

We proceed by two steps.

Step 1. Find all the zeros when $0 \leqslant n, m-n < p^k$.

By Proposition 1.6,

$$\binom{m}{n}^*_{p^k} = \binom{m}{n}^\dagger_{p^k} - \binom{m}{n} = p^k \sum_{p^k|(2t-(m-n)n)} p(m-n,n;t) - \binom{m}{n}.$$

Modulo p^k on both sides we get

$$\binom{m}{n}^*_{p^k} \equiv -\binom{m}{n} (\bmod\ p^k).$$

We first deal with the zeros.

If

$$\binom{m}{n}^*_{p^k} = 0,$$

we must have $v_p\binom{m}{n} \geqslant k$. Since $0 \leqslant n, m-n \leqslant p^k - 1$, we have $1 \leqslant n, m \leqslant 2(p^k-1) < p^{k+1} - 1$. By Corollary 2.4, $v_p\binom{m}{n} = k$ must be in the k-th generation.

Conversely, we show that $\binom{m}{n}^*_{p^k} = 0$ for all (m,n) belonging to the kth generation vanishing with $0 \leqslant n, m-n < p^k$. We do this by first proving

$$\binom{m}{n}^\bullet_{p^l} = 0$$

for every $0 \leqslant l \leqslant k$. First note that if $m \leqslant p^k - 1$, by Lemma 2.3, $v_p\binom{m}{n} = k$ never happens. So all such (m,n) satisfies that $1 \leqslant n \leqslant p^k - 1 < m \leqslant 2(p^k-1)$. And when n is added to $m-n$, there are carriers at every last k digits from the left. When $l = k$, suppose that $m = p^k + t$. We have $0 \leqslant t = m - p^k < m - n$. By Theorem 1.2, for v a primitive p^k-th root of unity,

$$\binom{m}{n}_v = \binom{p^k + t}{m-n}_v = \binom{1}{0} \cdot \binom{t}{m-n}_v = 0$$

for each p^k-th primitive root v. When $1 \leqslant l \leqslant k-1$, similarly, by Theorem 1.2, it's sufficient to show that

$$\binom{m (\bmod\ p^l)}{n (\bmod\ p^l)}_v = 0.$$

Since when n is added to $m-n$, there is a carrier at every digit, we have $m(\bmod\ p^l) < n(\bmod\ p^l)$, thus $\binom{m(\bmod\ p^l)}{n(\bmod\ p^l)}_v$ must be zero.

Therefore, by Proposition 1.6,

$$\binom{m}{n}^*_{p^k} = \sum_{l=0}^{k} \binom{m}{n}^\bullet_{p^l} = 0,$$

as desired.

Step 2. Finish the rest.

We write the p base expansion of m and n as

$$m = \sum_{i \geqslant 0} m_i p^i, n = \sum_{i \geqslant 0} n_i p^i.$$

Without loss of generality, say m and n all have t digits (we permits the starting digits to be 0 to make sure they have the same number of digits, for the sake of simplicity). By Proposition 1.6, we have

$$\begin{aligned}
\binom{m}{n}^*_{p^k} &= \sum_{i=1}^{k} \binom{m}{n}^\bullet_{p^i} \\
&= \binom{m_tp^t + \cdots m_1p}{n_tp^t + \cdots n_1p} \cdot \binom{m_0}{n_0}^\bullet_p \\
&+ \binom{m_tp^t + \cdots m_2p^2}{n_tp^t + \cdots n_2p^2} \cdot \binom{m_1p + m_0}{n_1p + n_0}^\bullet_{p^2} \\
&+ \cdots \\
&+ \binom{m_tp^t + \cdots m_kp^k}{n_tp^t + \cdots n_kp^k} \cdot \binom{m_{k-1}p^{k-1} + \cdots + m_1p + m_0}{n_{k-1}p^{k-1} + \cdots + n_1p + n_0}^\bullet_{p^k}.
\end{aligned}$$

Here we split into three cases.

Case 1. $\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}^\bullet_{p^k}$ vanishes.

In this case,

$$\binom{m}{n}^*_{p^k} = \sum_{i=1}^{k-1} \binom{m}{n}^\bullet_{p^i} = \binom{m}{n}^*_{p^{k-1}}.$$

If it is not zero, the valuation result follows by inductive assumption. If it is zero, by inductive assumption, (m, n) is in the $(k-1)$st generation, having the each last $(k-1)$ digits with a carrier. In this case, all $\binom{m_ip^i+\cdots+m_1p+m_0}{n_ip^i+\cdots+n_1p+n_0}^\bullet_{p^{i+1}}$ vanishes for all $i = 0, \ldots, k-1$. We have $\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}^*_{p^k} = 0$. By Theorem 1.2, we have

$$\binom{p^k + m_{k-1}p^{k-1} + \cdots + m_1p + m_0}{n_{k-1}p^{k-1} + \cdots + n_1p + n_0}^*_{p^k} = \binom{1}{0}\binom{m_{k-1}p^{k-1} + \cdots + m_1p + m_0}{n_{k-1}p^{k-1} + \cdots + n_1p + n_0}^*_{p^k} = 0.$$

By Step 1, $\binom{p^k+m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}$ is in the k-th generation. By Corollary 2.4, $m_i < n_i$ for all $i = 0, \ldots, k-1$. Again, by Corollary 2.4, $\binom{m}{n}$ is in the k-th generation, as desired.

Case 2. $\binom{m_ip^i+\cdots+m_1p+m_0}{n_ip^i+\cdots+n_1p+n_0}^\bullet_{p^{i+1}}$ vanishes for all $i = 0, 1, \cdots, k-2$, but not for $i = k-1$.

Now

$$\begin{aligned}
\binom{m_ip^i + \cdots + m_1p + m_0}{n_ip^i + \cdots + n_1p + n_0}^*_{p^k} &= \binom{m_ip^i + \cdots + m_1p + m_0}{n_ip^i + \cdots + n_1p + n_0}^\bullet_{p^k} \\
&= \binom{m_tp^t + \cdots m_kp^k}{n_tp^t + \cdots n_kp^k} \cdot \binom{m_{k-1}p^{k-1} + \cdots + m_1p + m_0}{n_{k-1}p^{k-1} + \cdots + n_1p + n_0}^\bullet_{p^k}.
\end{aligned}$$

Since $\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}^\bullet_{p^k}$ does not vanish, the k-th digit from the right side must not have carrier (otherwise it vanishes). Therefore,

$$v_p\binom{m}{n} = v_p\binom{m_tp^t + \cdots m_kp^k}{n_tp^t + \cdots n_kp^k} + v_p\binom{m_{k-1}p^{k-1} + \cdots + m_1p + m_0}{n_{k-1}p^{k-1} + \cdots + n_1p + n_0}.$$

It is sufficient to show

$$v_p\binom{m_{k-1}p^{k-1}\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}=v_p\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}^{\bullet}_{p^k}.$$

In this case, we only need to consider both m and n are smaller than p^k. Note again that by the assumption of Case 2,

$$\binom{m}{n}^*_{p^k}=\binom{m}{n}^{\bullet}_{p^k}.$$

The right hand side is a multiple of p^{k-1} and

$$\binom{m}{n}^*_{p^k}\equiv-\binom{m}{n}(\mathrm{mod}\ p^k)$$

by Proposition 1.6. Therefore, we must have $v_p\binom{m}{n}\geqslant k-1$, which holds, as shown in Corollary 2.4, only when the equality holds. Therefore, we have $v_p\binom{m}{n}^*_{p^k}=v_p\binom{m}{n}=k-1$ when both m,n are less than p^k. As desired.

In this case, $v_p\binom{m}{n}^*_{p^k}=k-1$ is not 0.

Case 3. $\binom{m_ip^i+\cdots+m_1p+m_0}{n_ip^i+\cdots+n_1p+n_0}^{\bullet}_{p^{i+1}}$ does not vanish for $i=k-1$ and is not zero for at least one of $i=1,2,\cdots,k-2$.

In this case, we prove that

$$\begin{aligned}v_p\binom{m}{n}^*_{p^{k-1}}&<v_p\binom{m}{n}^{\bullet}_{p^k}\\&=v_p\binom{m_tp^t+\cdots m_kp^k}{n_tp^t+\cdots n_kp^k}+v_p\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}^{\bullet}_{p^k}.\end{aligned}$$

By the inductive assumption,

$$v_p\binom{m}{n}^*_{p^{k-1}}=v_p\binom{m}{n}.$$

Again, there is no carriers at the k-th digit, we have

$$v_p\binom{m}{n}=v_p\binom{m_tp^t+\cdots m_kp^k}{n_tp^t+\cdots n_kp^k}+v_p\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}.$$

Again, we only need to show

$$v_p\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}<v_p\binom{m_{k-1}p^{k-1}+\cdots+m_1p+m_0}{n_{k-1}p^{k-1}+\cdots+n_1p+n_0}^{\bullet}_{p^k},$$

which is the case for both m,n less than p^k. Suppose that $m,n<p^k$, by Corollary 2.4, the left hand side is at most $k-1$.

If $v_p\binom{m}{n}=k-1$, by inductive assumption, $\binom{m}{n}$ is in the $(k-1)$st generation, so $\binom{m}{n}^*_{p^{k-1}}$ vanishes. In this case, all $\binom{m_ip^i+\cdots+m_1p+m_0}{n_ip^i+\cdots+n_1p+n_0}^{\bullet}_{p^{i+1}}$ vanishes when $i=0,\ldots,k-2$; this is a contradiction.

So $v_p\binom{m}{n}\leqslant k-2$. By Proposition 1.6, the right hand side is at least $k-1$. This inequality is proven.

Utilizing this bounding, we get

$$\binom{m}{n}^*_{p^k} = \binom{m}{n}^*_{p^{k-1}} + \binom{m}{n}^\bullet_{p^k}$$

is a sum of two integers with two different p valuations. The p valuation of the left side is the less p valuation on the other side, which is $v_p\binom{m}{n}^*_{p^{k-1}}$. So

$$v_p\binom{m}{n}^*_{p^k} = v_p\binom{m}{n}^*_{p^{k-1}} = v_p\binom{m}{n},$$

by inductive assumption, as desired.

Since all pairs (m, n) belong to one of the cases treated above, this completes the proof.

□

3. Further results

3.1. **Generalizations and axiomatic results.** Along with our main theorem on the generational behaviour of integral binomial coefficients, we have explored a few other directions. This section will outline a few of them and point out possible future directions. The first direction is the following axiomatization of combinatorial objects which display generational behaviour with respect to Pascal's triangle in the same way our coefficients do.

Definition 3.1. Call a function $f : (\mathbb{Z}^*)^3 \to \mathbb{Z}^*$ *p-generational* if

- $f(0, m, n) = \binom{m}{n}$.
- $f(k, m, n) = 0$ when $m < n$.
- $f(k, ap^k + m, bp^k + n) = \binom{a}{b} f(k, m, n)$ when $1 \leqslant m, n < p^k$.
- $p^{k-1} | f(k, m, n)$.
- $p^k | \sum_{i=0}^k f(i, m, n)$.

Theorem 3.2. *For any p-generational function f, $f(k, m, n) = 0$ if and only if $\binom{m}{n}$ is in the k-th generation. And*

$$v_p\left(\sum_{i=1}^{k} f(k, m, n)\right) = v_p\binom{m}{n}$$

otherwise.

Proof. The proof of this theorem is very similar to the proof of Theorem 2.2. All the main steps and techniques are identical. □

Our results from §2 prove that integral q-binomial coefficients at a p^kth root of unity serve as an example of a p-generational object as we define in Definition 3.1. We expect that future investigation will reveal other interesting examples of families of combinatorial objects satisfying these axioms.

3.2. **Zeroes of other constructions.** As a short addendum, our work on generational behaviour has also led us to results and conjectures on the zeros of the other coefficients we defined in Definition 1.5.

Theorem 3.3. *When q is an odd integer,*

$$\binom{m}{n}_q^\dagger = 0$$

holds if and only if n is odd, m is even, and $1 \leqslant n(m-n) \leqslant q-1$. When q is even,

$$\binom{m}{n}_q^\dagger = 0$$

holds if and only if n is odd and m is even.

To prove Theorem 3.3, we need to introduce the constant coefficient of Gaussian polynomials.

Definition 3.4. Let $\binom{m}{n}^G$ be the constant coefficient of Gaussian polynomial $\binom{m}{n}_v$.

From Theorem 1.4, $\binom{m}{n}^G$ can be easily interpreted as the number of partition inside the $(m-n)\times n$ rectangle occupying exactly half of its area, as shown in the following corollary.

Corollary 3.5.

$$\binom{m}{n}^G = p(m-n, n; (m-n)n/2).$$

Proof. By Definition 3.4 and Theorem 1.4, $\binom{m}{n}^G$ can be easily interpreted as the number of partition inside the $(m-n)\times n$ rectangle occupying exactly half of its area, and the corollary immediately follows. □

When (m, n) are small enough, $\binom{m}{n}^G$ nicely represent the value of $\binom{m}{n}_p^\dagger$.

Proposition 3.6. *For $1 \leqslant n(m-n) \leqslant p-1$,*

$$\binom{m}{n}_p^\dagger = p\binom{m}{n}^G.$$

Proof. The highest degree of the Gaussian polynomial $\binom{m}{n}_v$ is $n(m-n) < p$; similarly, the term with lowest degree has exponent at least $n(n-m) > -p$. When summing all roots of unity, the term with degree k is equal to $p[v^i]\binom{m}{n}_v$ if $p|k$ and 0 otherwise, where $\binom{m}{n}_v = \sum[v^i]\binom{m}{n}_v v^i$. Because the degree is less than p, the only nonzero term is equal to $p\binom{m}{n}^G$. □

Corollary 3.7. *If and only if m is even and n is odd,*

$$\binom{m}{n}^G = 0.$$

Proof. By Corollary 3.5, $\binom{m}{n}^G$ is the number of partitions with area $(m-n)n/2$ of a $(m-n)*n$ rectangle. If m is even and n is odd, $(m-n)n/2$ is not an integer, so no such partitions exist. Otherwise, there must be at least one permutation with half of the area: the left half of the rectangle or the lower half of the rectangle since $m-n$ and n has at least one even value. Thus, $\binom{m}{n}^G = 0$ if and only if m is even and n is odd, as desired. □

The following result summarizes the zeros of this triangle. Its proof is directly from the corollary above.

Proposition 3.8. *If and only if m is even and n is odd,*

$$\binom{m}{n}^G = 0.$$

Proof. We first recognize the fact that $\binom{m}{n}^G$ is the number of partitions with area $(m-n)n/2$ of a $(m-n)*n$ rectangle. If m is even and n is odd, $(m-n)n/2$ is not an integer, so no such partitions exist. Otherwise, there must be at least one permutation with half of the area: the left half of the rectangle or the lower half of the rectangle since $m-n$ and n has at least one even value. Thus, $\binom{m}{n}^G = 0$ if and only if m is even and n is odd, as desired. □

Now, we shall go back to the proof for Theorem 3.3.

Proof of Theorem 3.3. When q is odd, consider

$$b_k = [v^i]\binom{m}{n}_v \sum_{i=1}^{n} e^{\frac{2\pi i k}{q}},$$

the sum of the k^{th} term when summing all roots of unity. When n is not odd or m is not even, by Proposition 3.8, $\binom{m}{n}^G$ is greater than 0. So $\binom{m}{n}^\dagger$ is also greater than 0 (all the coefficients are non-negative). When $1 \leqslant n(m-n) \leqslant q-1$, the degree of the polynomial is at most $q-1$, and we only need to consider the constant term. By Lemma 3.8, $\binom{m}{n}^G = 0$ if and only if m is even and n is odd, we have $\binom{m}{n}_q^\dagger = 0$ if n is odd, m is even, and $1 \leqslant n(m-n) \leqslant q-1$. Conversely, if n is odd, m is even, but $n(m-n) \geqslant q$, we have $\binom{m}{n}_q^\dagger \geqslant p(n, m-n; (n(m-n)-p)/2) + p(n, m-n; (n(m-n)+p)/2)$. By the same argument as the proof of Lemma 3.8, $p(n, m-n; (n(m-n)-p)/2) > 0$ (such partition exist because $n(m-n)-p$ now is even). Similarly, when q is even, if m is even, and n is odd, a_k can be non-zero only if k is a multiple of q. However, with the combinatorial interpretation from Proposition 1.6, all terms with even degree must be zero, so $\binom{m}{n}_q^\dagger = 0$ if q is even, n is odd, and m is even. If m is odd or n is not odd, by Lemma 3.8, $\binom{m}{n}^G$ will not be zero, and thus $\binom{m}{n}_q^\dagger \geqslant \binom{m}{n}^G$ will not be zero. □

Theorem 3.9. *For odd prime p and positive integer k, when one of the following holds:*

- $n(\mathit{mod}\ p^k) \cdot ((m-n)(\mathit{mod}\ p^k))$ *is odd and is less than* p^{k-1}*;*
- $(q-1-m(\mathit{mod}\ p^k)) \cdot n(\mathit{mod}\ p^k)$ *is odd and is less than* p^{k-1}*;*
- $(q-1-m(\mathit{mod}\ p^k)) \cdot ((m-n)(\mathit{mod}\ p^k))$ *is odd and is less than* p^{k-1}*;*
- $n(\mathit{mod}\ p^k) > m(\mathit{mod}\ p^k))$*,*

we have

$$\binom{m}{n}_{p^k}^\bullet = 0.$$

Proof. If $n(\text{mod } p^k) \cdot (m-n)(\text{mod } p^k)$, we write

$$\binom{m}{n}_{p^k}^\bullet = \binom{m}{n}_{p^k}^\dagger - \binom{m}{n}_{p^{k-1}}^\dagger.$$

By Theorem 3.3, when $n(\mathrm{mod}\ p^k)\cdot(m-n)(\mathrm{mod}\ p^k)$ is odd and is less than p^{k-1}, both $\binom{m}{n}^{\dagger}_{p^k}$ and $\binom{m}{n}^{\dagger}_{p^{k-1}}$ are 0. So $\binom{m}{n}^{\bullet}_{p^k}=0$, as desired.

The other three cases are the same as the first case by the following proposition:

$$\left\|\binom{m}{n}^{\bullet}_{d}\right\|=\left\|\begin{bmatrix}m\\m-n\end{bmatrix}^{\bullet}_{d}\right\|=\left\|\begin{bmatrix}q-m+n-1\\n\end{bmatrix}^{\bullet}_{d}\right\|$$

$$=\left\|\begin{bmatrix}q-m+n-1\\q-m-1\end{bmatrix}^{\bullet}_{d}\right\|=\left\|\begin{bmatrix}q-n-1\\m-n\end{bmatrix}^{\bullet}_{d}\right\|=\left\|\begin{bmatrix}q-n-1\\q-m-1\end{bmatrix}^{\bullet}_{d}\right\|.$$

We first show that $\binom{m}{n}_v=\begin{bmatrix}m\\m-n\end{bmatrix}_v$. This is trivial since

$$\binom{m}{n}_v=\frac{[m]_v!}{[n]_v![m-n]_v!}=\begin{bmatrix}m\\m-n\end{bmatrix}_v$$

We then show $\left\|\binom{m}{n}_v\right\|=\left\|\begin{bmatrix}q-m+n-1\\n\end{bmatrix}_v\right\|$.

$$\|[k]_v\|=\left\|\frac{v^k-v^{-k}}{v-v^{-1}}\right\|=\left\|\frac{v^{q-k}-v^{k-q}}{v-v^{-1}}\right\|=\|[q-k]_v\|$$

Therefore,

$$\begin{aligned}\left\|\binom{m}{n}_v\right\|&=\left\|\frac{[m]_v!}{[n]_v![m-n]_v!}\right\|\\&=\left\|\frac{\prod_{i=m-n+1}^{m}[i]_v}{[n]_v!}\right\|\\&=\left\|\frac{\prod_{i=q-m}^{q-m+n-1}[i]_v}{[n]_v!}\right\|\\&=\left\|\frac{[q-m+n-1]_v!}{[n]_v![q-m-1]_v!}\right\|\\&=\left\|\begin{bmatrix}q-m+n-1\\n\end{bmatrix}_v\right\|.\end{aligned}$$

Manipulating the last three values with these two conclusions establishes the six values are equal. □

Conjecture 3.10. *If* $\binom{m}{n}^{\bullet}_{p^k}=0$, *one of the criteria in Theorem 3.9 is met.*

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